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Research paper

Measurement-based perturbation theory and differential equation parameter estimation with applications to satellite gravimetry

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ABSTRACT

The numerical integration method has been routinely used by major institutions worldwide, for example, NASA Goddard Space Flight Center and German Research Center for Geosciences (GFZ), to produce global gravitational models from satellite tracking measurements of CHAMP and/or GRACE types. Such Earth's gravitational products have found widest possible multidisciplinary applications in Earth Sciences. The method is essentially implemented by solving the differential equations of the partial derivatives of the orbit of a satellite with respect to the unknown harmonic coefficients under the conditions of zero initial values. From the mathematical and statistical point of view, satellite gravimetry from satellite tracking is essentially the problem of estimating unknown parameters in the Newton's nonlinear differential equations from satellite tracking measurements. We prove that zero initial values for the partial derivatives are incorrect mathematically and not permitted physically. The numerical integration method, as currently implemented and used in mathematics and statistics, chemistry and physics, and satellite gravimetry, is groundless, mathematically and physically. Given the Newton's nonlinear governing differential equations of satellite motion with unknown equation parameters and unknown initial conditions, we develop three methods to derive new local solutions around a nominal reference orbit, which are linked to measurements to estimate the unknown corrections to approximate values of the unknown parameters and the unknown initial conditions. Bearing in mind that satellite orbits can now be tracked almost continuously at unprecedented accuracy, we propose the measurement-based perturbation theory and derive global uniformly convergent solutions to the Newton's nonlinear governing differential equations of satellite motion for the next generation of global gravitational models. Since the solutions are global uniformly convergent, theoretically speaking, they are able to extract smallest possible gravitational signals from modern and future satellite tracking measurements, leading to the production of global high-precision, high-resolution gravitational models. By directly turning the nonlinear differential equations of satellite motion into the nonlinear integral equations, and recognizing the fact that satellite orbits are measured with random errors, we further reformulate the links between satellite tracking measurements and the global uniformly convergent solutions to the Newton's governing differential equations as a condition adjustment model with unknown parameters, or equivalently, the weighted least squares estimation of unknown differential equation parameters with equality constraints,

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for the reconstruction of global high-precision, high-resolution gravitational models from modern (and future) satellite tracking measurements.

1. Introduction

The history of geodesy has changed with the launch of the first artificial satellite Sputnik-1 on 4 October 1957 by the former Soviet Union. Satellites have brought a revolutionary change of the way we measure the Earth, both geometrically and physically. Sixty years of satellite gravimetry has since witnessed a profound advance in both satellite gravity theory and practical production of Earth's gravitational models, in particular, in the past decade or so, thanks to the launches of three dedicated satellite gravity missions: the Challenging Mini-satellite Payload (CHAMP) launched in 2000, the Gravity Recovery and Climate Experiment (GRACE) in 2002, and the Gravity field and steady-state Ocean Circulation Explorer (GOCE) in 2009. The dedicated satellite gravity missions have provided frontier challenges and science tools to explore and understand solid and fluid geophysical processes and dynamics of the Earth (see e.g., [19,78,87,94,104,109]). From the geodetic point of view, the most expected celebration and continuation of these sixty years of great achievement of satellite gravimetry may culminate in the launch of the GRACE Follow-on mission scheduled in 2017. For more information on GRACE Follow-on, the reader is referred to [15,22] and the website https://gracefo.jpl.nasa.gov/.

To compute the Earth's gravitational field from satellite tracking measurements, a number of mathematical methods have been proposed to establish the links between measurements and the force parameters of the Earth's gravitational field. The major classes of methods include: (i) linear perturbation methods; (ii) the dynamical numerical integration method; (iii) the orbit-energy-based method; (iv) the two-point (orbital) boundary value problem theory, which was first solved by Schneider [120,121] and further developed by Ilk et al. [40,41] (see also [74]); (v) the orbit-inverted acceleration approach. The idea of computing accelerations from GPS-derived coordinates was first proposed by Jekeli and Garcia [48] for airborne gravimetry and then applied to reconstruct the Earth's gravitational field from CHAMP mission by Reubelt et al. [85] (see also [10,21]); and (vi) satellite gradiometry (see e.g., [93,95]). Recently, Xu [111,114] proposed a measurement-based perturbation method, which is globally convergent uniformly.

In this paper, we will be mainly concerned with the first two types of methods to reconstruct the Earth's gravitational field from satellite tracking measurements, because Earth's gravitational models were first derived by using linear perturbation methods, and because the numerical integration method has now been routinely used to produce global Earth's gravitational models. Although the orbit-energy-based method is mathematically rigorous, as first proposed by Bjerhammer [7,8] and likely also independently by Wolff [110] and further modified by Jekeli [47] in order to account for technological advance of space observation, it is still not able to fully utilize the unprecedented accuracy of all modern space measurements (see also [36]). Neither the two-point boundary value problem nor the orbit-inverted acceleration approach have been used by any major institutions such as NASA Goddard Space Flight Center and GFZ to produce global Earth's gravitational models, though they are indeed used to compute gravitational models from GRACE and CHAMP measurements (see e.g., [10,21,74,85]). The two-point boundary value problem always has been based on short arcs and, as a result, can never fully utilize unprecedented accuracy and continuity of missions of GRACE/CHAMP types to their potential limit of measurement technology. Nevertheless, the method is only of quasi-linear accuracy but is well known to be difficult to implement, in particular, for sparse tracking data. Orbit-inverted accelerations can be unreliable and inaccurate, since the operator of differentiation is ill-posed in nature. As a direct consequence, position-inverted acceleration signals can be distorted and their noise can be substantially amplified, the extent of which depends on the noise level of satellite positions, the time interval used to derive accelerations and regularization. Whenever possible, one should avoid solving this intermediate inverse ill-posed problem and use its distorted signals of acceleration to further invert for global gravitational models. In the case of satellite gradiometry, since we directly measure the second gradients of the gravitational potential, the observational equations of the gravitational tensors are mathematically straightforward ([93,95]).

Linear perturbation methods are to find an approximate solution to nonlinear Lagrange's planetary equations and mathematically rigorous, which have been well developed and documented (see e.g., [11,12,34,49,52,60,102,122]). Soon after the launches of first artificial satellites in 1950s and 60s, with camera and Doppler tracking measurements, linear perturbation solutions and simplified variants for small and/or zero divisors were used to compute the flattening and/or the eccentricity of the Earth (see e.g., [13,45,54,75,77]), lumped and/or resonance-derived harmonic coefficients (see e.g., [1,16–18,55–57,119]), and gravitational models with low degrees and orders (see e.g., [17,18,30,31,34,46,49,52,61,62]). Combined solutions of satellite tracking measurements with terrestrial gravity data can be found, for example, in [24,25,50] and [23]. For more information on the early work on the determination of the Earth's gravitational field from satellite tracking measurements, the reader is referred to two excellent reviews by Kaula [51] and Kozai [63]. Although linear perturbation methods nowadays are mainly used for mission analysis, they could be revitalized to produce global satellite gravitational models by implementing the idea of measurement-based perturbation developed by Xu [111,114].

Estimating unknown differential equation parameters has been essential in many areas of science and engineering. A key mathematical component of such an estimation procedure has been to solve the derived differential equations of the partial derivatives with respect to the unknown differential equation parameters under the assumption of zero initial values for the partial derivatives, as originally published by Gronwall [28] almost 100 years ago (see also [26,37]). This estimation technique has found widest possible applications, for example, in mathematics, statistics, chemistry, physics and satellite gravimetry, which, however, has now been best known as the dynamical numerical integration method in geodesy. Actually, in the community of geoscience, this method seemed to be first hinted at by Anderle [2], likely independent of what had been published by then, since no references were made to the mathematical literature of [26,28] and [37]. The method was then (re-)published in a mathematical paper by Riley et al. [90] (see also [3,5,76]), again, without any reference to the above-mentioned mathematical literature.

The dynamical numerical integration method has gained wide spread acceptance without challenge with the publications of Anderle [2] and Riley et al. [90], and has since been routinely used by almost all major institutions worldwide to produce global gravitational models from satellite tracking measurements, likely partly attributed to the fact that NASA Goddard Space Flight Center used and implemented this numerical integration idea by Anderle [2] as the mathematical foundation to compute Earth's gravitational models (see e.g., [67,70]). Among the most important gravitational models before the dedicated satellite gravity missions CHAMP, GRACE and GOCE are the GEM series of gravitational models from the Goddard Space Flight Center (see e.g., [72,73]) and those from the joint German–French team (see e.g., [6,97]). For reviews on progress in satellite gravimetry before the launches of the dedicated satellite gravity missions, the reader is referred to [64] (see also [65]) for a brief progress report from 1958 to 1982 and to [80] for a summarized report, retrospective and prospective on gravity observation.

Although the numerical integration method has been widely accepted by almost all major institutions worldwide as a standard method for making global gravitational models from satellite tracking measurements, this is an unbelievable scientific fallacy, since the method is based on groundless claim and proved to be incorrect, mathematically and physically [112,115,116]. Unfortunately, the method has been used for almost 60 years by all important institutions worldwide to compute global gravitational models from satellite tracking measurements for use in geodesy, solid geophysics, ocean dynamics, hydrology, interaction of ocean and surface water with atmosphere, and far more beyond, bearing in mind that no technical documentation is available to provide a complete and mathematically rigorous support for the method, to my best knowledge, after an extensive search of literature.

In the 1965 U.S. Naval Weapons Laboratory technical report, Anderle [2] wrote two sentences in the section of Procedure to mean this method by saying "*Numerical integration ... was used to compute the orbit of the satellite. The partial derivatives of satellite position with respect to orbit and gravity parameters were also obtained by numerical integration of the perturbation equations*". No mathematical formulation was given to provide further technical support and explanation of the idea. In connection with the idea by Anderle [2], a complete publication about the mathematics of the method was given by Riley et al. [90] from Hughes Aircraft Company and Aerospace Corporation, which was soon well known and accepted among geodesists worldwide. Although Riley et al. [90] correctly derived the differential equations of the orbit and velocity of a satellite with respect to differential equation parameters in their mathematics paper, they simply claimed at the beginning of the second page of the paper "*The initial values ... in general will be zero if β_k is a differential equation parameter*". As a matter of fact, the differential equations for the partial derivatives with respect to the equation unknowns and the zero initial values for the partial derivatives dated back much earlier to Gronwall [28] and Ritt [91] (see also [26,37]). Mathematically, this zero-initial-value statement does not derive from the original differential equations but is nothing more than a claim. Obviously, a key element to decide the particular solution of differential equations is claimed without any mathematical/physical justification. In the technical report on Goddard Earth models (5 and 6), Lerch et al. [67] reported that they used the numerical integration method from the idea of Anderle [2]. More precisely, in the appendix of orbit theory for the software system GEODYN, Lerch et al. [67] wrote "*The partial derivatives ... are obtained by direct numerical integration of the variational equations*" on page A1–10 and "*Initially, ... the rest of the matrix (corresponding to the partial derivatives – notes added by the author) is zero*" on page A1–22. To support the production of global gravitational models from satellite tracking measurements, Goddard Space Flight Center, together with Computer Sciences Corporation, prepared a lengthy technical report of about 700 pages [70]. In Section 6.1.4 on page 6–11, although Long et al. [70] correctly realized that their differential Eqs. (6–49) required initial conditions, and even though the initial values are the key to solve the differential equations, they did not touch the issue of how to determine the initial values but chose to show how to numerically solve the differential equations as if the initial values had been given. The numerical integration method was followed by the joint German/French team as well (see e.g., [86]). Recently, Xu [112,115,116] mathematically proved rigorously that assigning zero values to the initial partial derivatives violates the physics of motion of celestial bodies. More will come in Section 2.2.

Profound technological advances in space observation have been achieved in comparison with those in 1950s and 1960s when linear perturbations were significantly developed and the numerical integration method was hinted at and published. Two most important features of these advances are: (i) low Earth orbiting (LEO) satellites can now be tracked and directly measured, precisely and almost continuously, by using Global Navigation Satellite Systems (GNSS). In other words, we have precise and continuous orbits of LEO satellites with an arc of arbitrary length; and (ii) tracking measurements are of unprecedented accuracy. With GNSS, the orbital precision of LEO satellites can now routinely reach the level of 1 cm (see e.g., [101]) and even the level of millimetres over a short period of time, as demonstrated by experiments on the ground [118]. The accuracy of inter-satellite tracking is now at the level of a few μm in rangings and 0.1 μm/s in range rates (see

e.g., [53]). Much higher accuracy can be expected when the new generations of laser instruments are operational (see e.g., [4,81,96,98,107]).

Both linear perturbation methods and the numerical integration method (even if it were correct) are not able to utilize a long orbital arc of continuous tracking and unprecedented accuracy of modern space observation technology. Linear perturbation methods are only valid in a small neighbourhood of mean orbital elements and almost break down in the case of small divisors such as resonances, critical inclinations and circular orbits. In the case of the numerical integration method, let us treat it as if it were correct for now, common practice is always to divide a long arc into many small pieces, say in hours or one day, because the modelling error will increase with time. Up to a certain epoch, the modelling error will dominate such that a gravitational solution would not be physically meaningful any more. To control the increase of modelling errors, one would have to divide a long arc into many short arcs. A direct consequence of this common practice is that we will not be able to extract small gravitational signals from satellite tracking measurements, since small gravitational signals would take time to show up their effects on the orbit. Thus, to summarize, we would conclude that both linear perturbation methods and the numerical integration method (as if it were correct) are too approximate to benefit from and do not match profound technological advances in modern and future space observation. Actually, Lambeck and Coleman [65] pointed out that further improvements both in satellite gravity theory and data evaluation methods were required before the next generation of satellite gravity missions was launched.

The purposes of this paper are threefold: (i) to briefly review the methods of linear perturbation and to prove that the numerical integration method is groundless, mathematically and physically, though the numerical integration method was essentially first published in the mathematical literature by Gronwall [28] (see also [26,37,90]) and then widely used in chemistry and physics [20,39,68], statistics [84,108], and likely independently developed and applied in satellite gravimetry [2,3,5,67,70,76,86,90]; (ii) given differential equations with unknown parameters and unknown initial conditions, we will construct the linearized solutions to the differential equations in terms of the unknown corrections to approximate values of the unknown differential equation parameters and the unknown initial conditions by extending Euler and modified Euler numerical integration methods. These linearized solutions are of local nature, since they are derived with a nominal reference trajectory. They are mathematically rigorous and require no assumption of zero initial values for the partial derivatives with respect to the unknown differential equation parameters, as otherwise incorrectly documented in the literature of mathematics, statistics, chemistry, physics and satellite gravimetry. These local solutions should help better understand the advantages, disadvantages and limitations/problems of linear perturbation methods and the foundational erroneousness of the numerical integration method; and (iii) to construct mathematically improved and global uniformly convergent solutions to the governing nonlinear differential equations of LEO satellite motion such that they can take full current and future technological advances of space observation to extract smallest possible gravitational signals from satellite tracking measurements. As a result, we expect to produce global high-precision high-resolution gravitational models, which can also be called the next generation of global gravitational models. From the mathematical point of view, the accuracy and resolution of the next generation of global gravitational models can be sufficiently high up to the limit that modern space observation can provide.

The paper is organized as follows. Section 2 will briefly review linear perturbation methods, with emphasis for the determination of gravitational models. Since the numerical integration method, though first published by Gronwall [28] almost 100 years ago (see also [26,37,90]), has been widely used by almost all major institutions worldwide to compute global gravitational models from satellite tracking measurements (of CHAMP and GRACE missions), we will first outline the method and then follow [112] to prove that assigning zero initial values to the partial derivatives of satellite position and velocity with respect to the gravitational unknown parameters, namely, the harmonic coefficients, is mathematically erroneous and physically not permitted. We will develop Euler and modified Euler numerical integration methods to solve nonlinear differential equations with unknown parameters and unknown initial conditions in Section 3. As a result, we can represent the linearized local solutions to the original nonlinear differential equations in terms of the unknown corrections to approximate values of the unknown parameters and the unknown initial conditions, which can then be used to estimate the unknown differential equation parameters from measurements. The content of this section should help understand correct implementation of numerical integration techniques for gravitational modelling. In Section 4, by assuming that LEO satellite orbits are precisely measured with GNSS, we will present a measurement-based perturbation theory, as originally developed by Xu [111], which guarantees mathematically global uniform convergence of the solutions to the Newton's differential equations of satellite motion for satellite orbits of arbitrary length. Finally, in Section 5, we will propose the method of measurement-based condition adjustment with unknown parameters to reconstruct global gravitational models from satellite tracking measurements.

2. Linear perturbation and the standard-implemented numerical integration method

2.1. Linear perturbation methods

The motion of artificial satellites is governed by Newton's law of gravitation. Almost all earlier works on satellite gravimetry and celestial mechanics are based on Lagrange's planetary equations:

$$\frac{da}{dt} = \frac{2}{na}\frac{dT}{dM} \tag{1a}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2e}\frac{\partial T}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial T}{\partial \omega} \tag{1b}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial T}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial T}{\partial e} \tag{1c}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial T}{\partial \omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial T}{\partial \Omega} \tag{1d}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial T}{\partial i} \tag{1e}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2e}\frac{\partial T}{\partial e} - \frac{2}{na}\frac{\partial T}{\partial a} \tag{1f}$$

(see e.g., [12,34,52,102]), where $\mathbf{K} = [a, e, \omega, i, \Omega, M]$ are the six Keplerian orbital elements, which stand for the semi-major axis of the orbital ellipse, the eccentricity, the argument of the perigee, the inclination of the orbital plane, the longitude of the ascending node and the mean anomaly, respectively; n is the mean motion, and T is the disturbing potential (of unknown force or equation parameters $\mathbf{p}$), which is usually a small quantity. In celestial mechanics of the solar system, T can come from disturbing planets of extremely small masses when compared with the solar mass (see e.g., [34]); in the case of artificial Earth's satellites, T can be mainly due to the disturbing potential of the Earth, celestial bodies of very large distances such as the Sun and the Moon, and/or other disturbing forces such as the solid earth and ocean tides, the radiation pressure of the Sun and the air drag of the atmosphere (see e.g., [47,52]). Since the general relativistic dragging, namely, Lense–Thirring effect, has been shown to be of significant impact on satellite orbits (see e.g., [42–44,88,89]), it should be fully taken into account in the future computation of high precision, high resolution gravitational models from tracking measurements. One may equivalently rewrite Lagrange's planetary Eqs. (1) in six other orbital elements (see e.g., [12,96,102]).

Without loss of generality, we denote the general solution to Lagrange's planetary Eqs. (1) by $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$, where $\mathbf{c}_k$ stands for six arbitrary integration constants. Different integration constants specify the motions of different satellites. In principle, if the general solution $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$ would be analytically available, given the parameters $\mathbf{p}$ and the six integration constants $\mathbf{c}_{k0}$ (or alternatively an initial point $\mathbf{K}_{t0}$ or any six independent values on $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$), one can then obtain the particular solution $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_{k0})$ and use it to compute and predict the orbit of the celestial body at any time t. On the other hand, given the general solution $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$ and a sufficient number of measurements on $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$, one can then estimate the (unknown) force parameters $\mathbf{p}$ from the measurements. Unfortunately, the general analytical solution $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$ can only be obtained for the idealized two-body problem in which a particle of negligible mass is attracted by another point mass (see e.g., [12,52,83,102]). In general, no analytical solution to (1) can be possible, for two reasons: (i) Lagrange's planetary Eqs. (1) are nonlinear and extremely difficult to solve analytically. Thus, perturbation theory has been playing a fundamental role in celestial mechanics and satellite gravimetry (see e.g., [11,14,34,52,60,102,123]), which attempts to construct an approximate solution to Lagrange's planetary Eqs. (1) through the procedure of successive approximation. Actually, there are two types of perturbation methods. One is to construct an approximate solution through the mathematical standard approach of small parameter perturbation (see e.g., [11,14,34,79,102,123]). The other method is to treat the variables on the right hand side of Lagrange's planetary Eqs. (1) as constants, except for the mean anomaly M, and then integrate the differential Eqs. (1) to construct an approximate solution. This latter approach has been widely applied in satellite geodesy (see e.g., [49,52,60]); and (ii) the disturbing potential function T itself may not be exact and/or sufficiently precise. Actually, perturbation theory, together with astronomical measurements, historically played a decisive role in correctly identifying an unknown disturbing celestial body to explain the deviations of the theoretical predictions from measurements by Adams (https://en.wikipedia.org/wiki/John_Couch_Adams), Le Verrier (https://en.wikipedia.org/wiki/Urbain_Le_Verrier) and Lowell [71] (see also https://en.wikipedia.org/wiki/Planets_beyond_Neptune), successfully leading to the great discovery of both Neptune and Pluto in 1846 and 1930, respectively (see e.g., [29,34,66]), though the data of Neptune given by Le Verrier and Adams are in large errors (see e.g., [38]).

As an inverse problem of celestial mechanics, satellite gravimetry is to reconstruct the unknown force parameters $\mathbf{p}$ in the nonlinear Lagrange's planetary Eqs. (1) with unknown initial conditions from a sufficient number of measurements on $\mathbf{K}(t, \mathbf{p}, \mathbf{c}_k)$. In this case, we assume that the disturbing potential function T itself is precisely given but can contain a number of unknown parameters $\mathbf{p}$, though part of T may be directly measured and corrected. We also implicitly assume that there exist no other unknown sources that can contribute to T in a non-negligible way. In satellite geodesy, the disturbing potential T is mainly attributed to the rotating Earth, the attraction by the Sun, the Moon and large planets, the solid Earth and ocean tides, the radiation pressure of the Sun, the air drag of the atmosphere and other conservative and non-conservative forces (see e.g., [47,52,86,96]). From a mathematical point of view, if parts of T can be directly observable, such effects can be treated as known or given; otherwise, they are modelled with unknown parameters. In physical geodesy, we usually write

the disturbing potential T of the Earth in the non-inertial earth-fixed reference frame as follows:

$$T = \frac{GM}{r}\sum_{l=2}^{N_{max}}\sum_{m=0}^{l}\left(\frac{R}{r}\right)^{l}[C_{lm}\cos(m\lambda) + S_{lm}\sin(m\lambda)]P_{lm}(\cos\theta), \tag{2}$$

(see e.g., [35,52,124]), where $N_{\max}$ is a maximum number of degrees and orders, R is the mean radius of the Earth, C_{lm} and S_{lm} are the unknown normalized, dimensionless harmonic coefficients which will be collected into the unknown vector $\mathbf{p}$ and to be estimated from satellite tracking measurements, λ and θ are the longitude and colatitude, respectively, and $P_{lm}(t)$ is the normalized Legendre function. However, in satellite gravimetry and celestial mechanics, the differential equations of motion of a satellite are almost always given in the inertial reference frame (see e.g., [52,96,102]). In this case, we will need the transformation of coordinates from the earth-fixed reference frame into the inertial reference frame through rotations (see e.g., [96]). More specifically, if we use spherical coordinate systems, then we need the following transformation:

$$\lambda = \alpha + \delta\alpha - \omega_e t, \tag{3a}$$

$$\theta = \zeta + \delta\zeta, \tag{3b}$$

(see e.g., [47]), where $\delta\alpha$ and $\delta\zeta$ are the corrections to α and ζ, which depend on precession, nutation, the Earth's rotation and polar motion and can be computed from theory and measurements (see e.g., [96]), α and ζ are the right ascension and co-declination in the inertial reference frame of epoch J2000.0, ω_e is the rate of the Earth's rotation. The disturbing potential T in the inertial reference frame is clearly a function of time.

To rigorously determine mathematically the unknown force parameters C_{lm} and S_{lm} from satellite tracking measurements, we have to first exactly solve Lagrange's planetary Eqs. (1), with the disturbing potential T given by (2), link the exact solution to the satellite tracking measurements and finally estimate C_{lm} and S_{lm}. Unfortunately, the nonlinear differential Eqs. (1) are too complicated to exactly solve analytically. Thus, perturbation theory is always used to construct an approximate solution to (1) (see e.g., [11,34,49,52,60,102,124]). The most complete perturbation theory was fully developed for the determination of C_{lm} and S_{lm} from satellite tracking measurements by Kaula [49,52], following the approach of Kozai [60] and given the representation of T expressed in terms of the six Keplerian orbital elements by Groves [124]. It is nowadays well known as *Kaula linear perturbation theory*. More specifically, since the exact analytical solution is generally hard or almost impossible to obtain, Kaula [49,52] derived the linear perturbation solution to Lagrange's planetary Eqs. (1) by treating all the orbital elements as constants and/or by replacing them with the mean orbital elements, except for the rapidly time-varying element of the mean anomaly M, and then integrating all the terms on the right hand side of (1).

The major advantage of Kaula's linear perturbation solution is its suitability to analyze the physical properties of the solution. Since T is expressed in terms of six Keplerian orbital elements, the physical features of T can be further classified into secular, long-periodic and short-periodic (see e.g., [49,52,60,124]). More precisely, the terms as a function of the mean anomaly M will change periodically (and rapidly) and are *short-periodic*; the terms as a function of ω but not M are *long-periodic*; and the terms irrelevant of ω nor M are *secular*. In other words, secular terms changes slowly but approximately linearly with time. In addition, one can also identify, in the linear perturbation solution, the physically interesting phenomenon of orbital mean-motion resonance when the rotation rate of the Earth and the mean motion of a satellite are commensurable (see e.g., [9,16,34,52,57,102,119]). Nevertheless, Kaula's linear perturbation solution is only valid locally around the neighbourhood of the mean orbital elements and will diverge with the increase of time. Thus, Kaula's linear perturbation solution will not be able to fully utilize a long orbital arc and high precision of modern space observation to estimate $\mathbf{p}$.

2.2. The standard-implemented numerical integration method

In the Cartesian coordinate system, the motion of an artificial satellite can also be mathematically written alternatively by the following nonlinear vector differential equation:

$$\ddot{\mathbf{x}} = \mathbf{a}_E(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) + \mathbf{a}_M(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}_M), \tag{4}$$

(see e.g., [12,52,96,102]), where $\mathbf{x}$ is the position vector of the satellite in the inertial reference frame, $\mathbf{a}_E(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p})$ is the Earth's gravitational attraction exerted on the satellite, $\mathbf{p}$ is the vector of unknown parameters, and $\mathbf{a}_M(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}_M)$ stands for all other forces which may include solid earth and ocean tides, atmospherical drag, solar radiation pressure and third-body effects (see e.g., [47,86,96,102]), with $\mathbf{p}_M$ standing for the unknown parameters (if any) of these force models. For the next generation of high precision, high resolution global gravitational models from satellite tracking, the force model $\mathbf{a}_M(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}_M)$ should include the general relativistic dragging, namely, the Lense–Thirring effect, which has been shown to have a significant effect on satellite orbits, satellite-to-satellite ranges and range-rates for a sufficiently lengthy arc (see e.g., [42–44,88,89]) but has not been considered up to the present in the production of global gravitational models.

In the remainder of this paper, we will use $\ddot{\mathbf{x}}$ and $\dot{\mathbf{x}}$ to stand for the first and second derivatives of $\mathbf{x}$ with time, respectively. (We use the notation $d\mathbf{x}/dt$ to stand for time derivative in (1), since the dot of i there does not look good) Since the acceleration $\mathbf{a}_E(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p})$ is independent of $\dot{\mathbf{x}}$, it can be rewritten as follows:

$$\mathbf{a}_E(t, \mathbf{x}, \mathbf{p}) = -\frac{GM}{r^3}\mathbf{x} + \frac{\partial T}{\partial \mathbf{x}}, \tag{5}$$

where GM is the product of the Earth's mass M and the gravitational constant G, $r = \|\mathbf{x}\|$, and T is the disturbing potential of the Earth's gravitational field with the parameters of harmonic coefficients. Because (4) is formulated in an inertial reference system (see e.g., [47,102]), the earth-fixed λ and θ in the disturbing potential T of (2) must first be transformed through (3) into the inertial reference frame (see e.g., [47]). Since the second term $\mathbf{a}_M(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}_M)$ of (4) adds no new mathematical difficulty to solve the nonlinear differential Eq. (4), we will limit ourselves to the Earth's gravitational field in the remainder of this paper. Thus, the nonlinear differential Eq. (4) can be simplified as follows:

$$\ddot{\mathbf{x}} = \mathbf{a}_E(t, \mathbf{x}, \mathbf{p}), \tag{6}$$

where $\mathbf{p}$ is an unknown vector of equation parameters. Initial conditions to (6) are unknown as well.

We should note, however, that if $\mathbf{a}_M(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}_M)$ exists but is neither estimated together with $\mathbf{p}$ nor corrected with a sufficiently precise model, the effect of $\mathbf{a}_M(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}_M)$ will be absorbed into the estimate of $\mathbf{p}$. Such an effect is theoretically systematic and of signal nature. It cannot be filtered out, since, in principle, a filter can only reduce the level of noise but would mess up or smear the true signal of concern, if the signals are not constant inside the window of the filter. In other words, filtering, as currently used in satellite gravimetry, will definitely distort the true signal of interest, because gravity signals are clearly not constant inside the window of filtering. To test the Lense–Thirring dragging from satellite measurements, for example, one should probably have to either estimate the effect together with the gravitational model or to develop statistical hypothesis testing methods for weak continuous functions; the latter is of theoretical interest by itself and deserves a separate full research.

Given a number of (geometrical) tracking measurements to the satellite such as positions of the satellite, ranges, range rates and directions to the satellite, denoted by $y_1, y_2, \ldots, y_n$ or in the vector form $\mathbf{y}$, the problem of satellite gravimetry is to use the tracking measurements $\mathbf{y}$ to determine the gravitational parameters $\mathbf{p}$. Mathematically, this is essentially the problem of estimating the unknown parameters $\mathbf{p}$ of the nonlinear differential Eq. (6) with unknown initial conditions from satellite tracking measurements. If there are a number of LEO satellites, say s satellites, the motion of each satellite being governed by the same differential equations of type (6) with the same gravitational parameters $\mathbf{p}$ but with different initial conditions or different integration constants. If we collect satellite tracking measurements $\mathbf{y}_i$ on the ith satellite, then we will have to combine all these measurements $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s$ together to solve for the parameters $\mathbf{p}$. In the following development of the method, without loss of generality, we will confine ourselves to one satellite.

To determine $\mathbf{p}$ from $\mathbf{y}$, one of the most important steps is to represent each y_i in terms of $\mathbf{p}$. Since a geometrical satellite tracking measurement y_i is generally a function of the position and velocity of the satellite at the ith epoch, in principle, we have to first solve the nonlinear differential Eq. (6). Let $\mathbf{x}(t, \mathbf{p}, \mathbf{c})$ denote the general solution to the nonlinear differential Eq. (6), with $\mathbf{c}$ standing for six arbitrary integration constants. These six integration constants $\mathbf{c}$ are mathematically independent of the equation parameters $\mathbf{p}$. In other words, the general solution $\mathbf{x}(t, \mathbf{p}, \mathbf{c})$ mathematically represents an infinite number of solutions to the differential Eq. (6), which can physically describe the motions of different satellites. As far as initial values are properly given, one can then use them to fix six arbitrary integration constants $\mathbf{c}$ and obtain the specific solution to uniquely describe the motion of the satellite. Mathematically, the vector $\mathbf{c}$ for this particular solution can now be expressed as the functions of $\mathbf{p}$ and the initial conditions. In satellite geodesy, initial values are often three initial position coordinates $\mathbf{x}_0$ and three initial velocity components $\mathbf{v}_0$ of the satellite at the initial epoch t_0. As a result, the orbital position solution of motion of the satellite can be implicitly written as $\mathbf{x}(t, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))$. We emphasize that six arbitrary integration constants $\mathbf{c}$ can also be alternatively determined from any six independent values on the solution, instead of the initial position and velocity conditions, since any specific value on the particular solution contains the information on $\mathbf{c}$. Except for the idealized two-body problem with the point mass model, it is almost impossible to obtain an analytical solution to the nonlinear differential Eq. (6) with the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$. Thus, we can only use the implicit orbit $\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))$ to develop observational equations for geometrical tracking measurements of any kind. Taking a (velocity-independent) geometrical tracking measurement y_i as an example, we can symbolically write its observational equation as follows:

$$y_i = f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))) + \epsilon_i, \tag{7}$$

where $f(\cdot)$ stands for a nonlinear functional and ϵ_i is the random error of the measurement y_i with zero mean.

Under the framework of the numerical integration method, as described in Lerch et al. [67], Long et al. [70] and Reigber [86] for satellite gravimetry, we will have to first linearize the observational Eq. (7). Given a set of approximate values $\mathbf{p}^0$, $\mathbf{x}_0^0$ and $\mathbf{v}_0^0$ for $\mathbf{p}$, $\mathbf{x}_0$ and $\mathbf{v}_0$, respectively, one can then numerically integrate the nonlinear differential Eq. (6) and obtain the approximate position of the satellite at time epoch t_i, which is denoted by $\mathbf{x}_i^0$. Thus, the observational Eq. (7) can be *formally* linearized as follows:

$$\delta y_i = \mathbf{a}_{ix}\Delta\mathbf{x}_0 + \mathbf{a}_{iv}\Delta\mathbf{v}_0 + \mathbf{a}_{ip}\Delta\mathbf{p} + \epsilon_i \tag{8a}$$

at the approximate values of $\mathbf{p}^0$, $\mathbf{x}_0^0$ and $\mathbf{v}_0^0$, where

$$\delta y_i = y_i - f(\mathbf{x}_i^0),$$

$$\Delta\mathbf{x}_0 = \mathbf{x}_0 - \mathbf{x}_0^0,$$

$$\Delta\mathbf{v}_0 = \mathbf{v}_0 - \mathbf{v}_0^0,$$

and

$$\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}^0.$$

The three row vectors $\mathbf{a}_{ix}$, $\mathbf{a}_{iv}$ and $\mathbf{a}_{ip}$ are all computed at the approximate values $\mathbf{p}^0$, $\mathbf{x}_0^0$ and $\mathbf{v}_0^0$, and defined, respectively, as follows:

$$\begin{aligned}\mathbf{a}_{ix} &= \frac{\partial f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0)))}{\partial \mathbf{x}_0^T} \\ &= \frac{\partial f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0)))}{\partial \mathbf{x}^T}\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0^T},\end{aligned} \tag{8b}$$

$$\begin{aligned}\mathbf{a}_{iv} &= \frac{\partial f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0)))}{\partial \mathbf{v}_0^T} \\ &= \frac{\partial f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0)))}{\partial \mathbf{x}^T}\frac{\partial \mathbf{x}}{\partial \mathbf{v}_0^T},\end{aligned} \tag{8c}$$

$$\begin{aligned}\mathbf{a}_{ip} &= \frac{\partial f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0)))}{\partial \mathbf{p}^T} \\ &= \frac{\partial f(\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0)))}{\partial \mathbf{x}^T}\frac{\partial \mathbf{x}}{\partial \mathbf{p}^T}.\end{aligned} \tag{8d}$$

As a key step to estimate $\mathbf{p}$ from $\mathbf{y}$, we have to compute the vectors $\mathbf{a}_{ix}$, $\mathbf{a}_{iv}$ and $\mathbf{a}_{ip}$. The common matrix of the partial derivatives $\partial f(\cdot)/\partial \mathbf{x}^T$ in (8b)–(8d) can be readily obtained, as widely available (see e.g., [49,52,67,70,105]). If we would treat $\mathbf{p}$ as if it were given for now, the problem is turned into a standard problem of statistical orbit determination with the nonlinear differential Eq. (6), the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$, and the measurements $\mathbf{y}$. In this case, computing the matrices of the partial derivatives $\partial \mathbf{x}/\partial \mathbf{x}_0^T$ and $\partial \mathbf{x}/\partial \mathbf{v}_0^T$ is theoretically equivalent to finding the state transition matrix for the state of position and velocity from the initial epoch t_0 to the current epoch t_i. This problem has been completely solved and well documented in, for example, [32,70,103,105].

Now, to complete the final construction of the observational Eq. (8a), the key issue is to compute the partial derivatives of $\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))$ with respect to $\mathbf{p}$ in (8d), namely, $\partial \mathbf{x}/\partial \mathbf{p}^T = \partial \mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))/\partial \mathbf{p}^T$ for conciseness of notations. Since we do not have an analytical solution $\mathbf{x}(t_i, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))$, it is not possible to directly compute its partial derivatives with respect to the parameters $\mathbf{p}$. Instead, one has attempted to obtain these partial derivatives through solving their differential equations.

To start with, let us collect the satellite position $\mathbf{x}$ and velocity $\mathbf{v}$ at time t in the vector $\mathbf{z}$ and denote the partial derivatives of $\mathbf{z}$ with respect to $\mathbf{p}$ by $\mathbf{S}(t, \mathbf{p})$, namely,

$$\mathbf{z} = (\mathbf{x}^T, \mathbf{v}^T)^T, \tag{9a}$$

$$\mathbf{S}(t, \mathbf{p}) = \frac{\partial \mathbf{z}}{\partial \mathbf{p}^T}. \tag{9b}$$

The partial derivatives $\partial \mathbf{x}/\partial \mathbf{p}^T$ is obviously part of a more general matrix $\mathbf{S}(t, \mathbf{p})$ of partial derivatives. It has been rigorously shown mathematically that $\mathbf{S}(t, \mathbf{p})$ can be directly derived from the original nonlinear differential Eq. (6) (see, e.g., [3,76,90]) and is governed by the following system of differential equations:

$$\frac{\partial \mathbf{S}(t, \mathbf{p})}{\partial t} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \partial \mathbf{a}_E(t, \mathbf{x}, \mathbf{p})/\partial \mathbf{x}^T & \mathbf{0} \end{pmatrix}\mathbf{S}(t) + \begin{pmatrix} \mathbf{0} \\ \partial \mathbf{a}_E(t, \mathbf{x}, \mathbf{p})/\partial \mathbf{p}^T \end{pmatrix}. \tag{10}$$

Obviously, the derived Eq. (10) of the partial derivatives do not mathematically add any new information on the original problem of satellite gravimetry. Actually, for a general differential equation with unknown parameters, the differential equations of type (10) were already given by Gronwall [28] and Ritt [91] almost 100 years ago (see also [26]).

Although we have the differential Eq. (10) for the partial derivatives $\mathbf{S}(t, \mathbf{p})$, they can still be useless, unless *the* initial conditions of $\mathbf{S}(t, \mathbf{p})$ at the time epoch t_0 are available. We use the italic font for "the" before "initial conditions" to emphasize that the initial conditions of $\mathbf{S}(t, \mathbf{p})$ cannot be arbitrarily given but must comply with the original problem. Unfortunately, the original problem of satellite gravimetry, namely, the governing differential Eq. (6) and the satellite tracking measurements $\mathbf{y}$, do not provide any direct hint/clue on what values $\mathbf{S}(t_0, \mathbf{p})$ can take on.

With no way out, the claims on $\mathbf{S}(t_0, \mathbf{p})$ were made and accepted in satellite geodesy, as cited verbatim from some of the publications in the introduction. Riley et al. [90] claimed that $\mathbf{S}(t_0, \mathbf{p})$ was generally zero. Lerch et al. [67] treated $\mathbf{S}(t_0, \mathbf{p})$ as zero for the software system GEODYN (see also [5]). Long et al. [70] mentioned the importance of the initial values $\mathbf{S}(t_0, \mathbf{p})$ for solving the differential Eq. (10) but without saying what they should be. Reigber [86] referred the reader to the earlier version of the report by Long et al. [70]. Others simply avoid mentioning the initial conditions for $\mathbf{S}(t, \mathbf{p})$ (see e.g., [32,92]). In the next subsection, we will use a counter example in [112] to prove that setting the initial values $\mathbf{S}(t_0, \mathbf{p})$ to zero is

mathematically incorrect and physically not permitted. It is unbelievable to see that the numerical integration method ends up on mathematically vain ground without rhyme or reason, though it has been widely used by almost all major institutions worldwide to produce global gravitational models from satellite tracking measurements, with widest possible applications in Earth Sciences.

2.3. No zero initial values for $\mathbf{S}(t_0, \mathbf{p})$ permitted mathematically and physically

In this part of the paper, we will prove that no zero initial values for $\mathbf{S}(t_0, \mathbf{p})$ can be permitted from both the mathematical and physical points of view. We will use a counter example reported in Xu [112,115] for this purpose, though one can readily construct many other counter examples. Then we will use strictly logical reasonings to explain why $\mathbf{S}(t_0, \mathbf{p})$ cannot be zero. For some of the arguments, the reader is referred to [112] for details.

As a general rule in mathematics, we need nothing more than a counter example to disprove that something is incorrect. Let us start with the second counter example in [112], which is rewritten as follows:

$$\ddot{y} + p_1^2 y - p_2 \cos(p_1 t) = 0, \tag{11}$$

where p_1 and p_2 are two equation (unknown) parameters. By directly solving the differential Eq. (11), we obtain the general solution:

$$y(t) = \frac{p_2}{2p_1^2}\{\cos(p_1 t) + p_1 t \sin(p_1 t)\} + c_1 \sin(p_1 t) + \frac{c_2}{p_1}\cos(p_1 t), \tag{12}$$

where c_1 and c_2 are two arbitrary integration constants. Mathematically, integration constants c_1 and c_2 are independent of p_1 and p_2. As far as c_1 and c_2 are given specific values, which can be implicitly defined, for example, through assuming two values of $y(t)$ at two different time epochs, we will then obtain the particular solution of (12). With the true general solution (12), we can easily compute and obtain the true values of the derivatives of $y(t)$ with respect to p_1 and p_2, which are simply given as follows:

$$\begin{aligned}\frac{\partial y(t)}{\partial p_1} &= -\frac{p_2}{p_1^3}\{\cos(p_1 t) + p_1 t \sin(p_1 t)\} + \frac{p_2}{2p_1} t^2 \cos(p_1 t) \\ &\quad + c_1 t \cos(p_1 t) - \frac{c_2}{p_1^2}\cos(p_1 t) - \frac{c_2 t}{p_1}\sin(p_1 t),\end{aligned} \tag{13a}$$

$$\frac{\partial y(t)}{\partial p_2} = \frac{1}{2p_1^2}\{\cos(p_1 t) + p_1 t \sin(p_1 t)\}. \tag{13b}$$

For an arbitrary t_0, the derivatives $\partial y(t)/\partial p_1$ of (13a) and $\partial y(t)/\partial p_2$ of (13b) clearly cannot be zero. Actually, if the derivatives (13a) and (13b) would be equal to zero at the time epoch t_0, we would readily have two equations for two unknowns p_1 and p_2 and would be able to solve them without any value of $y(t)$. For example, if the initial values for these derivatives would be allowed to be equal to zero, then the second derivative (13b) would be turned into the following equation:

$$\cos(p_1 t_0) + p_1 t_0 \sin(p_1 t_0) = 0, \tag{14}$$

for any non-zero p_1. By solving this equation, we could obtain the value(s) of p_1 (if solutions exist). Obviously, this is logically ridiculous, since this would indicate that we would be able to determine the unknown parameter p_1 in the differential Eq. (11) without any information on $y(t)$. In the case of $t_0 = 0$, (14) becomes $1 = 0$ — an even more ridiculous expression. The source of errors is clearly with the assumption of setting the values of partial derivatives (13a) and (13b) to zero at the initial epoch t_0.

For the satellite gravimetry problem (6) with the geometrical satellite tracking measurements $\mathbf{y}$, we will simply make some logical reasonings and explanations. The mathematical proof and physical explanations of no zero initial values for $\mathbf{S}(t_0, \mathbf{p})$ can be found in [112].

Remark 1: For the general orbit solution $\mathbf{x}(t, \mathbf{p}, \mathbf{c})$ of (6) without given initial conditions, the problem of satellite gravimetry is to determine both $\mathbf{p}$ and $\mathbf{c}$ from the geometrical satellite tracking measurements $\mathbf{y}$. The orbital position at the time epoch t_i can be mathematically written symbolically as $\mathbf{x}(t_i, \mathbf{p}, \mathbf{c})$. $\mathbf{x}(t_i, \mathbf{p}, \mathbf{c})$ is only a point of the general solution $\mathbf{x}(t, \mathbf{p}, \mathbf{c})$ of the satellite at the time epoch t_i. Logically, no orbital position at one time epoch is superior to any other orbital positions of the same orbit but at different time epochs. Since t_0 is arbitrary, any orbital position can equally serve as an initial condition. If $\mathbf{S}(t_0, \mathbf{p})$ could be set to zero, then all the other $\mathbf{S}(t_i, \mathbf{p})(t_i \neq t_0)$ could be treated in the same manner as zero from the mathematical point of view, implying physically that $\mathbf{x}(t, \mathbf{p}, \mathbf{c})$ would not be a function of $\mathbf{p}$. Obviously, this conclusion violates our starting differential Eq. (6) with $\mathbf{p}$. Thus, $\mathbf{S}(t_0, \mathbf{p})$ cannot be equal to zero.

Remark 2: According to [115,116], for the particular orbit solution $\mathbf{x}(t, \mathbf{p}, \mathbf{c}_{x0})$ of (6), with the integration constants $\mathbf{c}_{x0}$ fixed/given, for example, from two orbital positions at two different epochs, the partial derivatives $\mathbf{S}(t, \mathbf{p})$ at any time epoch t must be unique. If $\mathbf{S}(t_0, \mathbf{p}) = \mathbf{0}$, then we can compute $\mathbf{S}(t, \mathbf{p})$ by solving the differential Eq. (10). To emphasize the starting time t_0, we denote the values of $\mathbf{S}(t, \mathbf{p})$ by $\mathbf{S}(t, \mathbf{p}, t_0)$. Now let us assume a different starting time epoch, say $t_{01} \neq t_0$. Since the initial values for $\mathbf{S}(t, \mathbf{p})$ are assumed to be zero, we should have $\mathbf{S}(t_{01}, \mathbf{p}) = \mathbf{0}$ and obtain its corresponding values of the partial derivatives $\mathbf{S}(t, \mathbf{p}, t_{01})$. Following the same logic, let us assume another starting time epoch t_{0i}, which can be

arbitrarily different from either t_{01} or t_0. By the claim of Riley et al. [90] (see also [67,70]), we have $\mathbf{S}(t_{0i}, \mathbf{p}) = \mathbf{0}$, with which we can further obtain $\mathbf{S}(t, \mathbf{p}, t_{0i})$ by solving the differential Eq. (10). For three arbitrarily different time epochs t_0, t_{01} and t_{0i}, their corresponding partial derivatives at the same time epoch t, namely, $\mathbf{S}(t, \mathbf{p}, t_0)$, $\mathbf{S}(t, \mathbf{p}, t_{01})$ and $\mathbf{S}(t, \mathbf{p}, t_{0i})$, will not be equal to each other. This obviously contradicts the fact that $\mathbf{S}(t, \mathbf{p})$ is unique for this particular orbit. The source of errors again certainly comes from the incorrect assumption of zero initial values for $\mathbf{S}(t, \mathbf{p})$.

Remark 3: If $\mathbf{S}(t, \mathbf{p})$ could be set to zero at the initial epoch t_0, by following the same logical reasonings as in (13) and/or (14), we would be able to solve for $\mathbf{p}$ from the system of equations $\mathbf{S}(t_0, \mathbf{p}) = \mathbf{0}$, since the number of equations $\mathbf{S}(t_0, \mathbf{p}) = \mathbf{0}$ is exactly equal to that of the unknown parameters $\mathbf{p}$, implying that we could determine the unknown harmonic coefficients $\mathbf{p}$ without any satellite tracking measurements; this is again an unacceptable result. Actually, on the other hand, if $\mathbf{S}(t_0, \mathbf{p}) = \mathbf{0}$, then we could solve the differential equations (10) and obtain *the* orbital solution, denoted by $\mathbf{x}_{t0}(t, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))$, which implies that we do not need any initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$ to find the particular orbital solution. Since t_0 is arbitrary, we could obtain an infinite number of different solutions for the same satellite gravimetry problem. All these are certainly incorrect mathematically, again with the source of errors in the assumption of $\mathbf{S}(t_0, \mathbf{p}) = \mathbf{0}$.

The above counter example, together with all the mathematical, physical and logical reasonings, has all clearly nullified the claim by Riley et al. [90] that the initial values of the partial derivatives with respect to equation parameters are generally zero. Actually, this claim is also used as a starting point for the Goddard Space Flight Center software system GEODYN (see e.g., [67,70]) and in Europe (see e.g., [86]) for the production of global gravitational models from satellite tracking measurements. Bearing in mind that a variety of global gravitational model products has been widely used in and far more beyond geodesy such as solid geophysics, hydrology, continental water variation and so on, we believe that software systems must be first updated onto a solid mathematical foundation right now before continuing to produce and circulate such gravitational products from satellite tracking measurements. Finally, we state a theorem in [112] to conclude this subsection as follows:

Theorem 1. *Given the governing vector differential equations (6) with the unknown harmonic coefficients* $\mathbf{p}$ *in (2), then setting the initial values of the partial derivatives of the orbit and velocity with respect to the unknown harmonic coefficients* $\mathbf{p}$ *to zero at any specified initial time epoch* t_0 *is not permitted, mathematically and physically.*

Before closing this section, I should point out that zero initial partial derivatives with respect to the unknown parameters of differential equations has been routinely used beyond geodesy. Completely independent of the development in satellite geodesy, for simplicity but without loss of generality, given an ordinary differential equation $\dot{y} = f(t, y, p)$, Gronwall [28] correctly derived the differential equation of y with respect to the unknown equation parameter p, as in the case of (10) (see also [91]), but incorrectly claimed that its initial value is equal to zero without providing any reasons or arguments. The work of Gronwall [28] was then further spread through the book on ordinary differential equations by Goddington and Levinson [26]. Actually, the solution to the given ordinary differential equation can be symbolically written as follows:

$$y(t, p) = y(t_0, p) + \int_{t_0}^{t} f(t, y, p)dt, \tag{15}$$

from which we can only obtain the following identity:

$$\left.\frac{dy(t, p)}{dp}\right|_{t=t_0} = \frac{dy(t_0, p)}{dp}, \tag{16}$$

but certainly not the zero initial derivative, as we have proved in this paper. Since all the geodetic literature on satellite gravimetry has not cited or mentioned any of these mathematical publications, it seems that the claim of zero initial partial derivatives with respect to the equation parameters has been taken for granted everywhere for almost 100 years, though incorrectly, as we have proved in this paper and Xu [112]. The incorrect claim of zero initial derivatives now still continues to spread, as can be seen, for example, in [20,37,39,68,84,108].

Instead of solving a differential equation and using the solution to estimate the equation parameters, researchers have also chosen to use splines and/or basis function expansion to approximately represent the solution to the differential equation and then to estimate the equation parameters (see e.g., [69,84]). One may either first estimate the coefficients of the fitting basis functions and then further use the fitted solution to estimate the differential equation parameters or choose to simultaneously estimate both the unknown basis function coefficients and the unknown differential equation parameters. Nevertheless, the disadvantages of the basis function approach with a finite number of unknown coefficients could be threefold: (i) it generally does not satisfy the original differential equation; (ii) we need to estimate many more extra unknown coefficients of basis functions. If we simultaneously estimate both the unknown basis function coefficients and the unknown differential equation parameters from measurements, this new estimation can generally be nonlinear. Even worse, the total number of the unknown coefficients of basis functions and the unknown differential equation parameters may become larger than the number of measurements such that the new estimation problem becomes rank-deficient, though the number of measurements can be far more than sufficient to estimate the (original) differential equation parameters; and (iii) it will create modelling errors as a consequence of (i), whose extent would depend on the difference between the approximate solution as a finite series of basis functions and the (true) solution to the original differential equation. If the modelling errors are larger than the noise level of measurements, it would become impossible to extract maximum information on the equation parameters from measurements at the level of random measurement errors.

3. Linearization and numerical integration techniques for estimation of unknown differential equation parameters from measurements

If $\mathbf{S}(t_0, \mathbf{p}) \neq \mathbf{0}$ and is unknown, then the differential Eq. (10) are not useful. As a consequence, we are not able to compute $\mathbf{a}_{ip}$ of (8d) to complete the construction of the observational Eq. (8a). The question now is how we can properly implement numerical integration techniques to determine the Earth's gravitational field from satellite tracking measurements. In principle, given the satellite tracking measurements $\mathbf{y}$ with a corresponding weighting matrix $\mathbf{W}$, we can write the least squares objective function as follows:

$$\min: [\mathbf{y} - \mathbf{f}(\mathbf{x}(t_y, \mathbf{p}, \mathbf{c}))]^T \mathbf{W} [\mathbf{y} - \mathbf{f}(\mathbf{x}(t_y, \mathbf{p}, \mathbf{c}))] \tag{17}$$

subject to the equality constraint defined by the differential Eq. (6), where $\mathbf{f}(\cdot)$ are the theoretical values of the measurements $\mathbf{y}$, and each $\mathbf{x}(t_{yi}, \mathbf{p}, \mathbf{c})$ satisfies (6) and stands for the theoretical orbital position of the satellite at the time epoch t_{yi} when the tracking measurement y_i is collected. If initial conditions are available, the constants $\mathbf{c}$ can be alternatively expressed in terms of these initial conditions. If the tracking measurement y_i is involved with two satellites, we need the theoretical orbital positions (and likely also the velocities) of these two satellites to compute the theoretical value of y_i. For more details on this type of observational equations, the reader is referred to Xu [111].

Since the equality constraints are given in the form of differential equations, we cannot use conventional optimization methods to solve the minimization problem (17), subject to (6). We have to use numerical techniques to discretize the differential Eq. (6) such that we can represent $\mathbf{x}(t_{yi}, \mathbf{p}, \mathbf{c})$ in terms of the unknown differential equation parameters $\mathbf{p}$. As in the case of (8a), given some approximate values $\mathbf{p}^0$, $\mathbf{x}_0^0$ and $\mathbf{v}_0^0$ of $\mathbf{p}$, $\mathbf{x}_0$ and $\mathbf{v}_0$, respectively, we can obtain the numerical solution $\mathbf{x}^0(t, \mathbf{p}^0, \mathbf{c}(\mathbf{p}^0, \mathbf{x}_0^0, \mathbf{v}_0^0))$ (or simply $\mathbf{x}^0(t)$ for conciseness of notations) by numerically solving the following nonlinear differential equations:

$$\ddot{\mathbf{x}}^0 = \mathbf{a}_E(t, \mathbf{x}^0, \mathbf{p}^0) \tag{18}$$

under the given initial values of $\mathbf{x}_0^0$ and $\mathbf{v}_0^0$. Accordingly, the solution of $\mathbf{v}$ is denoted by $\mathbf{v}^0(t) (= \dot{\mathbf{x}}^0(t))$.

Since the differential Eq. (6) are nonlinear, we may attempt to find their approximate solutions in terms of $\mathbf{p}$ by either directly linearizing (6) or using numerical integration methods. For convenience, we rewrite the second order differential Eq. (6) as an equivalent system of first order differential equations:

$$\dot{\mathbf{z}}(t) = \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{a}_E(t, \mathbf{x}, \mathbf{p}) \end{bmatrix}. \tag{19}$$

As in the case of (6), both the equation parameters $\mathbf{p}$ and initial conditions to (19) are unknown.

3.1. The linearized local solution

Subtracting $\mathbf{z}^0(t)$ from (19), we have

$$\begin{aligned} \dot{\mathbf{z}}(t) - \dot{\mathbf{z}}^0(t) &= \begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}^0(t) \\ \dot{\mathbf{v}}(t) - \dot{\mathbf{v}}^0(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}(t) - \mathbf{v}^0(t) \\ \mathbf{a}_E(t, \mathbf{x}, \mathbf{p}) - \mathbf{a}_E(t, \mathbf{x}^0, \mathbf{p}^0) \end{bmatrix}. \end{aligned} \tag{20}$$

Denoting

$$\Delta\dot{\mathbf{z}}(t) = \dot{\mathbf{z}}(t) - \dot{\mathbf{z}}^0(t),$$

$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}^0(t),$$

$$\Delta\mathbf{v}(t) = \mathbf{v}(t) - \mathbf{v}^0(t),$$

and then linearizing the right hand side of (20), we have

$$\begin{aligned} \Delta\dot{\mathbf{z}}(t) &= \begin{bmatrix} \Delta\mathbf{v}(t) \\ \mathbf{F}_{ax}(t)\Delta\mathbf{x}(t) + \mathbf{F}_{ap}(t)\Delta\mathbf{p} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{F}_{ax}(t) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{v}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{ap}(t) \end{bmatrix} \Delta\mathbf{p} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{F}_{ax}(t) & \mathbf{0} \end{bmatrix} \Delta\mathbf{z}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{ap}(t) \end{bmatrix} \Delta\mathbf{p}, \end{aligned} \tag{21a}$$

which is a standard linear dynamical system of differential equations, where

$$\mathbf{F}_{ax}(t) = \left.\frac{\partial \mathbf{a}_E(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{x}^T}\right|_{\mathbf{x}=\mathbf{x}^0(t),\,\mathbf{p}=\mathbf{p}^0}, \tag{21b}$$

$$\mathbf{F}_{ap}(t) = \left.\frac{\partial \mathbf{a}_E(t,\mathbf{x},\mathbf{p})}{\partial \mathbf{p}^T}\right|_{\mathbf{x}=\mathbf{x}^0(t),\,\mathbf{p}=\mathbf{p}^0}, \tag{21c}$$

and $\mathbf{I}$ is a (3×3) identity matrix.

Given the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$ for the original problem of satellite gravimetry, we can have the corresponding initial conditions $\Delta\mathbf{z}_0$ for $\Delta\mathbf{z}(t)$. Thus, according to Stengel [99] and Grewal and Andrews [27], we can readily write the solution to the linear differential Eq. (21a) as follows:

$$\Delta\mathbf{z}(t) = \mathbf{\Phi}(t,t_0)\Delta\mathbf{z}_0 + \int_{t_0}^{t}\mathbf{\Phi}(t,\tau)\begin{bmatrix}\mathbf{0}\\ \mathbf{F}_{ap}(\tau)\end{bmatrix}d\tau\,\Delta\mathbf{p}, \tag{22}$$

or equivalently,

$$\mathbf{z}(t) = \mathbf{z}^0(t) + \mathbf{\Phi}(t,t_0)\Delta\mathbf{z}_0 + \int_{t_0}^{t}\mathbf{\Phi}(t,\tau)\begin{bmatrix}\mathbf{0}\\ \mathbf{F}_{ap}(\tau)\end{bmatrix}d\tau\,\Delta\mathbf{p}, \tag{23a}$$

where $\mathbf{\Phi}(t,\,t_0)$ is the state transition matrix and is equal to

$$\mathbf{\Phi}(t,t_0) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0), \tag{23b}$$

(see e.g., [27]), and the fundamental matrix $\mathbf{\Phi}(t)$ is the solution to the following matrix differential equations:

$$\dot{\mathbf{\Phi}}(t) = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{F}_{ax}(t) & \mathbf{0}\end{bmatrix}\mathbf{\Phi}(t), \tag{23c}$$

under the initial matrix conditions

$$\mathbf{\Phi}(t_0) = \mathbf{I}_6, \tag{23d}$$

with $\mathbf{I}_6$ is a (6×6) identity matrix. For more properties about $\mathbf{\Phi}(t)$, including its uniqueness and non-singularity, the reader is referred to [27].

It is clear that the solution (23a) of the satellite orbit and velocity is a linear vector function of the corrections $\Delta\mathbf{z}_0$ to the approximate initial values $[\mathbf{x}_0^0,\,\mathbf{v}_0^0]$ and the corrections $\Delta\mathbf{p}$ to the approximate values $\mathbf{p}^0$. Therefore, we can readily linearize the original satellite tracking measurement (7) with respect to $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$.

3.2. Numerical integration methods to construct local solutions

When numerical integration methods are required, one always assumes that the functions to be integrated are given and/or known, and the target is to use such methods to numerically compute the integration of the functions, as can be found in any standard textbooks on numerical analysis and numerical integration (see e.g., [100,106]). These techniques can be directly used to compute an approximate reference orbit of a satellite, given initial conditions $[\mathbf{x}_0^0,\,\mathbf{v}_0^0]$ and $\mathbf{p}^0$. However, in satellite gravimetry from satellite tracking measurements, since both initial conditions $[\mathbf{x}_0,\mathbf{v}_0]$ and the differential equation parameters $\mathbf{p}$ are unknown, it is impossible to exactly compute satellite orbits by directly implementing any well documented numerical integration methods.

In this part of the paper, unlike standard textbooks on numerical integration to compute the integral of a given function (without any unknown parameters) (see e.g., [100,106]), our basic idea is to construct a solution to nonlinear differential equations with unknown equation parameters and unknown initial conditions, with the aid of numerical integration methods. More precisely, in the case of satellite gravimetry from tracking measurements, we will represent the orbital solution $\mathbf{x}(t,\,\mathbf{p},\,\mathbf{c}(\mathbf{p},\,\mathbf{x}_0,\,\mathbf{v}_0))$ to the Newton's nonlinear differential Eq. (6) in terms of its approximate value, the unknown corrections $\Delta\mathbf{z}_0$ to the approximate initial values $[\mathbf{x}_0^0,\,\mathbf{v}_0^0]$ and the unknown corrections $\Delta\mathbf{p}$ of the harmonic coefficients by numerically solving the nonlinear differential Eq. (6) under the initial (unknown) conditions $\mathbf{x}_0$ and $\mathbf{v}_0$. Recall that for each measurement y_i at the time epoch t_{yi}, we obtain the nominal approximate orbit $\mathbf{x}^0(t_{yi},\mathbf{p}^0,\mathbf{c}(\mathbf{p}^0,\mathbf{x}_0^0,\mathbf{v}_0^0))$ by numerically solving the nonlinear differential Eq. (18) under the initial conditions $\mathbf{x}_0^0$ and $\mathbf{v}_0^0$ at the initial time epoch t_0. The procedure of numerical integration has to partition the time interval $[t_0,t_{yi}]$ into a number of sub-intervals, usually equidistant such that

$$t_j = t_0 + jh,\;\; j = 1,2,\ldots,m_{yi}$$

where $h = (t_{yi}-t_0)/m_{yi}$. One can then apply numerical integration methods to progressively compute all the nominal reference positions $\mathbf{x}^0(t_j,\mathbf{p}^0,\mathbf{c}(\mathbf{p}^0,\mathbf{x}_0^0,\mathbf{v}_0^0))$.

However, for the satellite gravimetry problem with tracking measurements, we do not have the true values of the satellite position and velocity at an initial time epoch t_0 but can only assume their approximate values. In addition, the harmonic coefficients $\mathbf{p}$ are unknown as well. Since the differential Eq. (6) are nonlinear, it is not likely to use analytical methods to directly derive a convergent, analytical representation of $\mathbf{x}(t,\,\mathbf{p},\,\mathbf{c}(\mathbf{p},\,\mathbf{x}_0,\,\mathbf{v}_0))$ in terms of $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$, unless one is satisfied

with a linearized, one-iteration solution. Thus, we will focus on explicit numerical integration methods to progressively solve the nonlinear differential Eq. (6) in the remainder of this section.

In what follows, we will use the Euler method and the modified Euler method to demonstrate the construction of $\mathbf{x}(t_j, \mathbf{p}, \mathbf{c}(\mathbf{p}, \mathbf{x}_0, \mathbf{v}_0))$ in terms of $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$. Other numerical integration methods such as Heun's method, Runge–Kutta methods of any order and/or the Newton–Cotes method can be treated in the same manner and will be omitted here. The interested reader can work them out by himself or herself. For conciseness of notations, we will denote the right hand side of (19) by $\mathbf{g}(t, \mathbf{z}(t), \mathbf{p})$ and rewrite (19) as follows:

$$\dot{\mathbf{z}}(t) = \mathbf{g}(t, \mathbf{z}(t), \mathbf{p}) \tag{24}$$

under the (unknown) initial conditions $\mathbf{z}_0$ (namely, $\mathbf{x}_0$ and $\mathbf{v}_0$).

To start the Euler method, we have

$$\mathbf{z}(t_1) = \mathbf{z}(t_0) + h\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}), \tag{25}$$

(see e.g., [100,106]). Linearizing the vector functions $\mathbf{g}(\cdot)$ at $(\mathbf{z}_0^0, \mathbf{p}^0)$ and bearing in mind the approximate orbit $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$, we can rewrite (25) into:

$$\mathbf{z}^0(t_1) + \Delta\mathbf{z}(t_1) = \mathbf{z}_0^0 + \Delta\mathbf{z}_0 + h\mathbf{g}(t_0, \mathbf{z}_0^0, \mathbf{p}^0) + h\mathbf{G}_{gz0}\Delta\mathbf{z}_0 + h\mathbf{G}_{gp0}\Delta\mathbf{p},$$

or equivalently,

$$\Delta\mathbf{z}(t_1) = \delta\mathbf{z}_{01}^0 + \left[\mathbf{I}_6 + h\mathbf{G}_{gz0}\right]\Delta\mathbf{z}_0 + h\mathbf{G}_{gp0}\Delta\mathbf{p}, \tag{26}$$

where

$$\delta\mathbf{z}_{01}^0 = \mathbf{z}_0^0 + h\mathbf{g}(t_0, \mathbf{z}_0^0, \mathbf{p}^0) - \mathbf{z}^0(t_1), \tag{27a}$$

$$\mathbf{G}_{gz0} = \left.\frac{\partial\mathbf{g}(t, \mathbf{z}(t), \mathbf{p})}{\partial\mathbf{z}^T}\right|_{\mathbf{z}(t)=\mathbf{z}_0^0,\ \mathbf{p}=\mathbf{p}^0}, \tag{27b}$$

and

$$\mathbf{G}_{gp0} = \left.\frac{\partial\mathbf{g}(t, \mathbf{z}(t), \mathbf{p})}{\partial\mathbf{p}^T}\right|_{\mathbf{z}(t)=\mathbf{z}_0^0,\ \mathbf{p}=\mathbf{p}^0}. \tag{27c}$$

To progress from t_1 to t_2, the Euler method takes the form of formula:

$$\mathbf{z}(t_2) = \mathbf{z}(t_1) + h\mathbf{g}(t_1, \mathbf{z}(t_1), \mathbf{p}). \tag{28}$$

Following the same procedure as described above, and neglecting all the terms of order h^2, we can rewrite the above formula as follows:

$$\Delta\mathbf{z}(t_2) = \delta\mathbf{z}_{12}^0 + \left[\mathbf{I}_6 + h\mathbf{G}_{gz1}\right]\Delta\mathbf{z}(t_1) + h\mathbf{G}_{gp1}\Delta\mathbf{p}, \tag{29}$$

where

$$\delta\mathbf{z}_{12}^0 = \mathbf{z}^0(t_1) + h\mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) - \mathbf{z}^0(t_2).$$

Inserting (26) into (29) and after some rearrangement, we have

$$\begin{aligned}\Delta\mathbf{z}(t_2) &= \delta\mathbf{z}_{12}^0 + \delta\mathbf{z}_{01}^0 + h\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 + \left[\mathbf{I}_6 + h\mathbf{G}_{gz0} + h\mathbf{G}_{gz1}\right]\Delta\mathbf{z}_0 + [\mathbf{G}_{gp0} + \mathbf{G}_{gp1}]\Delta\mathbf{p} \\ &= \delta\mathbf{z}_{02}^0 + h\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 + \left[\mathbf{I}_6 + h\sum_{j=0}^{1}\mathbf{G}_{gzj}\right]\Delta\mathbf{z}_0 + h\sum_{j=0}^{1}\mathbf{G}_{gpj}\Delta\mathbf{p},\end{aligned} \tag{30}$$

where

$$\delta\mathbf{z}_{02}^0 = \mathbf{z}_0^0 + h\sum_{j=0}^{1}\mathbf{g}(t_j, \mathbf{z}^0(t_j), \mathbf{p}^0) - \mathbf{z}^0(t_2).$$

The matrices $\mathbf{G}_{gz1}$ and $\mathbf{G}_{gp1}$ are computed in the same manner as in (27b) and (27c) but at the point of $\mathbf{z}^0(t_1)$.

Repeating the same procedure as described in the above, we can finally obtain the representation of the corrections $\Delta\mathbf{z}(t_{yi})$ as follows:

$$\Delta\mathbf{z}(t_{yi}) = \delta\mathbf{z}_{0t_{yi}}^0 + h\sum_{j=1}^{m_{yi}-1}\mathbf{G}_{gzj}\delta\mathbf{z}_{0j}^0 + \left[\mathbf{I}_6 + h\sum_{j=0}^{m_{yi}-1}\mathbf{G}_{gzj}\right]\Delta\mathbf{z}_0 + h\sum_{j=0}^{m_{yi}-1}\mathbf{G}_{gpj}\Delta\mathbf{p}, \tag{31}$$

where

$$\delta \mathbf{z}^0_{0t_{yi}} = \mathbf{z}^0_0 + h \sum_{j=0}^{m_{yi}-1} \mathbf{g}(t_j, \mathbf{z}^0(t_j), \mathbf{p}^0) - \mathbf{z}^0(t_{yi}).$$

In the similar manner, one can then work out the corrections for all the satellite tracking measurements $\mathbf{y}$, continue to linearize (7) and complete the construction of the observational equations for $\mathbf{y}$. Probably, we should note that the corrections $\Delta\mathbf{z}(t_{yi})$ of (31) contain constant calibrated terms, depending on $\delta\mathbf{z}^0_{0t_{yi}}$ and $\mathbf{G}_{gzj}\delta\mathbf{z}^0_{0j}$, plus the terms with the unknown orbital position and velocity corrections $\Delta\mathbf{z}_0$ and the unknown corrections $\Delta\mathbf{p}$ of the harmonic coefficients.

To further show that the representation of the corrections $\Delta\mathbf{z}(t_{yi})$ will change with different methods of numerical integration, we will now derive such a representation by using the modified Euler method. Given the initial (unknown) conditions $[\mathbf{x}_0, \mathbf{v}_0]$ and the unknown parameters $\mathbf{p}$, the modified Euler method formally starts with the following recursive formula:

$$\mathbf{z}(t_j) = \mathbf{z}(t_{j-1}) + \frac{h}{2}\Big[\mathbf{g}(t_{j-1}, \mathbf{z}(t_{j-1}), \mathbf{p}) + \mathbf{g}\{t_j, \mathbf{z}(t_{j-1}) + h\mathbf{g}(t_{j-1}, \mathbf{z}(t_{j-1}), \mathbf{p}), \mathbf{p}\}\Big], \tag{32}$$

for $j = 1, 2, \ldots, m_{yi}$ (see e.g., [106]), with the nominal reference orbit $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$.

Following the same technical procedure as in the case of the Euler method, we can finally obtain the representation of $\Delta\mathbf{z}(t_{yi})$ for the modified Euler method, as follows:

$$\begin{aligned}\Delta\mathbf{z}(t_{yi}) = {} & \delta\mathbf{z}^{0M}_{0t_{yi}} + \frac{h}{2}\sum_{j=1}^{m_{yi}-1}\Big[\mathbf{G}_{gzj} + \mathbf{G}_{gz(j+1)}\Big]\delta\mathbf{z}^{0M}_{0j} \\ & + \frac{h}{2}\sum_{j=0}^{m_{yi}-1}\mathbf{G}_{gz(j+1)}\delta\mathbf{z}^0_{j(j+1)} + \left[\mathbf{I}_6 + \frac{h}{2}\sum_{j=0}^{m_{yi}-1}\Big\{\mathbf{G}_{gzj} + \mathbf{G}_{gz(j+1)}\Big\}\right]\Delta\mathbf{z}_0 \\ & + \frac{h}{2}\sum_{j=0}^{m_{yi}-1}\Big[\mathbf{G}_{gpj} + \mathbf{G}_{gp(j+1)}\Big]\Delta\mathbf{p},\end{aligned} \tag{33}$$

where

$$\delta\mathbf{z}^{0M}_{0k} = \mathbf{z}^0_0 + \frac{h}{2}\sum_{j=0}^{k-1}\Big[\mathbf{g}(t_j, \mathbf{z}^0(t_j), \mathbf{p}^0) + \mathbf{g}\{t_{j+1}, \mathbf{z}^0(t_{j+1}), \mathbf{p}^0\}\Big] - \mathbf{z}^0(t_k),$$

and

$$\delta\mathbf{z}^0_{j(j+1)} = \mathbf{z}^0(t_j) + h\mathbf{g}(t_j, \mathbf{z}^0(t_j), \mathbf{p}^0) - \mathbf{z}^0(t_{j+1}).$$

The technical derivation of (33) is given in the appendix.

It is clear from (31) and (33) that different numerical integration methods will result in different representations of the orbital position and velocity corrections $\Delta\mathbf{z}(t_{yi})$, even though the formulae can be coded and the coefficients of both $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$ can be automatically computed. We should note that numerical integration schemes can be different for precisely computing the nominal orbital solution $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$ and for representing the corrections $\Delta\mathbf{z}(t_{yi})$ in terms of $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$. Precise numerical integration methods should be used to compute the nominal reference orbit of a satellite, given approximate initial conditions $[\mathbf{x}^0_0, \mathbf{v}^0_0]$ and a set of approximate harmonic coefficients $\mathbf{p}^0$. Implementations and interpretations of numerical integration methods in satellite geodesy are fundamentally different for computing the nominal orbital solution $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$ by solving the differential equation (18) under the initial conditions $[\mathbf{x}^0_0, \mathbf{v}^0_0]$ and for inverting for the unknown equation parameters $\mathbf{p}$ under the unknown initial conditions $[\mathbf{x}_0, \mathbf{v}_0]$ from satellite tracking measurements. The former is actually the problem of numerical orbit determination with given initial values and force parameters, but is only a first step towards the latter.

4. Measurement-based perturbation theory

Perturbation has been commonly carried out, either with a small parameter mathematically or around mean orbital elements in celestial mechanics (see e.g., [34,60,79,102]), though all the six orbital elements are the functions of time in reality. Although approximate perturbed solutions are useful to gain some physical insights into the orbit of a celestial body, they are too approximate to precisely invert for the unknown equation parameters from modern space observation. Kaula linear perturbation theory is a local approximate solution to Lagrange's planetary equations around mean orbital elements, which will be divergent with the increase of time and cannot utilize full advantages of unprecedented accuracy and continuity of tracking measurements from modern space observation technology. On the other hand, the numerical integration method, though widely used by major institutions worldwide to produce global Earth's gravitational models from satellite tracking measurements for highly multidisciplinary applications, has been proved to be groundless, mathematically and physically. To

fully use profound technological advance in space observation for the next generation of global gravitational models, mathematical solutions to the differential equations of motion of an LEO satellite must be sufficiently precise to extract small gravitational signals in modern space observation.

In this section, we will derive two perturbation solutions: one is local and the other is global. Our interest in constructing a local perturbation solution is mainly motivated to demonstrate how to properly use the nominal reference orbit $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$ to mathematically solve the governing differential Eq. (6) of motion of an LEO satellite, in addition to the approximate analytical and numerical integration solutions in Section 3. To take full advantages of modern and future (or next generation of) space observation technology, the key mathematics has to construct a global perturbation solution, which should meet the following two requirements: (i) the solution is either better than what modern and future space observation technology can provide physically, or at least, sufficiently precise at the noise level of such technology; and (ii) the solution is global uniformly convergent over arcs of any length. Since small gravitational signals will accumulate their effect on satellite orbits over time, this second condition will guarantee that we are able to extract smallest possible gravitational signals in modern and future space observation to its limit, and as a result, to produce high-precision, high-resolution global Earth's gravitational models.

As in the case of Xu [111], we will work out the perturbation solutions in the inertial reference frame. Nevertheless, to avoid any potential confusion of notations, we will now switch to the spherical coordinates (α, ζ) in the inertial reference frame, instead of continuing to use the notations (λ, θ) of [111]. Since the disturbing potential T of (34a) is in the spherical coordinate system, we will need coordinate transformation between the spherical coordinates (α, ζ, r) and the Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)^T$. To prepare for the derivations in the remainder of this section, we symbolically rewrite the disturbing potential (2), with (α, ζ) of (3) in the inertial reference frame, as follows:

$$T = \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} T_{lm}^c(\alpha, \zeta, r) C_{lm} + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} T_{lm}^s(\alpha, \zeta, r) S_{lm}, \tag{34a}$$

where

$$T_{lm}^c(\alpha, \zeta, r) = \frac{GM}{r}\left(\frac{R}{r}\right)^l \cos\{m(\alpha + \delta\alpha - \omega_e t)\} P_{lm}\{\cos(\zeta + \delta\zeta)\}, \tag{34b}$$

$$T_{lm}^s(\alpha, \zeta, r) = \frac{GM}{r}\left(\frac{R}{r}\right)^l \sin\{m(\alpha + \delta\alpha - \omega_e t)\} P_{lm}\{\cos(\zeta + \delta\zeta)\}. \tag{34c}$$

The partial derivatives of T with respect to $\mathbf{x}$ are given as follows:

$$\frac{\partial T}{\partial \mathbf{x}} = \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \mathbf{p}_{lm}^c(\alpha, \zeta, r) C_{lm} + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \mathbf{p}_{lm}^s(\alpha, \zeta, r) S_{lm}, \tag{35a}$$

where

$$\mathbf{p}_{lm}^c(\alpha, \zeta, r) = \mathbf{R}(\alpha, \zeta, r) \frac{\partial T_{lm}^c(\alpha, \zeta, r)}{\partial(\alpha, \zeta, r)^T}, \tag{35b}$$

$$\mathbf{p}_{lm}^s(\alpha, \zeta, r) = \mathbf{R}(\alpha, \zeta, r) \frac{\partial T_{lm}^s(\alpha, \zeta, r)}{\partial(\alpha, \zeta, r)^T}, \tag{35c}$$

$$\mathbf{R}(\alpha, \zeta, r) = \frac{\partial(\alpha, \zeta, r)}{\partial \mathbf{x}} = \begin{bmatrix} -\sin\alpha/(r\sin\zeta) & \cos\alpha/(r\sin\zeta) & 0 \\ \cos\alpha\cos\zeta/r & \sin\alpha\cos\zeta/r & -\sin\zeta/r \\ \cos\alpha\sin\zeta & \sin\alpha\sin\zeta & \cos\zeta \end{bmatrix}^T, \tag{35d}$$

$$\frac{\partial T_{lm}^c(\alpha, \zeta, r)}{\partial(\alpha, \zeta, r)^T} = -\frac{GM}{r}\left(\frac{R}{r}\right)^l \begin{bmatrix} m\sin\{m(\alpha + \delta\alpha - \omega_e t)\} P_{lm}\{\cos(\zeta + \delta\zeta)\} \\ \cos\{m(\alpha + \delta\alpha - \omega_e t)\}\sin(\zeta + \delta\zeta)\dot{P}_{lm}\{\cos(\zeta + \delta\zeta)\} \\ (l+1)\cos\{m(\alpha + \delta\alpha - \omega_e t)\} P_{lm}\{\cos(\zeta + \delta\zeta)\}/r \end{bmatrix}, \tag{35e}$$

$$\frac{\partial T_{lm}^s(\alpha, \zeta, r)}{\partial(\alpha, \zeta, r)^T} = -\frac{GM}{r}\left(\frac{R}{r}\right)^l \begin{bmatrix} -m\cos\{m(\alpha + \delta\alpha - \omega_e t)\} P_{lm}\{\cos(\zeta + \delta\zeta)\} \\ \sin\{m(\alpha + \delta\alpha - \omega_e t)\}\sin(\zeta + \delta\zeta)\dot{P}_{lm}\{\cos(\zeta + \delta\zeta)\} \\ (l+1)\sin\{m(\alpha + \delta\alpha - \omega_e t)\} P_{lm}\{\cos(\zeta + \delta\zeta)\}/r \end{bmatrix}, \tag{35f}$$

and $\dot{P}_{lm}(t)$ stands for the derivatives of the normalized Legendre function $P_{lm}(t)$ (see e.g., [59]).

4.1. Local perturbation around a nominal reference orbit

In Section 3, we have used linearization and numerical integration methods to construct local approximate solutions to (6). Given the approximate solution $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$, we will use the idea of Xu [111] to derive new local solutions by turning the nonlinear differential Eq. (6) into the equivalent nonlinear integral equations. More precisely, for convenience, we combine the nonlinear differential Eq. (6), the Earth's gravitational acceleration (5) and the Earth's disturbing potential T of (2), together with the unknown initial conditions $[\mathbf{z}_0, \mathbf{v}_0]$, and rewrite the complete system of nonlinear differential equations as follows:

$$\ddot{\mathbf{x}} = -\frac{GM}{r^3}\mathbf{x} + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{p}_{lm}^c(\alpha,\zeta,r)C_{lm} + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{p}_{lm}^s(\alpha,\zeta,r)S_{lm}, \tag{36}$$

under the initial conditions $[\mathbf{x}_0, \mathbf{v}_0]$.

The solution to (36) can be formally written as follows:

$$\begin{aligned}\mathbf{x}(t) = &-\int_{t_0}^{t}\int_{t_0}^{\eta}\frac{GM}{r^3(\tau)}\mathbf{x}(\tau)d\tau d\eta + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}C_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^c(\alpha(\tau),\zeta(\tau),r(\tau))d\tau d\eta \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}S_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^s(\alpha(\tau),\zeta(\tau),r(\tau))d\tau d\eta + \mathbf{v}_0(t-t_0) + \mathbf{x}_0,\end{aligned} \tag{37}$$

where the notations $[\alpha(\tau), \zeta(\tau), r(\tau)]$ are the same as those of $[\alpha, \zeta, r]$ but to explicitly emphasize that they are all the functions of time. Obviously, we have turned the nonlinear differential Eq. (36) into the nonlinear Volterra's integral Eq. (37) of the second kind.

Linearizing $\mathbf{x}(\tau)$ around the approximate orbit $\mathbf{x}^0(\tau)$ and bearing in mind that $\mathbf{x}^0(\tau)$ is essentially computed by integrating the same Eq. (37) with the initial values $[\mathbf{x}_0^0, \mathbf{v}_0^0, \mathbf{p}^0]$, we have

$$\begin{aligned}\Delta\mathbf{x}(t) = &-\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}^x(\mathbf{x}^0(\tau))\Delta\mathbf{x}(\tau)d\tau d\eta \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta C_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^c(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}C_{lm}^0\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}_{lm}^c(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))\Delta\mathbf{x}(\tau)d\tau d\eta \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta S_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^s(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}S_{lm}^0\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}_{lm}^s(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))\Delta\mathbf{x}(\tau)d\tau d\eta \\ &+\Delta\mathbf{v}_0(t-t_0) + \Delta\mathbf{x}_0,\end{aligned} \tag{38a}$$

where C_{lm}^0 and S_{lm}^0 are the approximate values of the harmonic coefficients used in computing the nominal reference orbit $\mathbf{x}^0(\tau)$, and

$$\mathbf{x}(\tau) = \mathbf{x}^0(\tau) + \Delta\mathbf{x}(\tau), \tag{38b}$$

$$\begin{aligned}\mathbf{A}^x(\mathbf{x}(\tau)) &= GM\frac{\partial}{\partial\mathbf{x}^T}\left(\frac{\mathbf{x}}{r^3}\right)\Bigg|_{\mathbf{x}=\mathbf{x}^0(\tau)} \\ &= GM\left\{\frac{1}{r_0^3(\tau)}\mathbf{I} - \frac{3}{r_0^5(\tau)}\mathbf{x}^0(\tau)\left[\mathbf{x}^0(\tau)\right]^T\right\},\end{aligned} \tag{38c}$$

$$r_0(\tau) = \sqrt{[\mathbf{x}^0(\tau)]^T\mathbf{x}^0(\tau)}, \tag{38d}$$

$$\mathbf{A}_{lm}^c(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau)) = \frac{\partial\mathbf{p}_{lm}^c(\alpha,\zeta,r)}{\partial\mathbf{x}^T}\Bigg|_{\mathbf{x}=\mathbf{x}^0(\tau)}$$

$$= \left.\frac{\partial \mathbf{p}^c_{lm}(\alpha,\zeta,r)}{\partial(\alpha,\zeta,r)}\right|_{\mathbf{x}=\mathbf{x}^0(\tau)} [\mathbf{R}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))]^T, \tag{38e}$$

$$\mathbf{A}^s_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau)) = \left.\frac{\partial \mathbf{p}^s_{lm}(\alpha,\zeta,r)}{\partial \mathbf{x}^T}\right|_{\mathbf{x}=\mathbf{x}^0(\tau)}$$
$$= \left.\frac{\partial \mathbf{p}^s_{lm}(\alpha,\zeta,r)}{\partial(\alpha,\zeta,r)}\right|_{\mathbf{x}=\mathbf{x}^0(\tau)} [\mathbf{R}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))]^T. \tag{38f}$$

The linearized Volterra's integral Eq. (38a) can be solved successively (see e.g., [33,58]). To start with, one can set $\Delta\mathbf{x}(\tau)$ on the right hand side of (38a) to zero and obtain the zeroth approximate (or quasi-linear) solution as follows:

$$\begin{aligned}\Delta\mathbf{x}(t) = &\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta C_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}^c_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta\\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta S_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}^s_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta\\ &+\Delta\mathbf{v}_0(t-t_0)+\Delta\mathbf{x}_0.\end{aligned} \tag{39}$$

Inserting the quasi-linear solution (39) into the right hand side of the integral Eq. (38a) and neglecting all the second order terms of the harmonic coefficients and the cross-product terms of the harmonic coefficients and $\Delta\mathbf{x}(\tau)$, we can derive the linear approximation solution as follows:

$$\begin{aligned}\Delta\mathbf{x}(t) = &-\int_{t_0}^{t}\int_{t_0}^{\eta}d\tau d\eta\mathbf{A}^x(\mathbf{x}^0(\tau))\Bigg\{\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{\tau}\int_{t_0}^{t_1}\Delta C_{lm}\mathbf{p}^c_{lm}(\alpha^0(t_2),\zeta^0(t_2),r^0(t_2))dt_2dt_1\\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{\tau}\int_{t_0}^{t_1}\Delta S_{lm}\mathbf{p}^s_{lm}(\alpha^0(t_2),\zeta^0(t_2),r^0(t_2))dt_2dt_1\\ &+\Delta\mathbf{v}_0(\tau-t_0)+\Delta\mathbf{x}_0\Bigg\}\\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{t}\int_{t_0}^{\eta}\Delta C_{lm}\mathbf{p}^c_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta\\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{t}\int_{t_0}^{\eta}\Delta S_{lm}\mathbf{p}^s_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta\\ &+\Delta\mathbf{v}_0(t-t_0)+\Delta\mathbf{x}_0,\end{aligned} \tag{40}$$

which, after some re-arrangement, becomes:

$$\Delta\mathbf{x}(t) = \mathbf{D}^x_v\Delta\mathbf{v}_0+\mathbf{D}^x_x\Delta\mathbf{x}_0+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}^{cx}_{lm}\Delta C_{lm}+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}^{sx}_{lm}\Delta S_{lm}, \tag{41a}$$

where

$$\mathbf{D}^x_v = \mathbf{I}\,(t-t_0)-\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}^x(\mathbf{x}^0(\tau))(\tau-t_0)d\tau d\eta, \tag{41b}$$

$$\mathbf{D}^x_x = \mathbf{I}-\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}^x(\mathbf{x}^0(\tau))d\tau d\eta, \tag{41c}$$

$$\begin{aligned}\mathbf{d}^{cx}_{lm} = &\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}^c_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta\\ &-\int_{t_0}^{t}\int_{t_0}^{\eta}d\tau d\eta\mathbf{A}^x(\mathbf{x}^0(\tau))\left\{\int_{t_0}^{\tau}\int_{t_0}^{t_1}\mathbf{p}^c_{lm}(\alpha^0(t_2),\zeta^0(t_2),r^0(t_2))dt_1dt_2\right\},\end{aligned} \tag{41d}$$

$$\begin{aligned}\mathbf{d}^{sx}_{lm} = &\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}^s_{lm}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau d\eta\\ &-\int_{t_0}^{t}\int_{t_0}^{\eta}d\tau d\eta\mathbf{A}^x(\mathbf{x}^0(\tau))\left\{\int_{t_0}^{\tau}\int_{t_0}^{t_1}\mathbf{p}^s_{lm}(\alpha^0(t_2),\zeta^0(t_2),r^0(t_2))dt_1dt_2\right\}.\end{aligned} \tag{41e}$$

If one is interested in constructing the second order solution of $\mathbf{x}(t)$, one will have to expand the nonlinear integral Eq. (37) into the Taylor series and truncate it up to the second order approximation. Then one can repeat the above procedure, insert the linear solution (41a) into the truncated second order integral equations, and finally obtain the second order solution of $\Delta\mathbf{x}(t)$ in terms of the unknown corrections $\Delta\mathbf{x}_0$, $\Delta\mathbf{v}_0$, ΔC_{lm} and ΔS_{lm}. Because the solutions derived in the above are only of local nature, we will not go further for the second order local solutions.

In the case of velocity of satellite motion, again bearing in mind that $\mathbf{v}^0(t)$ has been equivalently computed by integrating the right hand side of the following integral equations with $[\mathbf{x}_0^0, \mathbf{v}_0^0]$ and $\mathbf{p}^0$, we linearize the following integral equations of velocity:

$$\mathbf{v}(t) = -\int_{t_0}^{t} \frac{GM}{r^3(\tau)}\mathbf{x}(\tau)d\tau + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} C_{lm}\int_{t_0}^{t}\mathbf{p}_{lm}^{c}(\alpha(\tau),\zeta(\tau),r(\tau))d\tau + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} S_{lm}\int_{t_0}^{t}\mathbf{p}_{lm}^{s}(\alpha(\tau),\zeta(\tau),r(\tau))d\tau + \mathbf{v}_0, \tag{42}$$

around the nominal reference orbit and velocity $[\mathbf{x}^0(\tau),\mathbf{v}^0(\tau)]$ and obtain:

$$\begin{aligned}\Delta\mathbf{v}(t) = &-\int_{t_0}^{t}\mathbf{A}^x(\mathbf{x}^0(\tau))\Delta\mathbf{x}(\tau)d\tau \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta C_{lm}\int_{t_0}^{t}\mathbf{p}_{lm}^{c}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}C_{lm}^0\int_{t_0}^{t}\mathbf{A}_{lm}^{c}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))\Delta\mathbf{x}(\tau)d\tau \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta S_{lm}\int_{t_0}^{t}\mathbf{p}_{lm}^{s}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}S_{lm}^0\int_{t_0}^{t}\mathbf{A}_{lm}^{s}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))\Delta\mathbf{x}(\tau)d\tau + \Delta\mathbf{v}_0.\end{aligned} \tag{43}$$

As in the case of (39), by setting $\Delta\mathbf{x}(\tau)$ on the right hand side of (43) to zero, we obtain the zeroth order approximate (or quasi-linear) solution of the velocity as follows:

$$\begin{aligned}\Delta\mathbf{v}(t) = &\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta C_{lm}\int_{t_0}^{t}\mathbf{p}_{lm}^{c}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau \\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\Delta S_{lm}\int_{t_0}^{t}\mathbf{p}_{lm}^{s}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau + \Delta\mathbf{v}_0.\end{aligned} \tag{44}$$

To derive the linear solution of the velocity, by using the same approach as in deriving the linear solution for the orbit, we can insert the quasi-linear solution (39) into the right hand side of (43) and obtain the linear solution as follows:

$$\Delta\mathbf{v}(t) = \mathbf{D}_v^v\Delta\mathbf{v}_0 + \mathbf{D}_x^v\Delta\mathbf{x}_0 + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}_{lm}^{cv}\Delta C_{lm} + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}_{lm}^{sv}\Delta S_{lm}, \tag{45}$$

where

$$\mathbf{D}_v^v = \mathbf{I} - \int_{t_0}^{t}\mathbf{A}^x(\mathbf{x}^0(\tau))(\tau - t_0)d\tau, \tag{46a}$$

$$\mathbf{D}_x^v = -\int_{t_0}^{t}\mathbf{A}^x(\mathbf{x}^0(\tau))d\tau, \tag{46b}$$

$$\begin{aligned}\mathbf{d}_{lm}^{cv} = &\int_{t_0}^{t}\mathbf{p}_{lm}^{c}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau \\ &-\int_{t_0}^{t}d\tau\mathbf{A}^x(\mathbf{x}^0(\tau))\left\{\int_{t_0}^{\tau}\int_{t_0}^{t_1}\mathbf{p}_{lm}^{c}(\alpha^0(t_2),\zeta^0(t_2),r^0(t_2))dt_1dt_2\right\},\end{aligned} \tag{46c}$$

$$\mathbf{d}_{lm}^{sv} = \int_{t_0}^{t}\mathbf{p}_{lm}^{s}(\alpha^0(\tau),\zeta^0(\tau),r^0(\tau))d\tau$$

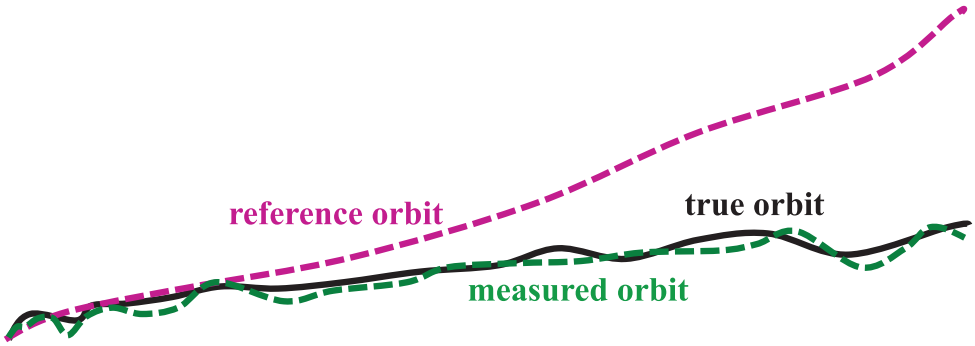

Fig. 1. Illustration of the relationship among the nominal reference orbit (pink-dashed line), the precisely measured orbit (green-dashed line) and the true (but unknown) orbit (black line) of an LEO satellite (modified after [114,116]).

$$-\int_{t_0}^{t} d\tau \mathbf{A}^x(\mathbf{x}^0(\tau))\left\{\int_{t_0}^{\tau}\int_{t_0}^{t_1} \mathbf{p}_{lm}^s(\alpha^0(t_2), \zeta^0(t_2), r^0(t_2))dt_1 dt_2\right\}. \tag{46d}$$

4.2. Global uniformly convergent measurement-based perturbation

With the technological advance in GNSS systems and GNSS receiver hardware, orbits of LEO satellites can now be measured almost continuously (at the sampling rate of 100 Hz or likely even 200 Hz in the near future) and precisely (at the cm and/or even mm level of accuracy). Thus, without loss of generality, we will assume a precisely measured orbit for an LEO gravity satellite, which is denoted analytically as $\{\mathbf{x}_o(\tau)\,|\,t_0 \le \tau \le t\}$ over the whole arc of orbit, with the subscript o standing for *observed*. This measured orbit $\mathbf{x}_o(\tau)$ is only slightly different from the true orbit $\mathbf{x}(\tau)$ at the level of random errors of measurements. Instead of using the nominal reference orbit $\{\mathbf{x}^0(\tau)\,|\,t_0 \le \tau \le t\}$ to derive local solutions for $\Delta\mathbf{x}(t)$ and $\Delta\mathbf{v}(t)$, we should certainly use the measured orbit $\mathbf{x}_o(\tau)$ to construct the solutions to the nonlinear differential Eq. (36), which will then never diverge with the increase of time. The relationship among the nominal reference orbit, the precisely measured orbit and the true but unknown orbit of an LEO satellite is illustrated in Fig. 1. In other words, since the measured orbit is very precise, all the solutions to be derived here are only different from the true orbit and velocity at the level of measurement noise and are guaranteed to converge globally uniformly, no matter how long an orbital arc can be.

For the nominal reference orbit $\mathbf{x}^0(\tau)$, the relative error $|\mathbf{x}(\tau) - \mathbf{x}^0(\tau)|/r$ will be unbounded with the increase of time. Thus, perturbation solutions with the approximate $\mathbf{x}^0(\tau)$ are only valid locally and will diverge with the increase of time. Since we have precisely measured orbits $\mathbf{x}_o(\tau)$, $|\mathbf{x}(\tau) - \mathbf{x}_o(\tau)|/r$ will remain small. Bearing in mind that the orbit of an LEO satellite can be geometrically measured at the cm and/or even mm level of accuracy with GNSS, for simplicity, say 1 cm, and by assuming that an LEO satellite is of altitude of 230 km, with 6371 km as the mean radius of the Earth, then the relative error $|\mathbf{x}(\tau) - \mathbf{x}_o(\tau)|/r$ would be roughly as small as 1.515×10^{-9}, irrelevant to the length of an orbital arc. In this case, the second and higher order terms can be negligible in the expansion of the nonlinear integral Eq. (37) around $\mathbf{x}_o(\tau)$.

To start with, we denote

$$\mathbf{x}(\tau) = \mathbf{x}_o(\tau) + \Delta\mathbf{x}(\tau). \tag{47}$$

Unlike $\mathbf{x}^0(\tau)$, $\mathbf{x}_o(\tau)$ is directly measured and does not satisfy the governing differential Eq. (36); thus, we cannot simply replace $\mathbf{x}^0(\tau)$ with $\mathbf{x}_o(\tau)$ in Section 4.1 to obtain the corresponding quasi-linear and linear solutions for $\mathbf{x}(t)$ and $\mathbf{v}(t)$. Instead, we linearize the nonlinear integral Eq. (37) around $\mathbf{x}_o(\tau)$ and obtain

$$\begin{aligned}
\mathbf{x}_o(t) + \Delta\mathbf{x}(t) = -\mathbf{x}_E^0(t) &- \int_{t_0}^{t}\int_{t_0}^{\eta} \mathbf{A}^x(\mathbf{x}_o)\Delta\mathbf{x}(\tau)d\tau d\eta \\
&+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{t}\int_{t_0}^{\eta} C_{lm}\mathbf{p}_{lm}^c(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))d\tau d\eta \\
&+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{t}\int_{t_0}^{\eta} C_{lm}\mathbf{A}_{lm}^c(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))\Delta\mathbf{x}(\tau)d\tau d\eta \\
&+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{t}\int_{t_0}^{\eta} S_{lm}\mathbf{p}_{lm}^s(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))d\tau d\eta \\
&+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{t}\int_{t_0}^{\eta} S_{lm}\mathbf{A}_{lm}^s(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))\Delta\mathbf{x}(\tau)d\tau d\eta
\end{aligned}$$

$$+\mathbf{v}_0^0(t-t_0)+\Delta\mathbf{v}_0(t-t_0)+\mathbf{x}_0^0+\Delta\mathbf{x}_0, \tag{48a}$$

where

$$\mathbf{x}_E^0(t)=\int_{t_0}^{t}\int_{t_0}^{\eta}\frac{GM}{r_o^3(\tau)}\mathbf{x}_o(\tau)d\tau d\eta. \tag{48b}$$

Other notations have been defined as in Section 4.1 but have to be computed by replacing the nominal orbit $\{\mathbf{x}^0(\tau)\,|\,t_0\le\tau\le t\}$ with the measured orbit $\{\mathbf{x}_o(\tau)\,|\,t_0\le\tau\le t\}$.

As in the case of (39), by setting $\Delta\mathbf{x}(\tau)$ on the right hand side of (48a) to zero, we obtain the zeroth order approximation or quasi-linear solution:

$$\begin{aligned}\Delta\mathbf{x}(t)&=\delta\mathbf{x}_0(t)+\Delta\mathbf{v}_0(t-t_0)+\Delta\mathbf{x}_0\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}C_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^c(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))d\tau d\eta\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}S_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^s(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))d\tau d\eta,\end{aligned} \tag{49}$$

where

$$\delta\mathbf{x}_0(t)=-\mathbf{x}_o(t)-\mathbf{x}_E^0(t)+\mathbf{v}_0^0(t-t_0)+\mathbf{x}_0^0.$$

Mathematically, the quasi-linear solution (49) is essentially equivalent to treating the measured orbit $\mathbf{x}_o(\tau)$ as the true (and given) orbit and substituting the unknown true orbit $\mathbf{x}(\tau)$ on the right hand side of (37) with this measured orbit $\mathbf{x}_o(\tau)$. Since all the integrals on the right hand side of (37) can be directly computed numerically with $\mathbf{x}_o(\tau)$, the solution $\mathbf{x}(t)$ on the left hand side of (37) can naturally be represented in terms of the unknown harmonic coefficients C_{lm} and S_{lm}. In other words, the solution (49) can be alternatively expressed as follows:

$$\begin{aligned}\mathbf{x}(t)&=-\int_{t_0}^{t}\int_{t_0}^{\eta}\frac{GM}{r_o^3(\tau)}\mathbf{x}_o(\tau)d\tau d\eta+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}C_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^c(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))d\tau d\eta\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}S_{lm}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{p}_{lm}^s(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))d\tau d\eta+\mathbf{v}_0(t-t_0)+\mathbf{x}_0.\end{aligned} \tag{50}$$

In a similar manner, by inserting (49) into (48a), we can then construct the linear perturbation solution. If we neglect the small terms $C_{lm}\Delta\mathbf{x}(\tau)$ and $S_{lm}\Delta\mathbf{x}(\tau)$ on the right hand side of (48a), we can obtain the linear perturbation solution of $\Delta\mathbf{x}(t)$ as follows:

$$\begin{aligned}\Delta\mathbf{x}(t)&=\mathbf{I}_0^x(t)+\mathbf{D}_v^x\Delta\mathbf{v}_0+\mathbf{D}_x^x\Delta\mathbf{x}_0\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}_{lm}^{cx}C_{lm}+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}_{lm}^{sx}S_{lm},\end{aligned} \tag{51}$$

where

$$\mathbf{I}_0^x(t)=\delta\mathbf{x}_0(t)-\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}^x(\mathbf{x}_o(\tau))\delta\mathbf{x}_0(\tau)d\tau d\eta,$$

$\mathbf{D}_v^x$, $\mathbf{D}_x^x$, $\mathbf{d}_{lm}^{cx}$ and $\mathbf{d}_{lm}^{sx}$ have been defined as in (41b)–(41e), respectively, but with $\mathbf{x}^0(\tau)$ there replaced by $\mathbf{x}_o(\tau)$ for use in (51).

Instead of completely neglecting the terms $C_{lm}\Delta\mathbf{x}(\tau)$ and $S_{lm}\Delta\mathbf{x}(\tau)$ altogether, since C_{20} is larger than other harmonic coefficients by an order of about 1,000, one may like to consider the term $C_{20}\Delta\mathbf{x}(\tau)$ to construct another linear perturbation solution. In this case, this new linear perturbation solution with the term $C_{20}\Delta\mathbf{x}(\tau)$ will become:

$$\begin{aligned}\Delta\mathbf{x}(t)&=\mathbf{I}_0^x(t)+\mathbf{D}_v^x\Delta\mathbf{v}_0+\mathbf{D}_x^x\Delta\mathbf{x}_0\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}_{lm}^{cx}C_{lm}+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\mathbf{d}_{lm}^{sx}S_{lm}\\&+C_{20}\int_{t_0}^{t}\int_{t_0}^{\eta}\mathbf{A}_{lm}^c(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))\Big\{\delta\mathbf{x}_0(\tau)+\Delta\mathbf{v}_0(\tau-t_0)+\Delta\mathbf{x}_0\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{\tau}\int_{t_0}^{t_1}C_{lm}\mathbf{p}_{lm}^c(\alpha_o(t_2),\zeta_o(t_2),r_o(t_2))dt_2dt_1\\&+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}\int_{t_0}^{\tau}\int_{t_0}^{t_1}S_{lm}\mathbf{p}_{lm}^s(\alpha_o(t_2),\zeta_o(t_2),r_o(t_2))dt_2dt_1\Big\}d\tau d\eta\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{l}_0^x(t) + \mathbf{D}_v^x \Delta\mathbf{v}_0 + \mathbf{D}_x^x \Delta\mathbf{x}_0 \\
&\quad + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \mathbf{d}_{lm}^{cx} C_{lm} + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \mathbf{d}_{lm}^{sx} S_{lm} \\
&\quad + C_{20} \int_{t_0}^{t} \int_{t_0}^{\eta} \mathbf{A}_{lm}^{c}(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau)) \delta\mathbf{x}_0(\tau) d\tau d\eta \\
&\quad + C_{20} \Delta\mathbf{v}_0 \int_{t_0}^{t} \int_{t_0}^{\eta} \mathbf{A}_{lm}^{c}(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))(\tau - t_0) d\tau d\eta \\
&\quad + C_{20} \Delta\mathbf{x}_0 \int_{t_0}^{t} \int_{t_0}^{\eta} \mathbf{A}_{lm}^{c}(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau)) d\tau d\eta \\
&\quad + C_{20} \int_{t_0}^{t} \int_{t_0}^{\eta} \mathbf{A}_{lm}^{c}(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau)) \Bigg\{ \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \int_{t_0}^{\tau} \int_{t_0}^{t_1} C_{lm} \mathbf{p}_{lm}^{c}(\alpha_o(t_2), \zeta_o(t_2), r_o(t_2)) dt_2 dt_1 \\
&\quad + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \int_{t_0}^{\tau} \int_{t_0}^{t_1} S_{lm} \mathbf{p}_{lm}^{s}(\alpha_o(t_2), \zeta_o(t_2), r_o(t_2)) dt_2 dt_1 \Bigg\} d\tau d\eta.
\end{aligned} \tag{52}
$$

We should note that although (52) is a linear perturbation solution to the nonlinear Volterra's integral Eq. (37) of the second kind, it is clearly nonlinear with respect to the unknown parameters, namely, $\Delta\mathbf{v}_0$, $\Delta\mathbf{x}_0$, and the harmonic coefficients C_{lm} and S_{lm}.

With the measured orbit $\mathbf{x}_o(\tau)$ in hands, we can also construct the global uniformly convergent quasi-linear and linear solutions to the velocity of satellite motion, which can be obtained by using the same approach as in deriving the solutions (49), (51) and (52). More precisely, by substituting the unknown (true) orbit $\mathbf{x}(\tau)$ (and equivalently, $[\alpha(\tau), \zeta(\tau), r(\tau)]$) on the right hand side of (42) with the measured orbit $\mathbf{x}_o(\tau)$, we can readily construct the quasi-linear perturbation solution of the velocity of satellite motion as follows:

$$
\begin{aligned}
\mathbf{v}(t) &= -\int_{t_0}^{t} \frac{GM}{r_o^3(\tau)} \mathbf{x}_o(\tau) d\tau + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} C_{lm} \int_{t_0}^{t} \mathbf{p}_{lm}^{c}(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau)) d\tau \\
&\quad + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} S_{lm} \int_{t_0}^{t} \mathbf{p}_{lm}^{s}(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau)) d\tau + \mathbf{v}_0^0 + \Delta\mathbf{v}_0.
\end{aligned} \tag{53}
$$

The quasi-linear solution (53) is obviously global uniformly convergent, since the measured $\mathbf{x}_o(\tau)$ $(t_0 \le \tau \le t)$ is a (precisely measured) realization of the unknown, true orbit $\mathbf{x}(\tau)$ $(t_0 \le \tau \le t)$, no matter how lengthy the arc of orbit is.

In a similar manner, we can linearize the integral Eq. (42) around the measured orbit $\mathbf{x}_o(\tau)$ $(t_0 \le \tau \le t)$, substitute the incremental $\Delta\mathbf{x}(\tau)$ with the quasi-linear solution (49), neglect the terms of $C_{lm}\Delta\mathbf{x}(\tau)$ and $S_{lm}\Delta\mathbf{x}(\tau)$, and finally obtain the linear perturbation solution of the velocity as follows:

$$
\mathbf{v}(t) = \mathbf{l}_0^v(t) + \mathbf{D}_v^v \Delta\mathbf{v}_0 + \mathbf{D}_x^v \Delta\mathbf{x}_0 + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \mathbf{d}_{lm}^{cv} C_{lm} + \sum_{l=2}^{N_{\max}} \sum_{m=0}^{l} \mathbf{d}_{lm}^{sv} S_{lm}, \tag{54}
$$

where

$$
\mathbf{l}_0^v(t) = -\int_{t_0}^{t} \frac{GM}{r_o^3(\tau)} \mathbf{x}_o(\tau) d\tau - \int_{t_0}^{t} \mathbf{A}^x(\mathbf{x}_o(\tau)) \delta\mathbf{x}_0(\tau) d\tau + \mathbf{v}_0^0,
$$

the coefficient matrices and vectors $\mathbf{D}_v^v$, $\mathbf{D}_x^v$, $\mathbf{d}_{lm}^{cv}$ and $\mathbf{d}_{lm}^{sv}$ have been defined as in (46a)–(46d), respectively, but computed with the measured orbit $\mathbf{x}_o(\tau)$ instead of the approximate nominal orbit $\mathbf{x}^0(\tau)$. If one would be interested in constructing the linear solution of the velocity with the term $C_{20}\Delta\mathbf{x}(\tau)$, one can follow the same approach as in the derivation of (52), which is omitted here, nevertheless.

In the previous two sections, i.e. Sections 3 and 4, we have solved for the orbital position and velocity solutions to the Newton's governing differential Eq. (36) of satellite motion and represented them in terms of the unknown equation parameters and the unknown initial conditions, given either the nominal reference orbit $\mathbf{x}^0(\tau)$ $(t_0 \le \tau \le t)$ or the measured orbit $\mathbf{x}_o(\tau)$ $(t_0 \le \tau \le t)$. The derived orbital and velocity solutions can then be used to establish the links between satellite tracking measurements and the unknown gravitational parameters and the corrections to the approximate initial values. For more details on observational equations of space measurements, including satellite tracking measurements and satellite-to-satellite tracking measurements, the reader is referred to [49,52,67,70,111]. By directly applying the least squares principle to the (nonlinear and/or linearized) observational equations, one can obtain the optimal estimate of the Earth's gravitational field. We may like to point out that in the case of quasi-linear perturbation with measured orbits, no iteration will be needed in the least squares estimation of the unknown gravitational parameters, since the perturbation solutions are linear with respect to the gravitational parameters and global uniformly convergent. Actually, we even do not need an initial force model, namely, initial approximate values of the unknown parameters $\mathbf{p}$.

5. Measurement-based condition adjustment with parameters

In this section, we will briefly outline an alternative method to estimate the gravitational field from measured orbits and satellite tracking measurements. The basic idea now is to treat the integral equations as natural equality constraints on the expectations of measurements and the unknown parameters, namely, the unknown harmonic coefficients and the unknown initial condition values. These equality constraints will automatically become a standard model of condition equations with unknown parameters, which can then naturally be solved by using the condition (LS) adjustment with parameters.

To start with, let us again assume the measured orbit $\mathbf{x}_o(\tau)$ $(t_0 \le \tau \le t)$ and a number of satellite tracking measurements y_i. These tracking measurements are assumed to be collected at different time epochs of $(t_{y1}, t_{y2}, \ldots, t_{yn}) \in [t_0, t]$ and, each of y_i is assumed, without loss of generality, to be the function of the satellite position (and likely, also velocity) at this particular epoch t_{yi}. If a measurement is involved with more than one LEO satellite, then the corresponding measurement is the function of the positions and velocities of all these satellites.

Under the above assumptions, we can rewrite the true position $\mathbf{x}(\tau)$ of a satellite as the measured position $\mathbf{x}_o(\tau)$ plus its correction $\boldsymbol{\xi}_{x\tau}$, namely,

$$\mathbf{x}(\tau) = \mathbf{x}_o(\tau) + \boldsymbol{\xi}_{x\tau}. \tag{55}$$

If the velocity of the satellite is not directly measured, then we can treat the velocity as an unknown vector. Thus, we have the equality constraint with unknowns as follows:

$$\begin{aligned}\mathbf{v}(t_{yi}) = &-\int_{t_0}^{t_{yi}} \frac{GM}{r_{o\xi}^3}[\mathbf{x}_o(\tau) + \boldsymbol{\xi}_{x\tau}]d\tau + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} C_{lm}\int_{t_0}^{t_{yi}} \mathbf{p}_{lm}^c(\alpha_{o\xi}, \zeta_{o\xi}, r_{o\xi})d\tau \\ &+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} S_{lm}\int_{t_0}^{t_{yi}} \mathbf{p}_{lm}^s(\alpha_{o\xi}, \zeta_{o\xi}, r_{o\xi})d\tau + \mathbf{v}_0,\end{aligned} \tag{56}$$

where $[\alpha_{o\xi}, \zeta_{o\xi}, r_{o\xi}]$ are transformed from $[\mathbf{x}_o(\tau) + \boldsymbol{\xi}_{x\tau}]$. In practice, $\mathbf{x}_o(\tau)$ $(t_0 \le \tau \le t)$ is only given in a densely discrete format. For simplicity, we assume that the orbit is sampled with an equal interval and denoted by $\mathbf{x}_o(t_i)$ $(0 \le i \le n_t)$. Thus, all the integrals on the right hand side of (56) can only be computed numerically by using numerical integration rules such as Newton-Cotes formulae or Gaussian integration rules. Since $\boldsymbol{\xi}_{x\tau}$ are small at the level of measurement noise, if we neglect all the terms of $C_{lm}\boldsymbol{\xi}_{x\tau}$ and $S_{lm}\boldsymbol{\xi}_{x\tau}$, the linearized version of (56) should then be equivalently written in the discretized form as follows:

$$\begin{aligned}\mathbf{v}(t_{yi}) = &-\int_{t_0}^{t_{yi}} \frac{GM}{r_o^3}\mathbf{x}_o(\tau)d\tau - \sum_{j=0}^{m_{yi}} w_j \mathbf{A}^x(\mathbf{x}_o(t_j))\boldsymbol{\xi}_{xt_j} + \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} C_{lm}\int_{t_0}^{t_{yi}} \mathbf{p}_{lm}^c(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))d\tau \\ &+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} S_{lm}\int_{t_0}^{t_{yi}} \mathbf{p}_{lm}^s(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))d\tau + \mathbf{v}_0^0 + \Delta\mathbf{v}_0,\end{aligned} \tag{57}$$

where w_j are positive coefficients, which are given, depending solely on the chosen rule of numerical integration. We still keep some integral notations in (57), mainly to emphasize that the integration rules for $\boldsymbol{\xi}_{xt_j}$ can be different from those integrals without $\boldsymbol{\xi}_{xt_j}$. For more details on numerical integration, the reader is referred to [82,100].

If the velocity of the satellite is also measured, then (57) should be replaced by

$$\begin{aligned}\mathbf{v}_o(t_{yi}) + \boldsymbol{\xi}_{vt_{yi}} = &-\int_{t_0}^{t_{yi}} \frac{GM}{r_o^3}\mathbf{x}_o(\tau)d\tau - \sum_{j=0}^{m_{yi}} w_j \mathbf{A}^x(\mathbf{x}_o(t_j))\boldsymbol{\xi}_{xt_j} \\ &+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} C_{lm}\int_{t_0}^{t_{yi}} \mathbf{p}_{lm}^c(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))d\tau \\ &+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} S_{lm}\int_{t_0}^{t_{yi}} \mathbf{p}_{lm}^s(\alpha_o(\tau), \zeta_o(\tau), r_o(\tau))d\tau + \mathbf{v}_0^0 + \Delta\mathbf{v}_0,\end{aligned} \tag{58}$$

where $\mathbf{v}_o(t_{yi})$ and $\boldsymbol{\xi}_{vt_{yi}}$ stand for the measurements of the velocity and the corrections at the time epoch t_{yi}, respectively.

For the measured orbital position $\mathbf{x}_o(t_i)$, we have the starting condition equations:

$$\begin{aligned}\mathbf{x}_o(t_i) + \boldsymbol{\xi}_{xt_i} = &-\int_{t_0}^{t_i}\int_{t_0}^{\eta} \frac{GM}{r_{o\xi}^3}[\mathbf{x}_o(\tau) + \boldsymbol{\xi}_{x\tau}]d\tau d\eta \\ &+ \sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} C_{lm}\int_{t_0}^{t_i}\int_{t_0}^{\eta} \mathbf{p}_{lm}^c(\alpha_{o\xi}, \zeta_{o\xi}, r_{o\xi})d\tau d\eta\end{aligned}$$

$$+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l} S_{lm}\int_{t_0}^{t_i}\int_{t_0}^{\eta}\mathbf{p}_{lm}^{s}(\alpha_{o\xi},\zeta_{o\xi},r_{o\xi})d\tau d\eta+\mathbf{v}_0(t_i-t_0)+\mathbf{x}_0. \tag{59}$$

As in the case of velocity, if we neglect all the terms of $C_{lm}\boldsymbol{\xi}_{x\tau}$ and $S_{lm}\boldsymbol{\xi}_{x\tau}$, then we can linearize the equality condition Eq. (59) and obtain the final linearized condition equations as follows:

$$\begin{aligned}\mathbf{x}_o(t_i)+\boldsymbol{\xi}_{xt_i} = &-\int_{t_0}^{t_i}\int_{t_0}^{\eta}\frac{GM}{r_o^3(\tau)}\mathbf{x}_o(\tau)d\tau d\eta\\ &-\sum_{j=0}^{i}w_j\sum_{k=0}^{j}w_k\mathbf{A}^x(\mathbf{x}_o(t_k))\boldsymbol{\xi}_{xt_k}\\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}C_{lm}\int_{t_0}^{t_i}\int_{t_0}^{\eta}\mathbf{p}_{lm}^{c}(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))d\tau d\eta\\ &+\sum_{l=2}^{N_{\max}}\sum_{m=0}^{l}S_{lm}\int_{t_0}^{t_i}\int_{t_0}^{\eta}\mathbf{p}_{lm}^{s}(\alpha_o(\tau),\zeta_o(\tau),r_o(\tau))d\tau d\eta\\ &+\mathbf{v}_0^0(t_i-t_0)+\Delta\mathbf{v}_0(t_i-t_0)+\mathbf{x}_0^0+\Delta\mathbf{x}_0.\end{aligned} \tag{60}$$

Satellite tracking measurements $\mathbf{y}$ of all types are functions of satellite positions and velocities. For ground-based tracking systems, tracking measurements are also functions of the positions of ground tracking stations, which are often assumed to be given. Since ground stations are actually derived *a priori*, they may also be treated as pseudo-measurements with random errors in satellite tracking systems. Thus, in its most general form, a tracking measurement y_i at the time epoch t_{yi} must theoretically satisfy the following physical and/or geometrical constraint, which can be symbolically written as:

$$E(y_i)=f(\mathbf{x}^s(t_{yi}),\mathbf{v}^s(t_{yi}),\mathbf{x}^g), \tag{61}$$

where $E(y_i)$ stands for the theoretical value of the measurement y_i (without biases), $\mathbf{x}^s(t_{yi})$ for the true position of the satellite, $\mathbf{v}^s(t_{yi})$ for the true velocity of the satellite, $\mathbf{x}^g$ for the true position of a ground tracking station, and $f(\cdot)$ is a nonlinear functional. By replacing the theoretical/true values of the quantities in (61) with the corresponding measurements plus corrections, we can readily turn the theoretical constraint (61) into a condition equation. For example, let us assume that except for $\mathbf{v}^s(t_{yi})$, all the other quantities in (61) are measured (and/or known *a priori* with random errors). As a result, we have the nonlinear condition equation:

$$y_i+\xi_{yi}=f(\mathbf{x}_o^s(t_{yi})+\boldsymbol{\xi}_{xt_{yi}}^s,\mathbf{v}^s(t_{yi}),\mathbf{x}_o^g+\boldsymbol{\xi}_x^g), \tag{62}$$

where ξ_{yi} stands for the correction to y_i, $\boldsymbol{\xi}_{xt_{yi}}^s$ for the corrections to the satellite orbital coordinates $\mathbf{x}_o^s(t_{yi})$, $\boldsymbol{\xi}_x^g$ for the corrections to the *a priori* coordinates $\mathbf{x}_o^g$ of the ground tracking station. If the velocity of the satellite is also measured, then we need to replace $\mathbf{v}^s(t_{yi})$ with $[\mathbf{v}_o^s(t_{yi})+\boldsymbol{\xi}_{vt_{yi}}^s]$. Very often, one would go ahead to linearize (62), which is technically straightforward and will be omitted here.

If a tracking measurement is involved with two satellites, then (61) can be alternatively written as follows:

$$E(y_i)=f(\mathbf{x}^{s1}(t_{yi}),\mathbf{v}^{s2}(t_{yi}),\mathbf{x}^{s2}(t_{yi}),\mathbf{v}^{s2}(t_{yi})), \tag{63}$$

where the superscripts *s1* and *s2* stand for satellites 1 and 2, respectively. By replacing the theoretical values with the corresponding measurements plus their corrections, we can construct the corresponding condition equation for (63) as follows:

$$y_i+\xi_{yi}=f(\mathbf{x}_o^{s1}(t_{yi})+\boldsymbol{\xi}_{xt_{yi}}^{s1},\mathbf{v}^{s1}(t_{yi}),\mathbf{x}_o^{s2}(t_{yi})+\boldsymbol{\xi}_{xt_{yi}}^{s2},\mathbf{v}^{s2}(t_{yi})), \tag{64}$$

if both $\mathbf{v}^{s1}(t_{yi})$ and $\mathbf{v}^{s2}(t_{yi})$ are unknown and not measured. If one or both of them are measured, we need to replace them with their corresponding measurements plus their corrections in (64).

Actually, the condition Eqs. (57), (58), (60), (62) and (64) apply to all types of satellite tracking measurements, including but not limited to orbital position measurements, satellite velocity measurements, Doppler measurements, directional measurements, ranges and range rates. By collecting the (linearized) condition equations for all tracking measurements of any types together, and by collecting all the corrections to measurements in the vector $\boldsymbol{\xi}$ and all the unknown parameters such as the unknown harmonic coefficients and the corrections to initial satellite position and velocity in the vector $\boldsymbol{\beta}$, we can symbolically write the final linearized condition equations for all measurements as follows:

$$\mathbf{A}\boldsymbol{\xi}+\mathbf{B}\boldsymbol{\beta}+\mathbf{u}=\mathbf{0}, \tag{65}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the known coefficient matrices, and $\mathbf{u}$ is the misclosure vector of measurements. If we further assume that the satellite tracking measurements $\mathbf{y}$ have the weighting matrix $\mathbf{W}$, then we can finally estimate the Earth's gravitational model, together with other nuisance unknown parameters, by solving the following minimization problem:

$$\text{min: } \boldsymbol{\xi}^T\mathbf{W}\boldsymbol{\xi} \tag{66}$$

subject to the equality constraints (65). If different types of satellite tracking measurements are used to reconstruct the Earth's gravitational field, we may also have to simultaneously estimate the different weighting factors of the measurements for this typical kind of ill-posed inverse problems in Earth Sciences (see e.g., [113,117]).

6. Concluding remarks

Differential equations with unknown parameters and the derived differential equations of the partial derivatives with respect to the unknown parameters were originally published by Gronwall [28] and Ritt [91] almost 100 years ago (see also [26,37]), which have been widely used for reconstruction of the unknown parameters of differential equations from measurements, with applications in many areas of science and engineering such as statistics, chemistry, physics, and satellite gravimetry (see e.g., [20,39,67,68,70,76,84,86,108]). The method was, likely independently, re-discovered by Anderle [2] and Riley et al. [90], now best known as the numerical integration method in geodesy and aerospace engineering, and has since become a standard technique in satellite gravimetry and been widely used routinely by major institutions worldwide to produce global gravitational models from satellite tracking measurements of CHAMP and/or GRACE types. A precise gravitational model can serve as a precise global static vertical datum surface (i.e. the geoid) in geodesy. Time-varying gravitational models from CHAMP and/or GRACE tracking measurements have found widest possible multidisciplinary applications in environmental monitoring, continental water variation, seismology, the structure and dynamics of the core and mantle, and ocean dynamics (see e.g., [19,78,80,104,109]). The most important element of the numerical integration method is to solve the nonlinear differential equations of the partial derivatives with respect to the unknown differential equation parameters such as the differential Eq. (10) in satellite gravimetry with the assumption of zero initial values (see e.g., [3,5,20,26,28,37,39,67,68,70,76,84,86,90,108]). Although the differential Eq. (10) are mathematically derived rigorously from the original Newton's governing differential Eq. (6), the zero initial values cannot be derived from (6) but are a claim without any mathematical/physical support.

We have proved that the numerical integration method is groundless, mathematically and physically. From the mathematical point of view, we have readily constructed counter examples to invalidate the assumption of zero initial values for the partial derivatives. Since an orbital position is nothing more than a mathematical point of the general solution, and since any epoch can serve as an initial epoch, if initial values of the partial derivatives in (10) could be set to zero, then all the partial derivatives could be logically set to zero as well. Obviously, all these mathematically erroneous consequences come from the source of setting the initial values of the partial derivatives to zero. From the physical point of view, if the initial values of the partial derivatives in (10) could be set to zero, this would imply that satellite tracking measurements would not contain any information on the Earth's gravitational field; this certainly contradicts with the fact that satellite tracking measurements indeed contain the physical information on the Earth's gravitational field and can be used to determine it. The effect of incorrectly setting the initial partial derivatives to zero on gravitational models produced by major institutions worldwide for the geoscience community remains unclear and should be further investigated in the future.

Given differential equations with unknown parameters and unknown initial conditions, and assuming a nominal reference orbit, we have developed three different methods, namely, linearization of the original nonlinear differential Eq. (6), Euler and modified Euler numerical integration methods with the unknown differential equation parameters and unknown initial conditions, and the integral equation approach, to derive local solutions to the differential Eq. (6), which, together with satellite tracking measurements, can be used to estimate the unknown differential equation parameters and unknown initial conditions, and as a result, to reconstruct global gravitational models. Unlike the numerical integration method, these new solutions require neither the differential equations of the partial derivatives nor the incorrect assumption of zero initial values for the partial derivatives. The solutions are represented in terms of the unknown corrections of parameters and initial conditions and are said to be local, since they are valid in the neighbourhood of a nominal reference orbit. The modelling errors will increase with time. In this case, if an orbital arc is sufficiently lengthy, one will have to iteratively solve for the unknown gravitational parameters. Deterministic global optimization methods can also be used to find the optimal solution [125,126]. I should note that these solution methods are generally applicable to any differential equations with unknown parameters, though they are developed here to solve problems of satellite gravimetry.

Orbits of LEO satellites can now be measured precisely and almost continuously, thanks to the profound advance of space observation technology and GNSS receiver hardware. Modern and next generation of space observation will become even more precise to an unprecedented level. With precisely and almost continuously measured orbits of LEO satellites, we have developed the measurement-based perturbation theory by turning the nonlinear differential Eq. (6) into the nonlinear Volterra's integral equations of the second kind and linearizing the nonlinear integral equations with precisely measured orbits of satellites. As a result, we have constructed different global uniformly convergent solutions. Theoretically speaking, the global uniformly convergent solutions are able to fully use unprecedented accuracy and continuity of modern and next generation of space observation and are a mathematical guarantee to extract smallest possible gravitational signals from satellite tracking measurements to their technological limit of noise level. Thus, the global uniformly convergent solutions can be used for high-precision high-resolution mapping of the Earth's gravitational field from satellite tracking measurements. One more important advantage of measurement-based perturbation theory is that no iteration will be needed to estimate the unknown gravitational parameters from satellite tracking measurements, if the perturbation solutions of position and velocity are used, because they are global uniformly convergent and linear with respect to the gravitational parameters. In this case, we do not need any initial approximate values of the harmonic coefficients $\mathbf{p}$ either. With precisely measured

orbits of LEO satellites, we have also reformulated the determination of the Earth's gravitational field from satellite tracking measurements as a standard condition adjustment with unknown parameters.

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Appendix. Derivation of $\Delta\mathbf{z}(t_{yi})$ for the modified Euler method

For convenience, we rewrite the recursive formula of the modified Euler method as follows:

$$\mathbf{z}(t_j) = \mathbf{z}(t_{j-1}) + \frac{h}{2}[\mathbf{g}(t_{j-1}, \mathbf{z}(t_{j-1}), \mathbf{p}) + \mathbf{g}\{t_j, \mathbf{z}(t_{j-1}) + h\mathbf{g}(t_{j-1}, \mathbf{z}(t_{j-1}), \mathbf{p}), \mathbf{p}\}], \tag{67}$$

for $j = 1, 2, \ldots, m$, with the nominal reference orbit $\mathbf{z}^0(t, \mathbf{z}^0(t), \mathbf{p}^0)$.

We will now derive the representation of $\Delta\mathbf{z}(t_{yi})$ in terms of the corrections $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$.

To start with, we set $j = 1$ in (67) and have

$$\mathbf{z}(t_1) = \mathbf{z}(t_0) + \frac{h}{2}[\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}) + \mathbf{g}\{t_1, \mathbf{z}(t_0) + h\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}), \mathbf{p}\}]. \tag{68}$$

Linearizing both $\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p})$ and $\mathbf{g}\{t_1, \mathbf{z}(t_0) + h\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}), \mathbf{p}\}$, and neglecting the terms of $h\Delta\mathbf{z}_0$ and $h\Delta\mathbf{p}$ (because of the coefficient $h/2$ before the brackets in (68)), we have

$$\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}) = \mathbf{g}(t_0, \mathbf{z}^0(t_0), \mathbf{p}^0) + \mathbf{G}_{gz0}\Delta\mathbf{z}_0 + \mathbf{G}_{gp0}\Delta\mathbf{p}, \tag{69a}$$

and

$$\begin{aligned}
&\mathbf{g}\{t_1, \mathbf{z}(t_0) + h\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}), \mathbf{p}\} \\
&= \mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}\{\mathbf{z}(t_0) + h\mathbf{g}(t_0, \mathbf{z}(t_0), \mathbf{p}) - \mathbf{z}^0(t_1)\} + \mathbf{G}_{gp1}\Delta\mathbf{p} \\
&= \mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}\{\mathbf{z}^0(t_0) + \Delta\mathbf{z}_0 + h\mathbf{g}(t_0, \mathbf{z}^0(t_0), \mathbf{p}^0) - \mathbf{z}^0(t_1)\} + \mathbf{G}_{gp1}\Delta\mathbf{p} \\
&= \mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 + \mathbf{G}_{gz1}\Delta\mathbf{z}_0 + \mathbf{G}_{gp1}\Delta\mathbf{p}.
\end{aligned} \tag{69b}$$

Inserting (69a) and (69b) into (68) yields

$$\begin{aligned}
\mathbf{z}(t_1) &= \mathbf{z}^0(t_1) + \Delta\mathbf{z}(t_1) \\
&= \mathbf{z}_0^0 + \Delta\mathbf{z}_0 + \frac{h}{2}[\mathbf{g}(t_0, \mathbf{z}^0(t_0), \mathbf{p}^0) + \mathbf{G}_{gz0}\Delta\mathbf{z}_0 + \mathbf{G}_{gp0}\Delta\mathbf{p}] \\
&\quad + \frac{h}{2}[\mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 + \mathbf{G}_{gz1}\Delta\mathbf{z}_0 + \mathbf{G}_{gp1}\Delta\mathbf{p}] \\
&= \mathbf{z}_0^0 + \frac{h}{2}[\mathbf{g}(t_0, \mathbf{z}^0(t_0), \mathbf{p}^0) + \mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0)] + \frac{h}{2}\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 \\
&\quad + \left[\mathbf{I}_6 + \frac{h}{2}(\mathbf{G}_{gz0} + \mathbf{G}_{gz1})\right]\Delta\mathbf{z}_0 + \frac{h}{2}(\mathbf{G}_{gp0} + \mathbf{G}_{gp1})\Delta\mathbf{p},
\end{aligned} \tag{70}$$

which can also be rewritten in terms of $\Delta\mathbf{z}(t_1)$ as follows:

$$\begin{aligned}
\Delta\mathbf{z}(t_1) &= \mathbf{z}_0^0 + \frac{h}{2}[\mathbf{g}(t_0, \mathbf{z}^0(t_0), \mathbf{p}^0) + \mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0)] + \frac{h}{2}\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 - \mathbf{z}^0(t_1) \\
&\quad + \left[\mathbf{I}_6 + \frac{h}{2}(\mathbf{G}_{gz0} + \mathbf{G}_{gz1})\right]\Delta\mathbf{z}_0 + \frac{h}{2}(\mathbf{G}_{gp0} + \mathbf{G}_{gp1})\Delta\mathbf{p} \\
&= \delta\mathbf{z}_{01}^{0M} + \frac{h}{2}\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^0 \\
&\quad + \left[\mathbf{I}_6 + \frac{h}{2}(\mathbf{G}_{gz0} + \mathbf{G}_{gz1})\right]\Delta\mathbf{z}_0 + \frac{h}{2}(\mathbf{G}_{gp0} + \mathbf{G}_{gp1})\Delta\mathbf{p},
\end{aligned} \tag{71}$$

where

$$\delta\mathbf{z}_{01}^{0M} = \mathbf{z}_0^0 + \frac{h}{2}[\mathbf{g}(t_0, \mathbf{z}^0(t_0), \mathbf{p}^0) + \mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0)] - \mathbf{z}^0(t_1).$$

For $j = 2$, we have

$$\mathbf{z}(t_2) = \mathbf{z}(t_1) + \frac{h}{2}[\mathbf{g}(t_1, \mathbf{z}(t_1), \mathbf{p}) + \mathbf{g}\{t_2, \mathbf{z}(t_1) + h\mathbf{g}(t_1, \mathbf{z}(t_1), \mathbf{p}), \mathbf{p}\}]. \tag{72}$$

In a similar manner, we linearize both $\mathbf{g}(t_1, \mathbf{z}(t_1), \mathbf{p})$ and $\mathbf{g}\{t_2, \mathbf{z}(t_1) + h\mathbf{g}(t_1, \mathbf{z}(t_1), \mathbf{p}), \mathbf{p}\}$ in the formula (72), neglect the terms of $h\Delta\mathbf{z}_0$ and $h\Delta\mathbf{p}$ (again due to the reason of the coefficient $h/2$) and obtain

$$\begin{aligned}
\mathbf{z}(t_2) &= \mathbf{z}^0(t_2) + \Delta\mathbf{z}(t_2) \\
&= \mathbf{z}^0(t_1) + \Delta\mathbf{z}(t_1) + \frac{h}{2}[\mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}\Delta\mathbf{z}(t_1) + \mathbf{G}_{gp1}\Delta\mathbf{p}] \\
&\quad + \frac{h}{2}\{\mathbf{g}(t_2, \mathbf{z}^0(t_2), \mathbf{p}^0) + \mathbf{G}_{gz2}[\mathbf{z}(t_1) + h\mathbf{g}(t_1, \mathbf{z}(t_1), \mathbf{p}) - \mathbf{z}^0(t_2)] + \mathbf{G}_{gp2}\Delta\mathbf{p}\} \\
&= \mathbf{z}^0(t_1) + \Delta\mathbf{z}(t_1) + \frac{h}{2}\{\mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}[\delta\mathbf{z}_{01}^{0M} + \Delta\mathbf{z}_0] + \mathbf{G}_{gp1}\Delta\mathbf{p}\} \\
&\quad + \frac{h}{2}\{\mathbf{g}(t_2, \mathbf{z}^0(t_2), \mathbf{p}^0) + \mathbf{G}_{gz2}[\delta\mathbf{z}_{12}^{0} + \delta\mathbf{z}_{01}^{0M} + \Delta\mathbf{z}_0] + \mathbf{G}_{gp2}\Delta\mathbf{p}\}.
\end{aligned} \tag{73}$$

Substituting $\Delta\mathbf{z}(t_1)$ of (71) into (73) and after some re-arrangement, we have

$$\begin{aligned}
\Delta\mathbf{z}(t_2) &= \mathbf{z}^0(t_1) - \mathbf{z}^0(t_2) + \delta\mathbf{z}_{01}^{0M} + \frac{h}{2}\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^{0} \\
&\quad + [\mathbf{I}_6 + \frac{h}{2}(\mathbf{G}_{gz0} + \mathbf{G}_{gz1})]\Delta\mathbf{z}_0 + \frac{h}{2}(\mathbf{G}_{gp0} + \mathbf{G}_{gp1})\Delta\mathbf{p} \\
&\quad + \frac{h}{2}\{\mathbf{g}(t_1, \mathbf{z}^0(t_1), \mathbf{p}^0) + \mathbf{G}_{gz1}\delta\mathbf{z}_{01}^{0M} + \mathbf{G}_{gz1}\Delta\mathbf{z}_0 + \mathbf{G}_{gp1}\Delta\mathbf{p}\} \\
&\quad + \frac{h}{2}\{\mathbf{g}(t_2, \mathbf{z}^0(t_2), \mathbf{p}^0) + \mathbf{G}_{gz2}[\delta\mathbf{z}_{12}^{0} + \delta\mathbf{z}_{01}^{0M} + \Delta\mathbf{z}_0] + \mathbf{G}_{gp2}\Delta\mathbf{p}\} \\
&= \delta\mathbf{z}_{01}^{0M} + \delta\mathbf{z}_{12}^{0M} + \frac{h}{2}\mathbf{G}_{gz1}\delta\mathbf{z}_{01}^{0} + \frac{h}{2}\mathbf{G}_{gz2}\delta\mathbf{z}_{12}^{0} + \frac{h}{2}[\mathbf{G}_{gz1} + \mathbf{G}_{gz2}]\delta\mathbf{z}_{01}^{0M} \\
&\quad + [\mathbf{I}_6 + \frac{h}{2}(\mathbf{G}_{gz0} + \mathbf{G}_{gz1})]\Delta\mathbf{z}_0 + \frac{h}{2}(\mathbf{G}_{gp0} + \mathbf{G}_{gp1})\Delta\mathbf{p} \\
&\quad + \frac{h}{2}(\mathbf{G}_{gz1} + \mathbf{G}_{gz2})\Delta\mathbf{z}_0 + \frac{h}{2}(\mathbf{G}_{gp1} + \mathbf{G}_{gp2})\Delta\mathbf{p} \\
&= \delta\mathbf{z}_{02}^{0M} + \frac{h}{2}\sum_{l=1}^{1}[\mathbf{G}_{gzl} + \mathbf{G}_{gz(l+1)}]\delta\mathbf{z}_{0l}^{0M} + \frac{h}{2}\sum_{l=0}^{1}\mathbf{G}_{gz(l+1)}\delta\mathbf{z}_{l(l+1)}^{0} \\
&\quad + \left[\mathbf{I}_6 + \frac{h}{2}\sum_{l=0}^{1}\{\mathbf{G}_{gzl} + \mathbf{G}_{gz(l+1)}\}\right]\Delta\mathbf{z}_0 \\
&\quad + \frac{h}{2}\sum_{l=0}^{1}\{\mathbf{G}_{gpl} + \mathbf{G}_{gp(l+1)}\}\Delta\mathbf{p},
\end{aligned} \tag{74}$$

where

$$\delta\mathbf{z}_{02}^{0M} = \mathbf{z}_0^0 + \frac{h}{2}\sum_{l=0}^{1}[\mathbf{g}(t_l, \mathbf{z}^0(t_l), \mathbf{p}^0) + \mathbf{g}(t_{l+1}, \mathbf{z}^0(t_{l+1}), \mathbf{p}^0)] - \mathbf{z}^0(t_2).$$

For $j = 3$, we simply list the representation of $\Delta\mathbf{z}(t_j)$ as follows:

$$\begin{aligned}
\Delta\mathbf{z}(t_3) &= \delta\mathbf{z}_{03}^{0M} + \frac{h}{2}\sum_{l=1}^{2}[\mathbf{G}_{gzl} + \mathbf{G}_{gz(l+1)}]\delta\mathbf{z}_{0l}^{0M} \\
&\quad + \frac{h}{2}\sum_{l=0}^{2}\mathbf{G}_{gz(l+1)}\delta\mathbf{z}_{l(l+1)}^{0} \\
&\quad + \left[\mathbf{I}_6 + \frac{h}{2}\sum_{l=0}^{2}\{\mathbf{G}_{gzl} + \mathbf{G}_{gz(l+1)}\}\right]\Delta\mathbf{z}_0 \\
&\quad + \frac{h}{2}\sum_{l=0}^{2}\{\mathbf{G}_{gpl} + \mathbf{G}_{gp(l+1)}\}\Delta\mathbf{p},
\end{aligned} \tag{75}$$

where

$$\delta\mathbf{z}_{03}^{0M} = \mathbf{z}_0^0 + \frac{h}{2}\sum_{l=0}^{2}[\mathbf{g}(t_l, \mathbf{z}^0(t_l), \mathbf{p}^0) + \mathbf{g}(t_{l+1}, \mathbf{z}^0(t_{l+1}), \mathbf{p}^0)] - \mathbf{z}^0(t_3).$$

Repeating the same procedure and by induction to summarize, we can finally obtain the representation of $\Delta\mathbf{z}(t_{yi})$ in terms of $\Delta\mathbf{z}_0$ and $\Delta\mathbf{p}$ as follows:

$$\begin{aligned}\Delta\mathbf{z}(t_{yi}) = {} & \delta\mathbf{z}^{0M}_{0t_{yi}} + \frac{h}{2}\sum_{j=1}^{m_{yi}-1}[\mathbf{G}_{gzj} + \mathbf{G}_{gz(j+1)}]\delta\mathbf{z}^{0M}_{0j} \\ & + \frac{h}{2}\sum_{j=0}^{m_{yi}-1}\mathbf{G}_{gz(j+1)}\delta\mathbf{z}^{0}_{j(j+1)} + \left[\mathbf{I}_6 + \frac{h}{2}\sum_{j=0}^{m_{yi}-1}\{\mathbf{G}_{gzj} + \mathbf{G}_{gz(j+1)}\}\right]\Delta\mathbf{z}_0 \\ & + \frac{h}{2}\sum_{j=0}^{m_{yi}-1}\{\mathbf{G}_{gpj} + \mathbf{G}_{gp(j+1)}\}\Delta\mathbf{p},\end{aligned} \tag{76}$$

where

$$\delta\mathbf{z}^{0M}_{0k} = \mathbf{z}^0_0 + \frac{h}{2}\sum_{j=0}^{k-1}[\mathbf{g}(t_j, \mathbf{z}^0(t_j), \mathbf{p}^0) + \mathbf{g}\{t_{j+1}, \mathbf{z}^0(t_{j+1}), \mathbf{p}^0\}] - \mathbf{z}^0(t_k),$$

and

$$\delta\mathbf{z}^0_{j(j+1)} = \mathbf{z}^0(t_j) + h\mathbf{g}(t_j, \mathbf{z}^0(t_j), \mathbf{p}^0) - \mathbf{z}^0(t_{j+1}).$$

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