A summary of “Asymptotic behaviors of random walks: application of heat kernel estimates”

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1.1 Introduction

This is a summary of the author’s thesis [9] entitled “Asymptotic behaviors or random walks; application of heat kernel estimates”. The main focus of the thesis is to analyze asymptotic behavior for random walks (RWs) on graphs. More specifically, we deal with the following two topics:

- Escape rates of random walks in random environments.
- Cutoff phenomena for lamplighter chains.

Heat kernel estimates (HKEs) are obtained for various classes of stochastic processes such as diffusions on $\mathbb{R}^d$, Riemannian manifolds, fractals, metric measure spaces and random walks on graphs, random environments, and so on. In this thesis, we discuss the above two topics as applications of the study of HKEs.

1.1.1 The random conductance model

The study of random walks in random environments has been one of the central topics in probability theory. The random conductance model, which is described below, is a specific class of random environments. The conductance model consists of a pair (G, ω) of a graph $G = (V(G), E(G))$ and a function $\omega : V(G) \times V(G) \to [0, \infty)$, $\omega \mapsto \omega_{xy}$ such that $\omega_{xy} = \omega_{yx}$ and $\omega_{xy} > 0$ iff $\{x, y\} \in E(G)$. The function ω is called the conductance. When ω is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the pair is called the random conductance model (RCM). For a given RCM, we consider discrete-time RW $\{X_n^\omega\}_{n\geq 0}$ and continuous-time RW $\{Y_t^\omega\}_{t\geq 0}$ whose transition probabilities are given by

$$P^\omega(x, y) = \frac{\omega_{xy}}{\omega_x}, \qquad \text{where } \omega_x = \sum_{y:\{x,y\}\in E(G)} \omega_{xy},$$

denoting the corresponding heat kernels for both discrete and continuous time RWs by

$$p_n^\omega(x, y) = \frac{P_x^\omega(X_n = y)}{\omega_y} \quad \text{and} \quad q_t^\omega(x, y) = \frac{P_x^\omega(Y_t = y)}{\omega_y}.$$

One of the most important models is the percolation on $\mathbb{Z}^d$, which descries how liquid percolates porous materials. The model is defined as follows: For each edge of $\mathbb{Z}^d$, flip a (possibly unfair) coin which takes head and tail with probability p and $1 - p$ respectively, and leave (resp. remove) the edge when the coin takes head (resp. tail). It is known that there exists $p_c = p_c(d)$ such that the resulting graphs have a unique infinite component for $p > p_c(d)$ and no infinite component for $p < p_c(d)$. The level sets of discrete Gaussian free field (DGFF) and the random interlacements are also interesting and important: The DGFF on $\mathbb{Z}^d (d \geq 3)$ is a family of centered Gaussian random variables $\{\varphi_x\}_{x\in\mathbb{Z}^d}$ such that the covariance is given by the Green function of a simple random walk on $\mathbb{Z}^d$. The level sets of the DGFF are the sets of the form $E_h := \{x \in \mathbb{Z}^d \mid \varphi_x \geq h\}$. The random interlacements is, roughly speaking, the random set visited by infinitely many independent random walks on $\mathbb{Z}^d$ $(d \geq 3)$. See [5] for more details of the random interlacements. We can regard these models as percolations with long range correlations.

Some properties of RWs on a certain class of the RCM for $G = \mathbb{Z}^d$ are similar to those of simple random walks on $\mathbb{Z}^d$. One of such properties is the long-time Gaussian HKEs. In fact, for a class of the RCM, there exists a family of random variables $\{N_x\}_{x\in V(G)}$ such that discrete-time RWs enjoy

$$p_n^\omega(x,y) \leq \frac{c}{n^{d/2}} \exp\left[-\frac{1}{c}\frac{d(x,y)^2}{n}\right], \tag{1.1}$$

$$p_n^\omega(x,y) + p_{n+1}^\omega(x,y) \geq \frac{1}{cn^{d/2}} \exp\left[-c\frac{d(x,y)^2}{n}\right] \tag{1.2}$$

for all n, $x, y \in V(G)$ with $d(x,y) \vee N_x \leq n$, and continuous-time RWs enjoy

$$q_t^\omega(x,y) \leq \begin{cases} \frac{c}{t^{d/2}} \exp\left[-\frac{1}{c}\left(\frac{d(x,y)^2}{t}\right)\right], & \text{if } t \geq d(x,y), \\ c\exp\left[-\frac{1}{c}d(x,y)\left(1 \vee \log\frac{d(x,y)}{t}\right)\right], & \text{if } t \leq d(x,y), \end{cases} \tag{1.3}$$

$$q_t^\omega(x,y) \geq \frac{1}{ct^{d/2}} \exp\left[-c\left(\frac{d(x,y)^2}{t}\right)\right] \tag{1.4}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G)$ and $t \geq 0$ with $d(x,y)^{1+\epsilon} \vee N_x(\omega) \leq t$, where d is the graph distance and $c \in (0,\infty)$ is a constant which is independent of ω. Moreover, the following form of estimate of tail probability of N_x is obtained for a class of RCMs:

$$\mathbb{P}(N_x \geq n) \leq f(n), \qquad \text{for some non-increasing function } f. \tag{1.5}$$

For the uniform elliptic case ($c^{-1} \leq \omega_{xy} \leq c$ for all $xy \in E(G)$ with some finite constant c), such results were obtained by Delmotte [4] for discrete and continuous time RWs with $N_x \equiv 0$. Later, Barlow [1] obtained the above HKEs for the super-critical percolation cluster with $f(n) = c\exp(-c^{-1}n)$ with a finite constant c. Barlow and Deuschel [3] obtained the coutinuous time HKEs in the case $\omega_{xy} \in [1,\infty)$ with $f(n) = c\exp(-c^{-1}n)$. Sapozhnikov [12] obtained the HKEs for the percolation cluster with long range correlation with $f(n) = c\exp(-c^{-1}(\log n)^{1+\delta})$ $(\delta > 0)$.

1.1.2 Lamplighter random walks on fractals

Suppose that each vertex of a graph G is equipped with a lamp ($= \{0,1\}$), and we consider a random walk that moves on the graph and also switches lamps uniformly at random before it moves to one of the nearest vertices of G. Such a random walk is called the lamplighter random walk, and is formulated on the wreath product $\mathbb{Z}_2 \wr G$.

The wreath product $\mathbb{Z}_2 \wr G$ is endowed with a group structure when G is a discrete group, and the lamplighter random walks have been studied in the context of random walks on discrete groups (see e.g. [13, 10, 7, 11] and the references therein).

The mixing time and the cutoff phenomenon are interesting topics in the study of finite Markov chains. To describe these two notions, let $\{H^{(N)}\}$ be a sequence of finite graphs, and $\{Y^{(N)}\}$ be irreducible and aperiodic Markov chains on $H^{(N)}$ with transition probability $P^{(N)}$. For each N, there exists a unique invariant distribution $\pi^{(N)}$: $\pi^{(N)}$ is the probability distribution on $V(H^{(N)})$ which satisfies

$$\pi^{(N)}P^{(N)} = \pi^{(N)},$$

(equivalently, $\sum_{y\in V(H^{(N)})}\pi^{(N)}(y)P^{(N)}(y,x) = \pi^{(N)}(x)$ for all $x \in V(H^{(N)})$). Moreover, $\pi^{(N)}$ is given by the limit of the distribution of the Markov chain:

$$\pi^{(N)}(y) = \lim_{n\to\infty}(P^{(N)})^n(x,y), \qquad \forall x,y \in V(H^{(N)}).$$

It is interesting to see the speed of the above convergence. The (ϵ-)mixing time is the first time that the total variation distance

$$\begin{aligned} d^{(N)}(n) &:= \max_{x\in V(H^{(N)})}\|P_x(Y_n^{(N)}=\cdot)-\pi^{(N)}(\cdot)\|_{\mathrm{TV}} \\ &= \frac{1}{2}\max_{x\in V(H^{(N)})}\sum_{y\in V(H^{(N)})}|P_x(Y_n^{(N)}=y)-\pi^{(N)}(y)| \end{aligned}$$

is less than ϵ, i.e.

$$T_{\mathrm{mix}}(H^{(N)};\epsilon) := \inf\{n\geq 0 \mid d^{(N)}(n)\leq\epsilon\}.$$

We say that the pair $(\{H^{(N)}\},\{Y^{(N)}\})$ has a cutoff if there exists a sequence $\{a_N\}$ such that

$$\lim_{N\to\infty}\frac{T_{\mathrm{mix}}(H^{(N)};\epsilon)}{a_N} = 1, \qquad \forall\epsilon\in(0,1).$$

In this thesis, we discuss the cutoff phenomena when $H^{(N)} = \mathbb{Z}_2 \wr G^{(N)}$ and $G^{(N)}$'s are finite fractal graphs.

It is known that, for a class of fractals, the number of vertices in a ball of radius r satisfies the d_f-set condition. Namely, there exists a positive and finite constant c such that, for any vertex x and $r > 0$,

$$c^{-1}r^{d_f} \leq \sharp B(x,r) \leq cr^{d_f}.$$

Another typical property of random walks on (countably infinite) fractal graphs is the sub-diffusivity. Namely, there exists d_w (called the escape rate or the walk dimension) and a constant $c \in (0,\infty)$ such that $c^{-1}n^{1/d_w} \leq E[d(X_0,X_n)] \leq cn^{1/d_w}$. For many fractals, $d_w > 2$ in contrast to $d_w = 2$ in the case of $\mathbb{Z}^d$. In fact, random walks on a class of fractal graphs enjoy the following HKEs:

$$p_n(x,y) \leq \frac{c}{n^{d_f/d_w}}\exp\left[-c^{-1}\left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right], \tag{1.6}$$

$$p_n(x,y)+p_{n+1}(x,y) \geq \frac{1}{cn^{d_f/d_w}}\exp\left[-c\left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right]. \tag{1.7}$$

Note that, for the case $d_w = 2$, (1.6) and (1.7) are called the Gaussian HKEs.

In the study of HKEs, it is also known that HKEs are equivalent to other conditions such as volume doubling property (VD), Poincaré inequality (PI), a cutoff Sobolev inequality (CS), and a parabolic Harnack inequality (PHI). In fact, the following conditions are known to be equivalent for a class of random walks (see [2]).

(a) (VD), (PI) and (CS),

(b) (HKE(d_w)),

(c) (PHI(d_w)).

1.2 Main results

The thesis [9] consists of three papers:

- Chapter 2 : T. Kumagai and C. Nakamura, Laws of the iterated logarithm for random walks on random conductance models. Stochastic analysis on large scale interacting systems, 141–156, RIMS Kôkyûroku Bessatsu, B59, Res. Inst. Math. Sci. (RIMS), Kyoto, 2016.
- Chapter 3: C. Nakamura, Rate functions for random walks on random conductance models and related topics. Kodai Math. J. 40 (2017), no. 2, 289–321.
- Chapter 4 : A. Dembo, T. Kumagai and C. Nakamura, Cutoff for lamplighter chains on fractals. Preprint.

We summarize the main results of the thesis.

1.2.1 Main results in Chapter 2

In Chapter 2 of [9], we study the laws of the iterated logarithm (LIL) for a discrete-time RW on the RCM.

The first result is the following (see [9, Theorem 2.1.2] for the precise statement).

Theorem 1.1 (Theorem 2.1.2 in Chapter 2)**.** *Suppose that a RW on the RCM enjoy long-time Gaussian HKEs* (1.1) (1.2)*, and $f(n)$ of* (1.5) *satisfies an integrability condition $\sum_n n^{2d} f(n) < \infty$. Then, for almost all environment $\omega \in \Omega$, there exist positive constants $C_1 = C_1(\omega)$ and $C_2 = C_2(\omega)$ such that the following hold.*

$$\limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/2}(\log\log n)^{1/2}} = C_1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G), \tag{1.8}$$

$$\liminf_{n\to\infty} \frac{\max_{0\le \ell\le n} d(X_0^\omega, X_\ell^\omega)}{n^{1/2}(\log\log n)^{-1/2}} = C_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G). \tag{1.9}$$

As we discussed in Section 1.1.1, the Gaussian HKEs are obtained for various examples, and the above results are applicable to those models.

Note that the constants C_1 and C_2 in Theorem 1.1 may depend on the random environment. The next result, which is motivated by [6, Sections 3 and 4], states that we can take the constants C_1 and C_2 independently from the random environment when we consider ergodic environments.

Theorem 1.2 (Theorem 2.1.4 in Chapter 2). *Suppose the same conditions as in Theorem 1.1. In addition, suppose that the random environment ω is ergodic w.r.t. the shifts on $\mathbb{Z}^d$. Then we can take C_1 and C_2 in Theorem 1.1 as deterministic constants (which do not depend on ω).*

We give a sketch of the proof (see [9, Chapter 2] for more details). By estimating $P_x(d(X_0^\omega, X_n^\omega) \geq \lambda n^{1/2}(\log\log n)^{1/2})$ from above and below by using HKEs, and then, using the Borel-Cantelli lemma we can deduce

$$c_1 \leq \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/2}(\log\log n)^{1/2}} \leq c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G).$$

Finally, by employing 0-1 law for the tail events, we conclude (1.8).

To show (1.9), we first deduce the following LIL for the first exiting time $\tau_{B(x,r)}$ from the ball $B(x,r)$:

$$c_1 \leq \limsup_{n\to\infty} \frac{\tau_{B(x,r)}}{r^2(\log\log r^2)} \leq c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G), \tag{1.10}$$

by estimating $P_x(\tau_{B(x,r)} \geq \lambda r^2(\log\log r^2))$ from above and below by using HKEs and using the Borel-Cantelli lemma. By putting $n = r^2(\log\log r^2)$ into (1.10) and by the 0-1 law for the tail events, we can deduce (1.9).

To show Theorem 1.2, we employ the notion of the random environment seen from the particle. Then the conclusion follows by the ergodic theorem.

1.2.2 Main results in Chapter 3

In Chapter 3 of [9], we study the escape rate of continuous-time RWs on a class of the RCMs.

The first result is the LIL for continuous-time RWs.

Theorem 1.3 (Theorem 3.1.5 in Chapter 3). *Suppose that a RW on the RCM enjoys the long-time Gaussian HKEs* (1.3) (1.4)*, and $f(n)$ in* (1.5) *satisfies the integrability condition $\sum_n n^{2d} f(n) < \infty$.*

(1) For almost all $\omega \in \Omega$ there exist positive numbers $c_1 = c_1^\omega$ and $c_2 = c_2^\omega$ such that

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/2}(\log\log t)^{1/2}} = c_1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G),$$
$$\limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/2}(\log\log t)^{1/2}} = c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G).$$

(2) For almost all $\omega \in \Omega$ there exists a positive number $c_3 = c_3^\omega$ such that

$$\liminf_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/2}(\log\log t)^{-1/2}} = c_3, \qquad P_x^\omega\text{-a.s. for all } x \in V(G).$$

The next result is concerning the escape rate for RWs on a class of RCM.

Theorem 1.4 (($d \geq 3$) Theorem 3.1.6 in Chapter 3)**.** *Let h be a non-increasing function, and suppose that a RW on a class of RCMs enjoys the long-time Gaussian HKEs* (1.3) *and* (1.4)*, and $f(n)$ in* (1.5) *satisfies the integrability condition $\sum_n n^d f(nh(n^2)) < \infty$. Then, either*

$$P_x^\omega \left(d(x, Y_t^\omega) \geq t^{1/2} h(t) \text{ for all sufficiently large } t \right) = 1$$

for almost all $\omega \in \Omega$ and all $x \in V(G)$, or

$$P_x^\omega \left(d(x, Y_t^\omega) \geq t^{1/2} h(t) \text{ for all sufficiently large } t \right) = 0,$$

for almost all $\omega \in \Omega$ and all $x \in V(G)$, according as $\int_1^\infty \frac{1}{t} h(t)^{d-2} dt < \infty$ or $= \infty$.

Theorems 1.3 and 1.4 are applicable to various examples such as a RW on the supercritical percolation cluster, the case $\omega_{xy} \in [1, \infty)$, the level sets of DGFF and the random interlacements.

As in the discrete-time case, the constants in Theorem 1.3 are deterministic when the random environment is ergodic.

Theorem 1.5 (Theorem 3.1.8 in Chapter 3)**.** *Suppose that the same assumptions as in Theorem 1.3 are fulfilled and suppose in addition that the random environment is ergodic w.r.t. the shifts on $\mathbb{Z}^d$. Then we can take c_1, c_2 and c_3 in Theorem 1.3 as deterministic constants (i.e. they do not depend on ω).*

The strategy of the proof of Theorem 1.3 is similar to that of Theorem 1.1, so we explain the strategy of the proof of Theorem 1.4. To prove Theorem 1.4, we need to estimate the probability that RW returns to balls centered around its starting point. In fact, we have the following estimates:

$$\frac{c_1^{-1} r^{d-2} t}{t^{d/2}} \leq P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t \right) \leq \frac{c_1 r^{d-2} t}{t^{d/2}},$$

where $d(x_0, x) \leq r$. By the above inequalities and the Borel-Cantelli lemma, we obtain the desired results.

1.2.3 Main Result in Chapter 4

In Chapter 4 of [9], we discuss the cutoff phenomena for the lamplighter random walks on fractals. Miller and Peres [8] gave a general framework for the cutoff phenomena with threshold $\frac{1}{2} T_{\text{cov}}(G^{(N)})$, where $T_{\text{cov}}(G^{(N)})$ is the expected cover time of the RW on $G^{(N)}$ (i.e. the expected time that a random walk visits all the vertices of $G^{(N)}$). A key assumption is uniform elliptic Harnack inequalities. We replace the uniform elliptic Harnack inequalities by uniform parabolic Harnack inequalities, and as a result, we derive a dichotomy result for the cutoff phenomena for lamplighter chains. Our main result is the following (see Theorem [9, Theorem 4.1.4] for the precise statement):

Theorem 1.6 (Theorem 4.1.4 in Chapter 4)**.** *Suppose that an increasing sequence of finite graphs $\{G^{(N)}\}_N$ satisfies (i) d_f-set condition and (ii) uniform parabolic Harnack inequalities with order d_w. Then,*

(a) if $d_f < d_w$, then the lamplighter random walks on $G^{(N)}$ do NOT have a cutoff.

(b) if $d_f > d_w$, then the lamplighter random walks on $G^{(N)}$ have a cutoff with threshold $a_N = \frac{1}{2}T_{\rm cov}(G^{(N)})$.

For examples, a sequence of finite graphs of a class of fractals satisfies the assumptions. As an analogue we discussed in Section 1.1.2, the uniform parabolic Harnack inequalities are equivalent to a finite analogue of HKEs.

The proof of Theorem 1.6 (b) is conducted by confirming the conditions given by [8]. To show Theorem 1.6 (a), we need to establish the following upper bound of the cover time:

$$\sup_{z \in V(G^{(N)})} \{P_z(\tau_{\rm cov}(G^{(N)}) > t)\} \le c_0 e^{-t/(c_0 T_N)}$$

for all t, N, where $T_N := (\mathrm{diam}\{G^{(N)}\})^{d_w}$. Moreover, we derive the following upper and lower bounds of total variation estimates:

$$\begin{aligned} c_1^{-1} e^{-c_1 t/T_N} - c_2(\mathrm{diam}\{G^{(N)}\})^{-d_f} &\le \max_{\boldsymbol{x} \in V(\mathbb{Z}_2 \wr G^{(N)})} \|P_t^*(\boldsymbol{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\rm TV} \\ &\le \max_{x \in V(G^{(N)})} P_x(\tau_{\rm cov}(G^{(N)}) > t) + \frac{\sqrt{S_N}}{2\sqrt{t}}, \end{aligned}$$

where $S_N := \max_{x,y \in V(G^{(N)})}\{R_{\rm eff}^{(N)}(x,y)\}$. Combining the above we obtain the desired result.

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