Asymptotic behaviors of random walks: application of heat kernel estimates

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Chapter 1

Introduction

1.1 General introduction

The main focus of this thesis to analyze asymptotic behavior for random walks (RWs) on graphs. More specifically, we deal with the following two topics:

- Escape rates of random walks in random environments.
- Cutoff phenomena for lamplighter chains.

Heat kernel estimates (HKEs) are obtained for various classes of stochastic processes such as diffusions on $\mathbb{R}^d$, Riemannian manifolds, fractals, metric measure spaces and random walks on graphs, random environments, and so on. In this thesis, we discuss the above two topics as applications of the study of HKEs.

1.1.1 The random conductance model

The study of random walks in random environments has been one of the central topics in probability theory. The random conductance model, which is described below, is a specific class of random environments. The conductance model consists of a pair (G, ω) of a graph $G = (V(G), E(G))$ and a function $\omega : V(G) \times V(G) \to [0, \infty)$, $\omega \mapsto \omega_{xy}$ such that $\omega_{xy} = \omega_{yx}$ and $\omega_{xy} > 0$ iff $\{x, y\} \in E(G)$. The function ω is called the conductance. When ω is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the pair is called the random conductance model (RCM). For a given RCM, we consider discrete-time RW $\{X_n^\omega\}_{n\geq 0}$ and continuous-time RW $\{Y_t^\omega\}_{t\geq 0}$

whose transition probabilities are given by

$$P^\omega(x,y) = \frac{\omega_{xy}}{\omega_x}, \qquad \text{where } \omega_x = \sum_{y:\{x,y\}\in E(G)} \omega_{xy},$$

denoting the corresponding heat kernels for both discrete and continuous time RWs by

$$p_n^\omega(x,y) = \frac{P_x^\omega(X_n = y)}{\omega_y} \quad \text{and} \quad q_t^\omega(x,y) = \frac{P_x^\omega(Y_t = y)}{\omega_y}.$$

One of the most important models is the percolation on $\mathbb{Z}^d$, which descries how liquid percolates porous materials. The model is defined as follows: For each edge of $\mathbb{Z}^d$, flip a (possibly unfair) coin which takes head and tail with probability p and $1-p$ respectively, and leave (resp. remove) the edge when the coin takes head (resp. tail). It is known that there exists $p_c = p_c(d)$ such that the resulting graphs have a unique infinite component for $p > p_c(d)$ and no infinite component for $p < p_c(d)$. The level sets of discrete Gaussian free field (DGFF) and the random interlacements are also interesting and important: The DGFF on $\mathbb{Z}^d(d \geq 3)$ is a family of centered Gaussian random variables $\{\varphi_x\}_{x\in\mathbb{Z}^d}$ such that the covariance is given by the Green function of a simple random walk on $\mathbb{Z}^d$. The level sets of the DGFF are the sets of the form $E_h := \{x \in \mathbb{Z}^d \mid \varphi_x \geq h\}$. The random interlacements is, roughly speaking, the random set visited by infinitely many independent random walks on $\mathbb{Z}^d$ ($d \geq 3$). See [26] for more details of the random interlacements. We can regard these models as percolations with long range correlations.

Some properties of RWs on a certain class of the RCM for $G = \mathbb{Z}^d$ are similar to those of simple random walks on $\mathbb{Z}^d$. One of such properties is the long-time Gaussian HKEs. In fact, for a class of the RCM, there exists a family of random variables $\{N_x\}_{x\in V(G)}$ such that discrete-time RWs enjoy

$$p_n^\omega(x,y) \leq \frac{c}{n^{d/2}} \exp\left[-\frac{1}{c}\frac{d(x,y)^2}{n}\right], \tag{1.1.1}$$

$$p_n^\omega(x,y) + p_{n+1}^\omega(x,y) \geq \frac{1}{cn^{d/2}} \exp\left[-c\frac{d(x,y)^2}{n}\right] \tag{1.1.2}$$

for all n, $x, y \in V(G)$ with $d(x,y) \vee N_x \le n$, and continuous-time RWs enjoy

$$q_t^\omega(x,y) \le \begin{cases} \frac{c}{t^{d/2}} \exp\left[-\frac{1}{c}\left(\frac{d(x,y)^2}{t}\right)\right], & \text{if } t \ge d(x,y), \\ c\exp\left[-\frac{1}{c}d(x,y)\left(1 \vee \log\frac{d(x,y)}{t}\right)\right], & \text{if } t \le d(x,y), \end{cases} \tag{1.1.3}$$

$$q_t^\omega(x,y) \ge \frac{1}{ct^{d/2}} \exp\left[-c\left(\frac{d(x,y)^2}{t}\right)\right] \tag{1.1.4}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G)$ and $t \ge 0$ with $d(x,y)^{1+\epsilon} \vee N_x(\omega) \le t$, where d is the graph distance and $c \in (0,\infty)$ is a constant which is independent of ω. Moreover, the following form of estimate of tail probability of N_x is obtained for a class of RCMs:

$$\mathbb{P}(N_x \ge n) \le f(n), \qquad \text{for some non-increasing function } f. \tag{1.1.5}$$

For the uniform elliptic case ($c^{-1} \le \omega_{xy} \le c$ for all $xy \in E(G)$ with some finite constant c), such results were obtained by Delmotte [24] for discrete and continuous time RWs with $N_x \equiv 0$. Later, Barlow [4] obtained the above HKEs for the supercritical percolation cluster with $f(n) = c\exp(-c^{-1}n)$ with a finite constant c. Barlow and Deuschel [12] obtained the coutinuous time HKEs in the case $\omega_{xy} \in [1,\infty)$ with $f(n) = c\exp(-c^{-1}n)$. Sapozhnikov [63] obtained the HKEs for the percolation cluster with long range correlation with $f(n) = c\exp(-c^{-1}(\log n)^{1+\delta})$ ($\delta > 0$).

1.1.2 Lamplighter random walks on fractals

Suppose that each vertex of a graph G is equipped with a lamp ($= \{0,1\}$), and we consider a random walk that moves on the graph and also switches lamps uniformly at random before it moves to one of the nearest vertices of G. Such a random walk is called the lamplighter random walk, and is formulated on the wreath product $\mathbb{Z}_2 \wr G$.

The wreath product $\mathbb{Z}_2 \wr G$ is endowed with a group structure when G is a discrete group, and the lamplighter random walks have been studied in the context of random walks on discrete groups (see e.g. [74, 58, 28, 62] and the references therein).

The mixing time and the cutoff phenomenon are interesting topics in the study of finite Markov chains. To describe these two notions, let $\{H^{(N)}\}$ be a sequence of finite graphs, and $\{Y^{(N)}\}$ be irreducible and aperiodic Markov chains on $H^{(N)}$ with transition probability $P^{(N)}$. For each N, there exists a unique invariant distribution $\pi^{(N)}$: $\pi^{(N)}$ is the probability distribution on $V(H^{(N)})$ which satisfies

$$\pi^{(N)}P^{(N)} = \pi^{(N)},$$

(equivalently, $\sum_{y \in V(H^{(N)})} \pi^{(N)}(y) P^{(N)}(y,x) = \pi^{(N)}(x)$ for all $x \in V(H^{(N)})$). Moreover, $\pi^{(N)}$ is given by the limit of the distribution of the Markov chain:

$$\pi^{(N)}(y) = \lim_{n \to \infty} (P^{(N)})^n(x,y), \qquad \forall x, y \in V(H^{(N)}).$$

It is interesting to see the speed of the above convergence. The (ϵ-)mixing time is the first time that the total variation distance

$$\begin{aligned} d^{(N)}(n) &:= \max_{x \in V(H^{(N)})} \|P_x(Y_n^{(N)} = \cdot) - \pi^{(N)}(\cdot)\|_{\mathrm{TV}} \\ &= \frac{1}{2} \max_{x \in V(H^{(N)})} \sum_{y \in V(H^{(N)})} |P_x(Y_n^{(N)} = y) - \pi^{(N)}(y)| \end{aligned}$$

is less than ϵ, i.e.

$$T_{\mathrm{mix}}(H^{(N)}; \epsilon) := \inf\{n \geq 0 \mid d^{(N)}(n) \leq \epsilon\}.$$

We say that the pair $(\{H^{(N)}\}, \{Y^{(N)}\})$ has a cutoff if there exists a sequence $\{a_N\}$ such that

$$\lim_{N \to \infty} \frac{T_{\mathrm{mix}}(H^{(N)}; \epsilon)}{a_N} = 1, \qquad \forall \epsilon \in (0,1).$$

In this thesis, we discuss the cutoff phenomena when $H^{(N)} = \mathbb{Z}_2 \wr G^{(N)}$ and $G^{(N)}$'s are finite fractal graphs.

It is known that, for a class of fractals, the number of vertices in a ball of radius r satisfies the d_f-set condition. Namely, there exists a positive and finite constant c such that, for any vertex x and $r > 0$,

$$c^{-1} r^{d_f} \leq \sharp B(x,r) \leq c r^{d_f}.$$

Another typical property of random walks on (countably infinite) fractal graphs is the sub-diffusivity. Namely, there exists d_w (called the escape rate or the walk dimension) and a constant $c \in (0,\infty)$ such that $c^{-1} n^{1/d_w} \leq E[d(X_0, X_n)] \leq c n^{1/d_w}$. For many fractals, $d_w > 2$ in contrast to $d_w = 2$ in the case of $\mathbb{Z}^d$. In fact, random walks on a class of fractal graphs enjoy the following HKEs:

$$p_n(x,y) \leq \frac{c}{n^{d_f/d_w}} \exp\left[-c^{-1} \left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right], \tag{1.1.6}$$

$$p_n(x,y) + p_{n+1}(x,y) \geq \frac{1}{c n^{d_f/d_w}} \exp\left[-c \left(\frac{d(x,y)^{d_w}}{n}\right)^{1/(d_w-1)}\right]. \tag{1.1.7}$$

Note that, for the case $d_w = 2$, (1.1.6) and (1.1.7) are called the Gaussian HKEs.

In the study of HKEs, it is also known that HKEs are equivalent to other conditions such as volume doubling property (VD), Poincaré inequality (PI), a cutoff Sobolev inequality (CS), and a parabolic Harnack inequality (PHI) (see Section 4.2.1 for these conditions.) In fact, the following conditions are known to be equivalent for a class of random walks (see [8]).

(a) (VD), (PI) and (CS),

(b) (HKE(d_w)),

(c) (PHI(d_w)).

1.2 Main results

This thesis consists of three papers:

- Chapter 2 : T. Kumagai and C. Nakamura, Laws of the iterated logarithm for random walks on random conductance models. Stochastic analysis on large scale interacting systems, 141–156, RIMS Kôkyûroku Bessatsu, B59, Res. Inst. Math. Sci. (RIMS), Kyoto, 2016.
- Chapter 3: C.Nakamura, Rate functions for random walks on random conductance models and related topics. Kodai Math. J. 40 (2017), no. 2, 289–321.
- Chapter 4: A. Dembo, T. Kumagai and C. Nakamura, Cutoff for lamplighter chains on fractals. Preprint.

We summarize the main results of this thesis.

1.2.1 Main results in Chapter 2

In Chapter 2, we study the laws of the iterated logarithm (LIL) for a discrete-time RW on the RCM.

The first result is the following (see Theorem 2.1.2 for the precise statement).

Theorem 1.2.1 (Theorem 2.1.2 in Chapter 2)**.** *Suppose that a RW on the RCM enjoy long-time Gaussian HKEs* (1.1.1) (1.1.2)*, and* $f(n)$ *of* (1.1.5) *satisfies an integrability* *condition $\sum_n n^{2d} f(n) < \infty$. Then, for almost all environment $\omega \in \Omega$, there exist positive constants $C_1 = C_1(\omega)$ and $C_2 = C_2(\omega)$ such that the following hold.*

$$\limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/2}(\log\log n)^{1/2}} = C_1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G), \tag{1.2.1}$$

$$\liminf_{n\to\infty} \frac{\max_{0\le\ell\le n} d(X_0^\omega, X_\ell^\omega)}{n^{1/2}(\log\log n)^{-1/2}} = C_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G). \tag{1.2.2}$$

As we discussed in Section 1.1.1, the Gaussian HKEs are obtained for various examples, and the above results are applicable to those models.

Note that the constants C_1 and C_2 in Theorem 1.2.1 may depend on the random environment. The next result, which is motivated by [27, Sections 3 and 4], states that we can take the constants C_1 and C_2 independently from the random environment when we consider ergodic environments.

Theorem 1.2.2 (Theorem 2.1.4 in Chapter 2)**.** *Suppose the same conditions as in Theorem 1.2.1. In addition, suppose that the random environment ω is ergodic w.r.t. the shifts on $\mathbb{Z}^d$. Then we can take C_1 and C_2 in Theorem 1.2.1 as deterministic constants (which do not depend on ω).*

We give a sketch of the proof (see Chapter 2 for more details). By estimating $P_x(d(X_0^\omega, X_n^\omega) \ge \lambda n^{1/2}(\log\log n)^{1/2})$ from above and below by using HKEs, and then, using the Borel-Cantelli lemma we can deduce

$$c_1 \le \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/2}(\log\log n)^{1/2}} \le c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G).$$

Finally, by employing 0-1 law for the tail events, we conclude (1.2.1).

To show (1.2.2), we first deduce the following LIL for the first exiting time $\tau_{B(x,r)}$ from the ball $B(x,r)$:

$$c_1 \le \limsup_{n\to\infty} \frac{\tau_{B(x,r)}}{r^2(\log\log r^2)} \le c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G), \tag{1.2.3}$$

by estimating $P_x(\tau_{B(x,r)} \ge \lambda r^2(\log\log r^2))$ from above and below by using HKEs and using the Borel-Cantelli lemma. By putting $n = r^2(\log\log r^2)$ into (1.2.3) and by the 0-1 law for the tail events, we can deduce (1.2.2).

To show Theorem 1.2.2, we employ the notion of the random environment seen from the particle. Then the conclusion follows by the ergodic theorem.

1.2.2 Main results in Chapter 3

In Chapter 3, we study the escape rate of continuous-time RWs on a class of the RCMs.

The first result is the LIL for continuous-time RWs.

Theorem 1.2.3 (Theorem 3.1.5 in Chapter 3)**.** *Suppose that a RW on the RCM enjoys the long-time Gaussian HKEs* (1.1.3) (1.1.4)*, and* $f(n)$ *in* (1.1.5) *satisfies the integrability condition* $\sum_n n^{2d} f(n) < \infty$.

(1) For almost all $\omega \in \Omega$ *there exist positive numbers* $c_1 = c_1^\omega$ *and* $c_2 = c_2^\omega$ *such that*

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/2}(\log\log t)^{1/2}} = c_1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G),$$
$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)}{t^{1/2}(\log\log t)^{1/2}} = c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G).$$

(2) For almost all $\omega \in \Omega$ *there exists a positive number* $c_3 = c_3^\omega$ *such that*

$$\liminf_{t\to\infty} \frac{\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)}{t^{1/2}(\log\log t)^{-1/2}} = c_3, \qquad P_x^\omega\text{-a.s. for all } x \in V(G).$$

The next result is concerning the escape rate for RWs on a class of RCM.

Theorem 1.2.4 (($d \ge 3$) Theorem 3.1.6 in Chapter 3)**.** *Let* h *be a non-increasing function, and suppose that a RW on a class of RCMs enjoys the long-time Gaussian HKEs* (1.1.3) *and* (1.1.4)*, and* $f(n)$ *in* (1.1.5) *satisfies the integrability condition* $\sum_n n^d f(nh(n^2)) < \infty$*. Then, either*

$$P_x^\omega\left(d(x, Y_t^\omega) \ge t^{1/2}h(t) \text{ for all sufficiently large } t\right) = 1$$

for almost all $\omega \in \Omega$ *and all* $x \in V(G)$*, or*

$$P_x^\omega\left(d(x, Y_t^\omega) \ge t^{1/2}h(t) \text{ for all sufficiently large } t\right) = 0,$$

for almost all $\omega \in \Omega$ *and all* $x \in V(G)$*, according as* $\int_1^\infty \frac{1}{t} h(t)^{d-2} dt < \infty$ *or* $= \infty$.

Theorems 1.2.3 and 1.2.4 are applicable to various examples such as a RW on the supercritical percolation cluster, the case $\omega_{xy} \in [1, \infty)$, the level sets of DGFF and the random interlacements.

As in the discrete-time case, the constants in Theorem 1.2.3 are deterministic when the random environment is ergodic.

Theorem 1.2.5 (Theorem 3.1.8 in Chapter 3)**.** *Suppose that the same assumptions as in Theorem 1.2.3 are fulfilled and suppose in addition that the random environment is ergodic w.r.t. the shifts on $\mathbb{Z}^d$. Then we can take c_1, c_2 and c_3 in Theorem 1.2.3 as deterministic constants (i.e. they do not depend on ω).*

The strategy of the proof of Theorem 1.2.3 is similar to that of Theorem 1.2.1, so we explain the strategy of the proof of Theorem 1.2.4. To prove Theorem 1.2.4, we need to estimate the probability that RW returns to balls centered around its starting point. In fact, we have the following estimates:

$$\frac{c_1^{-1} r^{d-2} t}{t^{d/2}} \leq P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t \right) \leq \frac{c_1 r^{d-2} t}{t^{d/2}},$$

where $d(x_0, x) \leq r$. By the above inequalities and the Borel-Cantelli lemma, we obtain the desired results.

1.2.3 Main Result in Chapter 4

In Chapter 4, we discuss the cutoff phenomena for the lamplighter random walks on fractals. Miller and Peres [54] gave a general framework for the cutoff phenomena with threshold $\frac{1}{2}T_{\text{cov}}(G^{(N)})$, where $T_{\text{cov}}(G^{(N)})$ is the expected cover time of the RW on $G^{(N)}$ (i.e. the expected time that a random walk visits all the vertices of $G^{(N)}$). A key assumption is uniform elliptic Harnack inequalities. We replace the uniform elliptic Harnack inequalities by uniform parabolic Harnack inequalities, and as a result, we derive a dichotomy result for the cutoff phenomena for lamplighter chains. Our main result is the following (see Theorem 4.1.4 for the precise statement):

Theorem 1.2.6 (Theorem 4.1.4 in Chapter 4)**.** *Suppose that an increasing sequence of finite graphs $\{G^{(N)}\}_N$ satisfies (i) d_f-set condition and (ii) uniform parabolic Harnack inequalities with order d_w. Then,*

(a) if $d_f < d_w$, then the lamplighter random walks on $G^{(N)}$ do NOT have a cutoff.

(b) if $d_f > d_w$, then the lamplighter random walks on $G^{(N)}$ have a cutoff with threshold $a_N = \frac{1}{2}T_{\text{cov}}(G^{(N)})$.

For examples, a sequence of finite graphs of a class of fractals satisfies the assumptions. As an analogue we discussed in Section 1.1.2, the uniform parabolic Harnack inequalities are equivalent to a finite analogue of HKEs.

The proof of Theorem 1.2.6 (b) is conducted by confirming the conditions given by [54]. To show Theorem 1.2.6 (a), we need to establish the following upper bound of the cover time:

$$\sup_{z \in V(G^{(N)})} \{P_z(\tau_{\rm cov}(G^{(N)}) > t)\} \le c_0 e^{-t/(c_0 T_N)}$$

for all t, N, where $T_N := (\mathrm{diam}\{G^{(N)}\})^{d_w}$. Moreover, we derive the following upper and lower bounds of total variation estimates:

$$\begin{aligned} c_1^{-1} e^{-c_1 t/T_N} - c_2(\mathrm{diam}\{G^{(N)}\})^{-d_f} &\le \max_{\boldsymbol{x} \in V(\mathbb{Z}_2 \wr G^{(N)})} \|P_t^*(\boldsymbol{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\rm TV} \\ &\le \max_{x \in V(G^{(N)})} P_x(\tau_{\rm cov}(G^{(N)}) > t) + \frac{\sqrt{S_N}}{2\sqrt{t}}, \end{aligned}$$

where $S_N := \max_{x,y \in V(G^{(N)})}\{R_{\rm eff}^{(N)}(x,y)\}$. Combining the above we obtain the desired result.

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Chapter 2

Laws of the iterated logarithm for random walks on random conductance models

We derive laws of the iterated logarithm for random walks on random conductance models under the assumption that the random walks enjoy long time sub-Gaussian heat kernel estimates.

2.1 Introduction

Random walks in random environments have been extensively studied for several decades in probability and mathematical physics. Random conductance model (RCM) is a specific class in that random walks on the RCMs are reversible, and that the class includes many important examples. Recently, there has been significant progress in the study of asymptotic behaviors of random walks on RCMs. In particular, asymptotic behaviors such as invariance principles and heat kernel estimates are obtained in the quenched sense, namely almost surely with respect to the randomness of the environments, even for degenerate cases. One of the typical examples is the random walk on the supercritical percolation cluster on $\mathbb{Z}^d$. In this case, Barlow [4] obtained quenched long time Gaussian heat kernel estimates such as (2.1.3) and (2.1.4) below with $\alpha = d, \beta = 2$. Soon after that, the quenched invariance principle was proved in [65] for $d \geq 4$ and later extended to all $d \geq 2$ in [16, 53]. Namely, for a simple random walk $\{Y_n^\omega\}_{n\geq 0}$ on the cluster, it was proved that $\varepsilon Y^\omega_{[t/\varepsilon^2]}$ converges as $\varepsilon \to 0$ to Brownian motion on $\mathbb{R}^d$ with covariance $\sigma^2 I$, $\sigma > 0$, for almost all environment ω. We note that the proof for $d \geq 3$ uses the heat kernel estimates given in [4].

The RCM on a graph is a family of non-negative random variables indexed by edges of the graph. Supercritical bond percolation cluster is a typical (degenerate) RCM which endows each edge of $\mathbb{Z}^d$ with i.i.d. Bernoulli random variable. The quenched Gaussian heat kernel estimates are established for various other RCMs, for example

(a) uniformly elliptic conductances ([24]),

(b) i.i.d. unbounded conductances bounded from below by a strictly positive constant ([12]),

(c) i.i.d. conductances bounded from above and some tail condition near 0 ([15]),

(d) random walks on the level sets of Gaussian free fields and the framework of random interlacements ([63]),

(e) positive conductances with some integrability condition ([2]).

Note that conductances in (a), (d), (e) are not necessarily i.i.d. Note also that, while (b)-(d) are discussed on $\mathbb{Z}^d$, (a) and (e) are discussed for more general graphs with some analytic properties. Quenched invariance principles for the random walks on RCMs are also established extensively. For more details, see [17, 49] and the references therein.

We are interested in further quenched asymptotic behaviors of the random walks on RCMs. The aim of this paper is to establish the laws of the iterated logarithm (LILs) for the sample paths of the random walk such as (2.1.6) and (2.1.7) below in the quenched level. In fact, for the random walk on the supercritical percolation cluster, Duminil-Copin [27] obtained the standard LIL (limsup version as in (2.1.6)) by using the results of [4]. Also, in [48] the LIL is obtained for a class of transient random walk in random environments. The novelty of this paper is twofold.

- We establish another law of the iterated logarithm (liminf version as in (2.1.7)).
- We establish quenched LILs for random walks on much more general RCMs.

Our approach is through the heat kernel estimates. Namely, we assume the quenched heat kernel estimates (Assumption 2.1.1) and establish the quenched LILs (Theorem 2.1.2). Since the quenched heat kernel estimates are established for many RCMs, our theorem applies for those examples as we discuss in Section 2.1.2.

The organization of the paper is as follows. We first explain the framework and main results of this paper. In Section 2.2, we give the preliminary estimates to prove the main results. In Section 2.3 we prove the LIL and in Section 2.4 we prove another LIL. Finally in Section 2.5, we assume the ergodicity of the media when $G = \mathbb{Z}^d$ and prove that the constants appearing in the limsup and liminf in the LILs are deterministic.

2.1.1 Framework and main results

Let $G = (V, E)$ be the countably infinite, locally finite and connected graph. We can define the graph distance $d : V \times V \to [0, \infty)$ in the usual way, i.e. the shortest length of path in G. Write $B(x, r) = \{y \in V(G) \mid d(x, y) \leq r\}$. Throughout this paper we assume that there exist $\alpha \geq 1$ and $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \sharp B(x, r) \leq c_2 r^\alpha \tag{2.1.1}$$

holds for all $x \in V(G)$ and $r \geq 1$.

We assume that the graph G is endowed with the non-negative weights (or conductance) $\omega = \{\omega(e) \mid e \in E\}$ which are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We write $\omega(e) = \omega_e = \omega_{xy}$ if $e = xy$. We take the base point x_0 of G and set $V(G^\omega) = \{v \in V(G) \mid x_0 \overset{\omega}{\longleftrightarrow} v\}$, where $x_0 \overset{\omega}{\longleftrightarrow} v$ means that there exists a path $\gamma = e_1 e_2 \cdots e_k$ from x_0 to v such that $\omega(e_i) > 0$ for all $i = 1, 2, \cdots, k$. We also define $\mathcal{C}(\omega)$ as the set of all vertices x which satisfy $x \overset{\omega}{\longleftrightarrow} \infty$, i.e. there exists an infinite length and self-avoiding path $\gamma = e_1 e_2 \cdots$ starting at x which satisfies $\omega(e_i) > 0$ for all i. Note that if each weight $\omega(e)$ is strictly positive, then $V(G^\omega) = \mathcal{C}(\omega) = V(G)$. Let $\mu^\omega(x) = \sum_{y; y \sim x} \omega_{xy}$ be the weight of x, $V^\omega(A) = \displaystyle\sum_{y \in A \cap V(G^\omega)} \mu^\omega(y)$ be the volume of $A \subset V(G)$ and $V^\omega(x, r) = V^\omega(B(x, r))$ be the volume of the ball $B(x, r)$. We also denote $B^\omega(x, r) = B(x, r) \cap V(G^\omega)$.

Next we define the random walk on the weighted graph. Let $\{X_n^\omega\}_{n \geq 0}$ be the discrete time random walk on $V(G^\omega)$ whose transition probability is given by $P^\omega(x, y) = \dfrac{\omega_{xy}}{\mu^\omega(x)}$. We write $P_n^\omega(x, y) = P_x^\omega(X_n^\omega = y)$. The heat kernel is denoted by $p_n^\omega(x, y) = \dfrac{P_n^\omega(x, y)}{\mu^\omega(y)}$.

For our main results, we assume the following conditions. Note that $\alpha \geq 1$ is the same as in (2.1.1).

Assumption 2.1.1. *There exist $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$, positive constants $c_{1.1}, c_{1.2}, \cdots, c_{1.6}, \beta, \epsilon$, with $\epsilon + 1 < \beta$ and random variables $N_{x,\epsilon}(\omega)$ $(x \in V(G), \omega \in \Omega_0)$ such that the following hold.*

(1) For all $\omega \in \Omega_0$, $x \in V(G^\omega)$ and $r \geq N_{x,\epsilon}(\omega)$, it holds that

$$c_{1.1} r^\alpha \leq V^\omega(x, r) \leq c_{1.2} r^\alpha. \tag{2.1.2}$$

(2) For all $\omega \in \Omega_0$, $\{X_n^\omega\}_{n \geq 0}$ enjoys the following heat kernel estimates;

$$p_n^\omega(x, y) \leq \frac{c_{1.3}}{n^{\alpha/\beta}} \exp\left[-c_{1.4} \left(\frac{d(x, y)}{n^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \tag{2.1.3}$$

for $d(x, y) \vee N_{x,\epsilon}(\omega) \leq n$, and

$$p_n^\omega(x, y) + p_{n+1}^\omega(x, y) \geq \frac{c_{1.5}}{n^{\alpha/\beta}} \exp\left[-c_{1.6} \left(\frac{d(x, y)}{n^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \tag{2.1.4}$$

for $d(x, y)^{1+\epsilon} \vee N_{x,\epsilon}(\omega) \leq n$.

(3) There exists a non-increasing function $f_\epsilon(n)$ which satisfies

$$\mathbb{P}(N_{x,\epsilon} \geq n) \leq f_\epsilon(n) \quad \text{and} \quad \sum_{n \geq 1} n^{\alpha\beta} f_\epsilon(n) < \infty. \tag{2.1.5}$$

Now we state the main result of this paper.

Theorem 2.1.2. *Suppose that Assumption 2.1.1 holds. Then for almost all environment $\omega \in \Omega$ there exist positive constants $C_1 = C_1(\omega)$ and $C_2 = C_2(\omega)$ such that the following hold.*

$$\limsup_{n \to \infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta} (\log \log n)^{1-1/\beta}} = C_1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega), \tag{2.1.6}$$

$$\liminf_{n \to \infty} \frac{\max_{0 \leq \ell \leq n} d(X_0^\omega, X_\ell^\omega)}{n^{1/\beta} (\log \log n)^{-1/\beta}} = C_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega). \tag{2.1.7}$$

We note that we can replace $d(X_0^\omega, X_n^\omega)$ in (2.1.6) to $\max\limits_{0 \leq \ell \leq n} d(X_0^\omega, X_\ell^\omega)$ with possibly different C_1. We also note that if the random walk can be embedded into Brownian motion in some strong sense (which seems plausible in various concrete models), then (2.1.6),(2.1.7) can be shown as a consequence ([18]). It would be very interesting to prove such a strong approximation theorem.

The constants C_i above may depend on the environment ω. In order to guarantee that they are deterministic constants, we need to assume the ergodicity of the media.

For the purpose, we now consider the case $G = \mathbb{Z}^d$. In this case, we can define the shift operators $\tau_x : \Omega \to \Omega$ $(x \in \mathbb{Z}^d)$ as

$$(\tau_x \omega)_{yz} = \omega_{y+x,z+x}.$$

We assume the following ergodicity of the media.

Assumption 2.1.3. *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies the following conditions;*

(1) $\mathbb{P}$ is ergodic with respect to the translation operators τ_x, i.e. $\mathbb{P} \circ \tau_x = \mathbb{P}$ and for any $A \in \mathcal{F}$ with $\tau_x(A) = A$ for all $x \in \mathbb{Z}^d$ then $\mathbb{P}(A) = 0$ or 1.

(2) For almost all environment ω, $\mathcal{C}(\omega)$ contains an unique infinite connected component.

Theorem 2.1.4. *Suppose that Assumption 2.1.1 and Assumption 2.1.3 hold. Then we can take C_1, C_2 in Theorem 2.1.2 as deterministic constants (which do not depend on ω).*

Remark 2.1.5. *In this paper, we only consider discrete time Markov chains, but similar results hold for continuous time Markov chains (constant speed random walks and variable speed random walks); see [57].*

2.1.2 Examples

In this subsection, we give examples for which our results hold.

Example 2.1.6 (Bernoulli supercritical percolation cluster)**.** *Barlow [4, Theorem 1] proved that heat kernels of simple random walks on the super-critical percolation cluster for $\mathbb{Z}^d$, $d \geq 2$ satisfy Assumption 2.1.1 with $\alpha = d$, $\beta = 2$ and $f_\epsilon(n) = c\exp(-c'n^\delta)$ for some $c, c', \delta > 0$. (In [4], heat kernels for continuous time random walk were obtained. See the remark after [4, Theorem 1] and [16, Section A] for discrete time modifications.) Since the media is i.i.d. and there exists an unique infinite connected component, we can obtain the LILs* (2.1.6) *and* (2.1.7) *with deterministic constants. Note that* (2.1.6) *for the supercritical percolation cluster was already obtained by [27, Theorem 1.1].*

Example 2.1.7 (Uniform elliptic case)**.** *Suppose the graph $G = (V, E)$ endowed with weight 1 on each edge satisfies* (2.1.1) *and the scaled Poincaré inequalities. Put random conductance on each edge so that $c_1 \leq \omega(e) \leq c_2$ for all $e \in E$ and for almost all ω, where $c_1, c_2 > 0$ are deterministic constants. Then Assumption 2.1.1 holds with $\beta = 2$ and $N_{x,\epsilon} \equiv 1$. So the LILs* (2.1.6) *and* (2.1.7) *hold.*

Example 2.1.8 (Gaussian free fields and random interlacements)**.** *Sapozhnikov [63, Theorem 1.15] proved that for* $\mathbb{Z}^d$, $d \geq 3$, *the random walks on (i) certain level sets of Gaussian free fields; (ii) random interlacements at level* $u > 0$*; (iii) vacant sets of random interlacements for suitable level sets, satisfy our Assumption 2.1.1 with* $\alpha = d$, $\beta = 2$ *and the tail estimates of* $N_{x,\epsilon}(\omega)$ *as* $f_\epsilon(n) = c\exp(-c'(\log n)^{1+\delta})$ *for some* $c, c', \delta > 0$*. This subexponential tail estimate is sufficient for Assumption 2.1.1* (3)*. Since the media is ergodic and there is an unique infinite connected components (see [60], [66, Corollary 2.3] and [71, Theorem 1.1]), the LILs* (2.1.6) *and* (2.1.7) *hold with deterministic constants.*

Example 2.1.9 (Uniform elliptic RCM on fractals)**.** *Let* $a_1 = (0,0), a_2 = (1,0), a_3 = (1/2, \sqrt{3}/2)$, $I = \{1,2,3\}$ *and set* $F_i(x) = (x - a_i)/2 + a_i$ *for* $i \in I$*. Define*

$$V = \bigcup_{n\in\mathbb{N}} \Big(2^n \bigcup_{i,i_1,\cdots,i_n \in I} F_{i_n} \circ \cdots \circ F_{i_1}(a_i) \Big), \quad E = \bigcup_{n\in\mathbb{N}} \Big(2^n \bigcup_{i_1,\cdots,i_n \in I} F_{i_n} \circ \cdots \circ F_{i_1}(B_0) \Big),$$

where $B_0 = \{\{x,y\} : x \neq y \in \{a_1,a_2,a_3\}\}$*.* $G = (V,E)$ *is called the 2-dimensional pre-Sierpinski gasket. Put random conductance on each edge so that* $c_1 \leq \omega(e) \leq c_2$ *for all* $e \in E$ *and almost all* ω*, where* $c_1, c_2 > 0$ *are deterministic constants. Then Assumption 2.1.1 holds with* $\alpha = \log 3/\log 2$, $\beta = \log 5/\log 2 > 2$ *and* $N_{x,\epsilon} \equiv 1$*. (In fact, this can be generalized to the uniform finitely ramified graphs for some* $\alpha \geq 1$ *and* $\beta \geq 2$*; see [37].) So the LILs* (2.1.6) *and* (2.1.7) *hold.*

We note that among the examples mentioned at the beginning of this paper, (b), (c) and (e) are for continuous time Markov chains, so the LILs will be discussed in [57].

2.2 Consequences of Assumption 2.1.1

In this section, we prepare the preliminary results of Assumption 2.1.1.

2.2.1 Consequences of heat kernel estimates

We first give consequences of the heat kernel estimates (2.1.3) and (2.1.4).

Lemma 2.2.1. *(1) There exist* $c_1, c_2 > 0$ *such that for almost all* $\omega \in \Omega$*,*

$$P_y^\omega \left(\max_{0\leq j\leq n} d(x, X_j^\omega) \geq 3r \right) \leq c_1 \exp\left(-c_2 \left(\frac{r^\beta}{n} \right)^{\frac{1}{\beta-1}} \right)$$

holds for all $n \geq 1, r \geq 1$ *and* $x, y \in V(G^\omega)$ *with* $\max_{z \in B(y,2r)} N_{z,\epsilon}(\omega) \leq r$ *and* $d(x,y) \leq r$.

(2) There exist $c_3, c_4, R_0 > 0$ *such that for almost all* $\omega \in \Omega$,

$$P_x^\omega\left(\max_{0 \leq j \leq n} d(X_0^\omega, X_j^\omega) \leq r\right) \leq c_3 \exp\left(-c_4 \frac{n}{r^\beta}\right)$$

holds for all $n \geq 1, r \geq R_0$ *and* $x \in V(G^\omega)$ *with* $\max_{y \in B(x,r)} N_{y,\epsilon}(\omega) \leq 2r$.

(3) Suppose $\epsilon + 1 < \beta$*. Then there exist* $c_5, c_6 > 0$ *and* $\eta \geq 1$ *such that for almost all* $\omega \in \Omega$,

$$P_x^\omega\left(\max_{0 \leq j \leq n} d(X_0^\omega, X_j^\omega) \leq r\right) \geq c_5 \exp\left(-c_6 \frac{n}{r^\beta}\right)$$

holds for all $x \in V(G^\omega)$ *and* $n \geq 1, r \geq 1$ *with* $\max_{z \in B(x,3\eta r)} N_{z,\epsilon}(\omega) \leq r^{1/\beta}$.

Since the computations are standard, we omit the proof. Indeed, (1) can be proved by simple modifications of [3, Lemma 3.9], and (2) can be proved similarly to [51, Lemma 3.2]. (3) is simple modification of [51, Proposition 3.3] respectively.

Let $c_5, c_6 > 0$ be as in Lemma 2.2.1 (3). Define $a_k, b_k, \lambda_k, u_k, \sigma_k$ as follows:

$$a_k^\beta = e^{k^2},\ b_k^\beta = e^k,\ \lambda_k = c_6^{-1}\log(c_5(1+k)^{2/3}),\ u_k = \lambda_k a_k^\beta,\ \sigma_k = \sum_{i=1}^{k-1} u_i. \qquad (2.2.1)$$

Corollary 2.2.2 (Corollary of Lemma 2.2.1 (3))**.** *Let* $\eta \geq 1$ *be as in Lemma 2.2.1 (3). Then the following holds for almost all* $\omega \in \Omega$*, all* $x \in V(G^\omega)$ *and* $k \geq 1$ *with* $\max_{z \in B(x,4\eta a_k)} N_{z,\epsilon}(\omega) \leq a_k^{1/\beta}$,

$$\min_{z \in B^\omega(x,a_k)} P_z^\omega\left(\max_{0 \leq s \leq u_k} d(X_0^\omega, X_s^\omega) \leq a_k\right) \geq \frac{1}{(1+k)^{2/3}}.$$

The heat kernel estimates (2.1.3) and (2.1.4) also give the triviality of tail events.

Theorem 2.2.3 ($0-1$ law for tail events)**.** *For almost all* $\omega \in \Omega$*, the following holds; Let* A^ω *be a tail event, i.e.* $A^\omega \in \bigcap_{n=0}^{\infty} \sigma\{X_k^\omega : k \geq n\}$*. Then either* $P_x^\omega(A^\omega) = 0$ *for all* x *or* $P_x^\omega(A^\omega) = 1$ *for all* x *holds.*

The proof of Theorem 2.2.3 is quite similar to that of [14, Proposition 2.3], so we omit the proof.

2.2.2 Consequences of the tail estimate (2.1.5)

We next give simple consequences of the tail estimate (2.1.5). Recall the notations in (2.2.1), and set $\Phi(q) = q^{1/\beta}(\log\log q)^{1-1/\beta}$.

Lemma 2.2.4. *(1) Suppose that $f_\epsilon(n)$ satisfies $\sum_n n^\alpha f_\epsilon(n) < \infty$. Then for any $\gamma_1, \gamma_2 > 0$ and for almost all $\omega \in \Omega$, there exists $L_{x,\epsilon,\gamma_1,\gamma_2}(\omega) > 0$ such that the following hold for all $n \geq L_{x,\epsilon,\gamma_1,\gamma_2}(\omega)$,*

$$\gamma_1 a_n \geq \max_{z \in B(x,\gamma_2 a_n)} N_{z,\epsilon}(\omega), \quad \gamma_1 b_n \geq \max_{z \in B(x,\gamma_2 b_n)} N_{z,\epsilon}(\omega).$$

(2) Suppose that $f_\epsilon(n)$ satisfies $\sum_n n^\alpha f_\epsilon(n) < \infty$. Then for any $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$, there exists $L_{x,\epsilon,\gamma_1,\gamma_2,q}(\omega) > 0$ such that the following hold for all $n \geq L_{x,\epsilon,\gamma_1,\gamma_2,q}(\omega)$,

$$\gamma_1 \Phi(q^n) \geq \max_{z \in B(x,\gamma_2 \Phi(q^n))} N_{z,\epsilon}(\omega), \quad \gamma_1 q^{(n-1)/\beta} \geq \max_{z \in B(x,\gamma_2 q^{(n-1)/\beta})} N_{z,\epsilon}(\omega).$$

(3) Suppose that $f_\epsilon(n)$ satisfies $\sum_n n^{\alpha\beta} f_\epsilon(n) < \infty$. Then for all $\gamma_1, \gamma_2 > 0$ and for almost all $\omega \in \Omega$, there exists $K_{x,\epsilon,\gamma_1,\gamma_2}(\omega) > 0$ such that the following holds for all $n \geq K_{x,\epsilon,\gamma_1,\gamma_2}(\omega)$,

$$\gamma_1 a_n^{1/\beta} \geq \max_{z \in B(x,\gamma_2 a_n)} N_{z,\epsilon}(\omega).$$

Proof. We only prove the first inequality in (1). It is easy to see that

$$\mathbb{P}\left(\max_{z \in B(x,\gamma_2 n)} N_{z,\epsilon} > \gamma_1 n\right) \leq \sum_{z \in B(x,\gamma_2 n)} \mathbb{P}\left(N_{z,\epsilon} \geq \gamma_1 n\right) \leq c_1 (\gamma_2 n)^\alpha f_\epsilon(\gamma_1 n).$$

The assumption implies $\sum_n n^\alpha f_\epsilon(\gamma_1 n) < \infty$, so the conclusion follows by the Borel-Cantelli Lemma. □

2.3 Proof of LIL

In this section, we prove (2.1.6) in Theorem 2.1.2. We continue to use the notation $\Phi(q) = q^{1/\beta}(\log\log q)^{1-1/\beta}$ in this section.

Theorem 2.3.1. *Suppose that Assumption 2.1.1 holds. Then there exists $c_+ > 0$ such that the following holds for almost all $\omega \in \Omega$,*

$$\limsup_{n\to\infty} \frac{\max_{0\le k\le n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} \le c_+, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).$$

Proof. By Lemma 2.2.1 (1) we have

$$P_x^\omega\left(\max_{0\le k\le q^n} d(X_0^\omega, X_k^\omega) \ge \eta\Phi(q^n)\right) \le c_1 \exp\left[-c_2\left(\frac{(\eta\Phi(q^n))^\beta}{q^n}\right)^{\frac{1}{\beta-1}}\right]$$
$$= c_1 \exp\left[-c_2\eta^{\beta/(\beta-1)}\log\log q^n\right] = c_1\left(\frac{1}{n\log q}\right)^{c_2\eta^{\beta/(\beta-1)}}$$

for all $q \ge 1$, almost all ω and n with $\max_{z\in B(x,2\Phi(q^n))} N_{z,\epsilon}(\omega) \le \Phi(q^n)$. Therefore the above estimate holds for $n \ge L_{x,\epsilon,1,2,q}(\omega)$ by Lemma 2.2.4 (2).

So taking $\eta > 0$ large enough and using the Borel-Cantelli Lemma, we have

$$\limsup_{n\to\infty} \frac{\max_{0\le k\le q^n} d(X_0^\omega, X_k^\omega)}{\Phi(q^n)} \le \eta.$$

We can easily obtain the conclusion from the above inequality. □

Theorem 2.3.2. *Suppose that Assumption 2.1.1 holds. Then there exists $c_- > 0$ such that the following holds for almost all $\omega \in \Omega$,*

$$\limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} \ge c_-, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).$$

Proof. Note that $d(X_0^\omega, X_{q^n}^\omega) \ge d(X_{q^{n-1}}^\omega, X_{q^n}^\omega) - d(X_0^\omega, X_{q^{n-1}}^\omega)$ for any $q > 1$. By Theorem 2.3.1, for almost all $\omega \in \Omega$ and P_x^ω-a.s. there exists a constant M_x such that

$$\frac{d(X_0^\omega, X_{q^{n-1}}^\omega)}{\Phi(q^n)} = \frac{d(X_0^\omega, X_{q^{n-1}}^\omega)}{\Phi(q^{n-1})}\frac{\Phi(q^{n-1})}{\Phi(q^n)} \le \frac{2c_+}{q^{1/\beta}}$$

holds for any $n \ge M_x$, where c_+ is as in Theorem 3.1. The right hand side of the above inequality can be small enough by taking q sufficiently large. So it is enough to show that there exists a positive constant c_- independent of q such that the following holds,

$$\limsup_{n\to\infty} \frac{d(X_{q^{n-1}}^\omega, X_{q^n}^\omega)}{\Phi(q^n)} \ge c_-. \tag{2.3.1}$$

We may and do take $q \geq 2$. To prove (2.3.1), let $\mathcal{F}_n^\omega = \sigma\left(X_k^\omega \mid k \leq n\right)$ and $t_n = q^n - q^{n-1}$. Set $\kappa > 0$ so that $c_{1.1}\kappa^\alpha - c_{1.2} \geq 1$. Let $\lambda > 0$ be a small constant so that $\kappa\lambda < 1$. By Theorem 2.3.1 there exists a constant c'_+ such that $d(X_0^\omega, X_{q^{n-1}}^\omega) \leq c'_+\Phi(q^{n-1})$ for almost all ω and for sufficiently large n. We first note that

$$\begin{aligned}
&P_x^\omega\left(d(X_{q^{n-1}}^\omega, X_{q^n}^\omega) \geq \lambda\Phi(q^n)\middle|\mathcal{F}_{q^{n-1}}^\omega\right)\\
&\geq P_x^\omega\left(d(X_{q^{n-1}}^\omega, X_{q^n}^\omega) \geq \lambda\Phi(q^n), d(X_0^\omega, X_{q^{n-1}}^\omega) \leq c'_+\Phi(q^{n-1})\middle|\mathcal{F}_{q^{n-1}}^\omega\right)\\
&= 1_{\left\{d(X_0^\omega, X_{q^{n-1}}^\omega) \leq c'_+\Phi(q^{n-1})\right\}} P_{X_{q^{n-1}}^\omega}^\omega\left(d(X_0^\omega, X_{t_n}^\omega) \geq \lambda\Phi(q^n)\right)\\
&\geq \left(\min_{y \in B^\omega(x, c'_+\Phi(q^{n-1}))} P_y^\omega\left(d(X_0^\omega, X_{t_n}^\omega) \geq \lambda\Phi(q^n)\right)\right) 1_{\left\{d(X_0^\omega, X_{q^{n-1}}^\omega) \leq c'_+\Phi(q^{n-1})\right\}}. \qquad (2.3.2)
\end{aligned}$$

We estimate the first term of (2.3.2). For any n with $\lambda\Phi(q^n) \geq N_{y,\epsilon}(\omega)$, using (2.1.2) we have

$$\mu^\omega(B(y, \kappa\lambda\Phi(q^n)) \setminus B(y, \lambda\Phi(q^n))) \geq c_{1.1}(\kappa\lambda\Phi(q^n))^\alpha - c_{1.2}(\lambda\Phi(q^n))^\alpha \geq (\lambda\Phi(q^n))^\alpha.$$

So for such n and for $y \in B^\omega(x, c'_+\Phi(q^{n-1}))$ we have

$$\begin{aligned}
&P_y^\omega\left(\lambda\Phi(q^n) \leq d(X_0^\omega, X_{t_n}^\omega) \leq \kappa\lambda\Phi(q^n)\right) \geq \sum_{z \in B^\omega(y, \kappa\lambda\Phi(q^n)) \setminus B^\omega(y, \lambda\Phi(q^n))} p_{t_n}^\omega(y, z)\mu^\omega(z)\\
&\geq \frac{c_{1.5}}{t_n^{\alpha/\beta}} \exp\left[-c_{1.6}\left(\frac{(\kappa\lambda\Phi(q^n))^\beta}{t_n}\right)^{\frac{1}{\beta-1}}\right] \mu^\omega\left(B(y, \kappa\lambda\Phi(q^n)) \setminus B(y, \lambda\Phi(q^n))\right)\\
&\geq c_1\left(\frac{1}{n}\right)^{c_2(\kappa\lambda)^{\beta/(\beta-1)}},
\end{aligned}$$

where we can take c_1, c_2 as the constants which do not depend on q. Therefore for any n with $\max_{y \in B(x, c'_+\Phi(q^{n-1}))} N_{y,\epsilon}(\omega) \leq \lambda\Phi(q^n)$ we have

$$\min_{y \in B^\omega(x, c'_+\Phi(q^{n-1}))} P_y^\omega\left(d(X_0^\omega, X_{t_n}^\omega) \geq \lambda\Phi(q^n)\right) \geq c_1\left(\frac{1}{n}\right)^{c_2(\kappa\lambda)^{\beta/(\beta-1)}}.$$

By Lemma 2.2.4 (2), $\max_{y \in B(x, c'_+\Phi(q^{n-1}))} N_{y,\epsilon}(\omega) \leq \lambda\Phi(q^n)$ for all $n \geq L_{x,\epsilon,\lambda,c'_+,q}(\omega)$. As we mentioned before, $d(X_0^\omega, X_{q^{n-1}}^\omega) \leq c'_+\Phi(q^{n-1})$ for sufficiently large n. Thus for sufficiently small λ we have

$$\sum_n P_x^\omega\left(d(X_{q^{n-1}}^\omega, X_{q^n}^\omega) \geq \lambda\Phi(q^n)\middle|\mathcal{F}_{q^{n-1}}^\omega\right) = \infty.$$

Hence by the second Borel-Cantelli lemma, we have

$$\limsup_{n\to\infty} \frac{d(X^\omega_{q^{n-1}}, X^\omega_{q^n})}{\Phi(q^n)} \geq \lambda.$$

We thus complete the proof. □

By Theorem 2.2.3, Theorem 2.3.1 and Theorem 2.3.2, we complete the proof of (2.1.6) in Theorem 2.1.2.

2.4 Proof of another LIL

In this section, we prove (2.1.7) of Theorem 2.1.2.

Theorem 2.4.1. *Suppose that Assumption 2.1.1 holds. Then for almost all $\omega \in \Omega$ there exists $c = c(\omega) > 0$ such that the following holds,*

$$\liminf_{n\to\infty} \frac{\max_{0<\ell\leq n} d(X^\omega_0, X^\omega_\ell)}{n^{1/\beta}(\log\log n)^{-1/\beta}} = c, \qquad P^\omega_x\text{-a.s. for all } x \in V(G^\omega). \tag{2.4.1}$$

Proof. We follow the strategy in [46]. It is enough to prove that there exist positive constants $c_1, c_2 > 0$ such that the following holds,

$$c_1 \leq \limsup_{r\to\infty} \frac{\tau^\omega_{B(x,r)}}{r^\beta(\log\log r^\beta)} \leq c_2, \qquad P^\omega_x\text{-a.s. for all } x \in V(G^\omega), \tag{2.4.2}$$

where $\tau^\omega_{B(x,r)} = \inf\{n \geq 0 \mid X^\omega_n \notin B(x,r)\}$. Indeed, putting $n = r^\beta(\log\log r^\beta)$ into (2.4.2) and using Theorem 2.2.3, we can easily obtain (2.4.1). In the following, we use the notation in (2.2.1).

Lower bound of (2.4.2); It is enough to show that there exist constants $\eta > 0$ and $J(\omega) > 0$ such that

$$P^\omega_x\left(\max_{a_m\leq r\leq a_{2m}} \frac{\tau^\omega_{B(x,r)}}{r^\beta(\log\log r^\beta)} \leq \eta\right) \leq \exp(-m^{1/4}) \tag{2.4.3}$$

holds for all $m \geq J(\omega)$, since the lower bound of (2.4.2) follows by (2.4.3) and the Borel-Cantelli Lemma.

First, we estimate the left hand side of (2.4.3) as follows,

$$
\begin{aligned}
&P_x^\omega\left(\max_{2a_m\le r\le 2a_{2m}} \frac{\tau^\omega_{B(x,r)}}{r^\beta(\log\log r^\beta)} \le \eta\right) \le P_x^\omega\left(\max_{m\le k\le 2m} \frac{\tau^\omega_{B(x,2a_k)}}{u_k} \le 1\right)\\
&\le P_x^\omega\left(\max_{m\le k\le 2m} \frac{\tau^\omega_{B(x,2a_k)}}{\sigma_k} \le 1\right) \le P_x^\omega\left(\bigcap_{m\le k\le 2m}\left\{\max_{0\le s\le \sigma_{k+1}} d(X_0^\omega, X_s^\omega) \ge 2a_k\right\}\right)\\
&= P_x^\omega(A_m^\omega), \qquad (2.4.4)
\end{aligned}
$$

where we define $D_k^\omega = \left\{\max_{0\le s\le\sigma_{k+1}} d(X_0^\omega, X_s^\omega) \ge 2a_k\right\}$ and use $A_m^\omega = \bigcap_{k=m}^{2m} D_k^\omega$ in the last equation. In order to estimate $P_x^\omega(A_m^\omega)$, set

$$
\begin{aligned}
G_k^\omega &= \left\{\max_{\sigma_k\le s\le \sigma_{k+1}} d(X^\omega_{\sigma_k}, X_s^\omega) > a_k, d(X_0^\omega, X^\omega_{\sigma_k}) < a_k\right\},\\
H_k^\omega &= \left\{\max_{0\le s\le\sigma_k} d(X_0^\omega, X_s^\omega) \ge a_k\right\}.
\end{aligned}
$$

We can easily see $D_k^\omega \subset G_k^\omega \cup H_k^\omega$. Let $\eta \ge 1$ be as in Corollary 2.2.2. For any k with $\max_{z\in B(x,4\eta a_k)} N_{z,\epsilon}(\omega) \le a_k^{1/\beta}$, we have

$$
\begin{aligned}
P_x^\omega(G_k^\omega) &= E_x^\omega\left[1_{\{d(x,X^\omega_{\sigma_k})<a_k\}} P^\omega_{X^\omega_{\sigma_k}}\left(\max_{0\le s\le u_k} d(X_0^\omega, X_s^\omega) > a_k\right)\right]\\
&\le \max_{z\in B^\omega(x,a_k)} P_z^\omega\left(\max_{0\le s\le u_k} d(z, X_s^\omega) > a_k\right)\\
&= 1 - \min_{z\in B^\omega(x,a_k)} P_z^\omega\left(\max_{0\le s\le u_k} d(z, X_s^\omega) \le a_k\right)\\
&\le 1 - \frac{1}{(1+k)^{2/3}} \le \exp\left(-c_3 k^{-2/3}\right),
\end{aligned}
$$

where we use Corollary 2.2.2 in the forth inequality. So, it holds that

$$
\max_{z\in B^\omega(x,a_k)} P_z^\omega(G_k^\omega) \le \exp\left(-c_3 k^{-2/3}\right) \qquad (2.4.5)
$$

for any k with $\max_{z\in B(x,5\eta a_k)} N_{z,\epsilon}(\omega) \le a_k^{1/\beta}$. Hence, by Lemma 2.2.4 (3), (2.4.5) holds for $k \geq m \geq K_{x,\epsilon,1,5\eta}(\omega)$. For any $k \geq m \geq L_{x,\epsilon,2/3,1/3}(\omega)$ we have

$$\begin{aligned}P_x^\omega(H_k^\omega) &\leq c_4 \exp\left[-c_5\left(\frac{a_k^\beta}{\sigma_k}\right)^{1/(\beta-1)}\right] \\ &\leq c_6 \exp\left[-c_7\left(\frac{a_k^\beta}{(k-1)\lambda_{k-1}a_{k-1}^\beta}\right)^{1/(\beta-1)}\right] \\ &\leq c_8 \exp\left[-c_9\left(\frac{e^{2k}}{k\log k}\right)^{1/(\beta-1)}\right], \qquad (2.4.6)\end{aligned}$$

where we use Lemma 2.2.1 (1) and Lemma 2.2.4 (1) in the first inequality. We can easily see

$$A_m^\omega \subset \left(\bigcap_{k=m}^{2m} G_k^\omega\right) \cup \left(\bigcup_{k=m}^{2m} H_k^\omega\right).$$

Using the Markov property, (2.4.5) and (2.4.6) we have

$$\begin{aligned}P_x^\omega(A_m^\omega) &\leq \prod_{k=m}^{2m} \exp(-c_3 k^{-2/3}) + c_8 \sum_{k=m}^{2m} \exp\left[-c_9\left(\frac{e^{2k}}{k\log k}\right)^{1/(\beta-1)}\right] \\ &\leq \exp(-c_{10} m^{1/4}) \qquad (2.4.7)\end{aligned}$$

for any $m \geq K_{x,\epsilon,1,5\eta}(\omega) \vee L_{x,\epsilon,2/3,1/3}(\omega)$. By (2.4.4) and (2.4.7) we obtain

$$\sum_m P_x^\omega\left(\max_{2a_m \leq r \leq 2a_{2m}} \frac{\tau_{B(x,r)}^\omega}{r^\beta(\log\log r^\beta)} \leq \eta\right) < \infty$$

and thus by the Borel-Cantelli lemma, we obtain the lower bound of (2.4.3).

Upper bound; Define $B_k^\omega = \left\{\max_{b_k \leq r \leq b_{k+1}} \frac{\tau_{B(x,r)}^\omega}{r^\beta(\log\log r^\beta)} \geq \eta\right\}$. Then by Lemma 2.2.1 (2) and Lemma 2.2.4 (1), for any $k \geq L_{x,\epsilon,2,1}(\omega)$ we have

$$\begin{aligned}P_x^\omega(B_k^\omega) &\leq P_x^\omega\left(\tau_{B(x,b_{k+1})}^\omega \geq \eta b_k^\beta \log\log b_k^\beta\right) \\ &\leq P_x^\omega\left(\max_{0\leq s\leq \eta b_k^\beta \log\log b_k^\beta} d(X_0^\omega, X_s^\omega) \leq b_{k+1}\right) \\ &= P_x^\omega\left(\max_{0\leq s\leq \frac{\eta}{e} b_{k+1}^\beta \log k} d(X_0^\omega, X_s^\omega) \leq b_{k+1}\right) \leq \left(\frac{c_{11}}{k}\right)^{c_{12}\eta/e}.\end{aligned}$$

Since the right hand side of the above is summable for sufficient large η, by the Borel-Cantelli lemma we have

$$\limsup_{k\to\infty} \max_{b_k \le r \le b_{k+1}} \frac{\tau^{\omega}_{B(x,r)}}{r^{\beta}(\log\log r^{\beta})} \le \eta, \qquad P^{\omega}_x\text{-a.s.}$$

We can easily obtain the upper bound of (2.4.2) from the above inequality. We thus complete the proof. □

2.5 Ergodic media

In this section, we consider the case $G = (V, E) = \mathbb{Z}^d$ and obtain Theorem 2.1.4 under Assumption 2.1.1 and Assumption 2.1.3.

2.5.1 Ergodicity of the shift operator on $\Omega^{\mathbb{Z}}$

Let $\Omega = [0, \infty)^E$ and define $\mathscr{B}$ as the natural σ-algebra (generated by coordinate maps). We write $\mathcal{X} = \Omega^{\mathbb{Z}}$, $\mathscr{X} = \mathscr{B}^{\otimes \mathbb{Z}}$ and denote a shift operator by τ_x, i.e. $(\tau_x\omega)_e = \omega_{x+e}$. If each conductance may take the value 0, we regard 0 as the base point and define $\mathcal{C}_0(\omega) = \{x \in \mathbb{Z}^d \mid 0 \overset{\omega}{\longleftrightarrow} x\}$, where $0 \overset{\omega}{\longleftrightarrow} x$ means that there exists a path $\gamma = e_1e_2\cdots e_k$ from 0 to x such that $\omega(e_i) > 0$ for all $i = 1, 2, \cdots, k$. Define $\Omega_0 = \{\omega \in \Omega \mid \sharp\mathcal{C}_0(\omega) = \infty\}$ and $\mathbb{P}_0 = \mathbb{P}(\cdot \mid \Omega_0)$.

Next we consider the Markov chain on the random environment (called the environment seen from the particle) according to Kipnis and Varadhan [47]. Let $\omega_n(\cdot) = \omega(\cdot + X^{\omega}_n) = \tau_{X^{\omega}_n}\omega(\cdot) \in \Omega$. We can regard this Markov chain $\{\omega_n\}_{n\ge 0}$ as being defined on $\mathcal{X} = \Omega^{\mathbb{Z}}$. We define a probability kernel $Q : \Omega_0 \times \mathscr{B} \to [0, 1]$ as

$$Q(\omega, A) = \frac{1}{\sum_{e':|e'|=1} \omega_{e'}} \sum_{v:|v|=1} \omega_{0v} 1_{\{\tau_v\omega \in A\}}.$$

This is nothing but the transition probability of the Markov chain $\{\omega_n\}_{n\ge 0}$.

Next we define the probability measure on $(\mathcal{X}, \mathscr{X})$ as

$$\mu\left((\omega_{-n}, \cdots, \omega_n) \in B\right) = \int_B \mathbb{P}_0(d\omega_{-n})Q(\omega_{-n}, d\omega_{-n+1})\cdots Q(\omega_{n-1}, d\omega_n).$$

By the above definition, $\{\tau_{X^{\omega}_k}\omega\}_{k\ge 0}$ has the same law in $\mathbb{E}_0(P^{\omega}_0(\cdot))$ as $(\omega_0, \omega_1, \cdots)$ has in μ, that is,

$$\mathbb{E}_0\left[P^{\omega}_0(\{\tau_{X^{\omega}_k}\omega\}_{k\ge 0} \in B)\right] = \mu((\omega_0, \omega_1, \cdots) \in B) \tag{2.5.1}$$

holds for any $B \in \mathscr{X}$.

We need the following theorem to derive Theorem 2.1.4. Let $T : \mathcal{X} \to \mathcal{X}$ be a shift operator of $\mathcal{X}$, that is,

$$(T\omega)_n = \omega_{n+1}.$$

Theorem 2.5.1. *Under Assumption 2.1.3, T is ergodic with respect to μ.*

The proof is similar to [16, Proposition 3.5], so we omit it.

2.5.2 The Zero-One law

The purpose of this subsection is to give the proof of Theorem 2.1.4. We need the following version of the 0-1 law. Let $a \geq 0$ and $A_1^\omega(a), A_2^\omega(a)$ be the events

$$A_1^\omega(a) = \left\{ \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} > a \right\},$$
$$A_2^\omega(a) = \left\{ \liminf_{n\to\infty} \frac{\max_{0\le k\le n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{-1/\beta}} > a \right\}.$$

Define

$$\tilde{A}_i(a) = \{\omega \in \Omega \mid \ A_i^\omega(a) \text{ holds for } P_x^\omega\text{-a.s. and for all } x \in \mathcal{C}_0(\omega)\}.$$

Proposition 2.5.2. $\mathbb{P}_0(\tilde{A}_i(a))$ *is either* 0 *or* 1.

Proof. We follow the proof of [27, Corollary 3.2]. Let $F_i : \Omega \to [0,1]$ be $F_i(\omega) = P_0^\omega(A_i^\omega(a))$. By the Markov property of $\{\omega_n = \tau_{X_n^\omega}(\omega)\}_n$ we have

$$P_0^\omega(A_i^\omega(a) \mid \mathcal{F}_n^\omega) = F_i(\omega_n),$$

where $\mathcal{F}_n^\omega = \sigma(X_k^\omega \mid k \le n)$. So $\{F_i(\omega_n)\}_n$ is $\mathcal{F}_n^\omega$-martingale. By the martingale convergence theorem we see

$$F_i(\omega_n) \to 1_{A_i^\omega(a)} \qquad P_0^\omega\text{-a.s.}$$

Therefore

$$\mathbb{E}_0\left[P_0^\omega\left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} F_i(\omega_n) = 1_{A_i^\omega(a)} \right) \right] = 1.$$

Next we define $\tilde{F}_i : \Omega^{\mathbb{Z}} \to [0,1]$ by $\tilde{F}_i(\bar{\omega}) = F_i(\bar{\omega}_0)$. Since T is ergodic w.r.t. μ, Birkhoff's ergodic theorem gives

$$\mu\left(\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\tilde{F}_i\circ T^n = \int \tilde{F}_i d\mu\right) = 1.$$

By (2.5.1) we see

$$1_{A_i^{\omega}(a)} = \int \tilde{F}_i d\mu.$$

So, either $A_i^{\omega}(a)$ holds almost surely or it does not hold almost surely. We thus complete the proof. $\square$

Theorem 2.1.2 and Proposition 2.5.2 immediately give Theorem 2.1.4.

Chapter 3

Rate functions for random walks on Random conductance models and related topics

We consider laws of the iterated logarithm and the rate function for sample paths of random walks on random conductance models under the assumption that the random walks enjoy long time sub-Gaussian heat kernel estimates.

3.1 Introduction

The random conductance model (RCM) is a pair of a graph and a family of non-negative random variables (random conductances) which are indexed by edges of the graph. The RCM includes various important examples such as the supercritical percolation cluster, whose random conductances are i.i.d. Bernoulli random variables. In the recent progress on the RCM, various asymptotic behaviors of random walks are obtained on a class of RCM such as invariance principle, functional CLT, local CLT and long time heat kernel estimates. Here is a partial list of examples of the RCM;

1. Uniform elliptic case [24],
2. The supercritical percolation cluster [4],
3. I.i.d. unbounded conductance bounded from below [12],
4. I.i.d. bounded conductance under some tail conditions near 0 [15],

5. The level sets of Gaussian free field and the random interlacements [63].

We refer to [16], [53], [65] for the invariance principle for random walks on the supercritical percolation cluster, [13] for the local limit theorem for random walks on the supercritical percolation cluster, [1] for the invariance principle on general i.i.d. RCMs, [2] for the Gaussian heat kernel upper bound on the possibly degenerate RCMs. We also refer to [17] and [49] for more details about the RCM.

In [50], we discussed the laws of the iterated logarithms (LILs) for discrete time random walks on a class of RCM under the assumption on long time heat kernel estimates. The aims of this paper are to establish the laws of the iterated logarithm and to describe the rate functions for the sample paths of continuous time random walks on the RCM.

The LILs describe the fluctuation of stochastic processes, which was originally obtained by Khinchin [42] for a random walk. We establish the LIL w.r.t. both $\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)$ and $d(Y_0^\omega, Y_t^\omega)$, and another LIL, which describes liminf behavior of $\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)$, where $\{Y_t^\omega\}_{t\ge0}$ is a continuous time random walk on the random environment ω.

The rate function describes the sample path ranges of stochastic processes. For d-dimensional Brownian motion $B=\{B_t\}_{t\ge0}$, the Kolmogorov test tells us that

$$\mathbb{P}\left(|B_t| \ge t^{1/2}h(t) \text{ for sufficiently large } t \right) = \begin{cases} 1 \\ 0, \end{cases}$$
$$\text{according as} \quad \int_1^\infty \frac{1}{t} h(t)^d e^{-\frac{h(t)^2}{2}} dt \begin{cases} < \infty \\ = \infty, \end{cases}$$

where $h(t)$ is a positive function such that $h(t) \nearrow \infty$ as $t\to\infty$. For $d\ge3$, the Dvoretzky and Erdős test tells us that

$$\mathbb{P}\left(|B_t| \ge t^{1/2}h(t) \text{ for sufficiently large } t \right) = \begin{cases} 1 \\ 0, \end{cases}$$
$$\text{according as} \quad \int_1^\infty \frac{1}{t} h(t)^{d-2} dt \begin{cases} < \infty \\ = \infty, \end{cases} \tag{3.1.1}$$

where $h(t)$ is a positive function such that $h(t) \searrow 0$ as $t\to\infty$. These results were extended to various frameworks such as symmetric stable processes on $\mathbb{R}^d$, Brownian motions on Riemannian manifolds, symmetric Markov chains on weighted graphs and β stable like processes ($\beta \geq 2$). We refer to [38], [43], [44], [69], [70] for stable processes on $\mathbb{R}^d$, [31], [35] for Brownian motions on Riemannian manifolds, [39], [40] for symmetric Markov chains on weighted graphs, [64] for β stable like processes. We establish an analogue of (3.1.1) w.r.t. random walks on the RCM.

Our approach is as follows; We assume quenched heat kernel estimates and establish both quenched LILs and an analogue of the Dvoretzky and Erdős test. As we will see in Section 3.1.2, our results are applicable for various models since heat kernel estimates are obtained for random walks on various RCMs. The concrete examples are given in Section 3.1.2.

The organization of this paper is as follows. First, we give the framework and main results of this paper in Section 3.1.1 and examples in Section 3.1.2. In Section 3.2 we establish some preliminary results. In Section 3.3 we give the proof of the LILs. In Section 3.4 we establish an analogue of (3.1.1). Finally in Section 3.5 we discuss the case where $G = \mathbb{Z}^d$ and the media is ergodic.

In this paper, we use the following notation.

Notation 3.1.1. *(1) We use $c, C, c_1, c_2, \cdots$ as the deterministic positive constants. These constants do not depend on the random environment ω, time parameters $t, s \cdots$, distance parameters $r, \cdots$, and vertices of graphs.*

(2) We define $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

3.1.1 Framework and Main results

Let $G = (V, E) = (V(G), E(G))$ be a countable and connected graph of bounded degree, i.e. $M := \sup_{x \in V(G)} \deg x < \infty$. We write $x \sim y$ if $(x, y) \in E(G)$. A sequence $\ell_{xy} : x = x_0, x_1, \cdots, x_n = y$ on G is called a path from x to y if $x_i \sim x_{i+1}$ for all $i = 0, 1, \cdots, n-1$. We write $d(\cdot, \cdot)$ as the usual graph distance, that is, the length of a shortest path in G, and denote $B(x, r) = \{y \in V(G) \mid d(x, y) \leq r\}$.

Throughout of this paper we assume that there exist $\alpha \geq 1$, $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \sharp B(x, r) \leq c_2 r^\alpha \tag{3.1.2}$$

for any $x \in V(G)$ and $r \geq 1$.

We introduce the random conductance model below. Let $\omega = \{\omega_e = \omega_{xy}\}_{e=(x,y)\in E(G)}$ be a family of non-negative weight which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We call ω the random conductance. For non-negative weights $\omega = \{\omega_e\}_e$, we define $\pi^\omega(x) = \sum_{y; y \sim x} \omega_{xy}$ and $\nu^\omega(x) = 1$. We fix a base point $x_0 \in V(G)$, and define graphs

$G^\omega = (V(G^\omega), E(G^\omega))$ as

$$V(G^\omega) = \left\{ y \in V(G) \,\middle|\, \begin{array}{l} \text{There exists a path } \ell_{x_0y} : x_0, x_1, \cdots, x_n = y \text{ such that} \\ \omega_{x_i x_{i+1}} > 0 \text{ for all } i = 0, 1, \cdots, n-1. \end{array} \right\},$$
$$E(G^\omega) = \{e = (x, y) \in E(G) \mid x, y \in V(G^\omega) \text{ and } \omega_{xy} > 0\}.$$

We denote $d^\omega(\cdot, \cdot)$ as the graph distance of G^ω. Note that $G^\omega = G$ and $d^\omega = d$ if conductance ω is strictly positive.

We will consider two types of random walks, constant speed random walk (CSRW) and variable speed random walk (VSRW) associated to $\omega \in \Omega$. Both CSRW and VSRW are continuous time random walks whose transition probability is given by $P^\omega(x, y) = \dfrac{\omega_{xy}}{\pi^\omega(x)}$. For the CSRW, the holding time distribution at $x \in V(G^\omega)$ is Exp (1), whereas for the VSRW, the holding time distribution at $x \in V(G^\omega)$ is Exp $(\pi^\omega(x))$. We write $\mathcal{L}^\omega_\theta$ for the generator which is given by

$$\mathcal{L}^\omega_\theta f(x) = \frac{1}{\theta^\omega(x)} \sum_{y; y\sim x} (f(y) - f(x))\omega_{xy},$$

and we also write the corresponding heat kernel as

$$q^\omega_t(x, y) = \frac{P^\omega(x, y)}{\theta^\omega(y)},$$

where $\theta^\omega = \pi^\omega$ for the CSRW case and $\theta^\omega \equiv 1$ for the VSRW case. We write $Y^\omega = \{Y^\omega_t\}_{t\geq 0}$ as either the CSRW or the VSRW, P^ω_x as the law of the random walk Y^ω which starts at x, and

$$\begin{aligned} \tau_F = \tau^\omega_F = \inf\{t \geq 0 \mid Y^\omega_t \notin F\}, \quad \sigma_F = \sigma^\omega_F = \inf\{t \geq 0 \mid Y^\omega_t \in F\}, \\ \sigma^+_F = \sigma^{+\omega}_F = \inf\{t > 0 \mid Y^\omega_t \in F\}. \end{aligned} \tag{3.1.3}$$

We denote $F^\omega = F \cap V(G^\omega)$, $V^\omega(F) = \sum_{y\in F\cap V^\omega(G)} \theta^\omega(y)$ for $F \subset V(G)$ and $V^\omega(x, r) = V^\omega(B(x, r))$. We write $T^\omega_0 = 0$ and $T^\omega_{n+1} = \inf\{t > T^\omega_n \mid Y^\omega_t \neq Y^\omega_{T^\omega_n}\}$, and introduce a discrete time random walk $\{X^\omega_n := Y^\omega_{T^\omega_n}\}_{n\geq 0}$.

First, we state the results about LILs. To do this, we need the following assumptions.

Assumption 3.1.2. *There exist positive constants ϵ, β such that $\epsilon < \beta + 1$ and a family of non-negative random variables $\{N_x = N_{x,\epsilon}\}_{x\in V(G)}$ such that the following hold;*

(1) There exist positive constants $c_{1.1}, c_{1.2}, c_{1.3}, c_{1.4}$ such that

$$q_t^\omega(x,y) \le \begin{cases} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left(-c_{1.2}\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right), & \text{if } t \ge d(x,y), \\ c_{1.3}\exp\left(-c_{1.4} d(x,y)\left(1 \vee \log\frac{d(x,y)}{t}\right)\right), & \text{if } t \le d(x,y), \end{cases} \tag{3.1.4}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$ and $t \ge N_x(\omega)$.

(2) There exist positive constants $c_{2.1}, c_{2.2}$ such that

$$q_t^\omega(x,y) \ge \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left(-c_{2.2}\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right) \tag{3.1.5}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$ and $t \ge 0$ with $d(x,y)^{1+\epsilon} \vee N_x(\omega) \le t$.

(3) There exist positive constants $c_{3.1}, c_{3.2}$ such that

$$c_{3.1} r^\alpha \le V^\omega(x,r) \le c_{3.2} r^\alpha \tag{3.1.6}$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$ and $r \ge N_x(\omega)$.

(4) There exist positive constants $c_{4.1}, c_{4.2}, c_{4.3}, c_{4.4}, c_{4.5}$ such that

$$q_t^\omega(x,y) \le \begin{cases} \frac{c_{4.1}}{\sqrt{\theta^\omega(x)\theta^\omega(y)}} \exp\left(-c_{4.2}\frac{d(x,y)^2}{t}\right), & \text{if } t \ge c_{4.3} d(x,y), \\ \frac{c_{4.4}}{\sqrt{\theta^\omega(x)\theta^\omega(y)}} \exp\left(-c_{4.5} d(x,y)\left(1 \vee \log\frac{d(x,y)}{t}\right)\right), & \text{if } t \le c_{4.3} d(x,y), \end{cases} \tag{3.1.7}$$

for almost all $\omega \in \Omega$, all $t > 0$ and $x, y \in V(G^\omega)$ with $d(x,y) \ge N_x(\omega) \wedge N_y(\omega)$.

Note that (3.1.4) holds for $t \ge N_x(\omega)$ while (3.1.7) holds for all $t > 0$. (3.1.7) is called the Carne-Varopoulos bound. This type of bound were originally obtained by [19], [75]. It is known that (3.1.7) holds under general conditions which will be described in the following Proposition (see [29, Theorems 2.1, 2.2]).

Proposition 3.1.3. *Let $\{N_x\}$ be as in Assumption 3.1.2 and $d_\theta^\omega(\cdot,\cdot)$ be a metric on $G^\omega = (V(G^\omega), E(G^\omega))$ which satisfies*

$$\frac{1}{\theta^\omega(x)} \sum_{y \in V(G^\omega)} d_\theta^\omega(x,y)^2 \omega_{xy} \le 1. \tag{3.1.8}$$

If there exists a positive constant c such that $d_\theta^\omega(x,y) \ge c d(x,y)$ for all $x, y \in V(G^\omega)$ with $d(x,y) \ge N_x(\omega) \wedge N_y(\omega)$, then (3.1.7) holds.

Next we assume the following three types of integrability conditions.

Assumption 3.1.4. *Let $\{N_x\}_{x\in V(G)}$ be as in Assumption 3.1.2 and define $f(t) = f_\epsilon(t) = \mathbb{P}(N_x \geq t)$. We impose one of the following three types of integrability conditions on $f(t)$.*

(1) $\displaystyle\sum_{n\geq 1} n^\alpha f(n) < \infty,$

(2) $\displaystyle\sum_{n\geq 1} n^{\alpha\beta} f(n) < \infty,$

(3) For positive and non-increasing function $h(t)$, $\displaystyle\sum_n n^\alpha f(nh(n^\beta)) < \infty.$

We now state the main results of this paper.

Theorem 3.1.5. *(1) Under Assumption 3.1.2 (1) (2) (3) and Assumption 3.1.4 (1), for almost all $\omega \in \Omega$ there exists positive numbers $c_1 = c_1^\omega, c_2 = c_2^\omega$ such that*

$$\begin{aligned}\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} &= c_1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega),\\ \limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} &= c_2, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).\end{aligned} \tag{3.1.9}$$

(2) Under Assumption 3.1.2 (1) (2) (3) and Assumption 3.1.4 (2), for almost all $\omega \in \Omega$ there exist a positive number $c_3 = c_3^\omega$ such that

$$\liminf_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{-1/\beta}} = c_3, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega). \tag{3.1.10}$$

Theorem 3.1.6. *Suppose Assumption 3.1.2 (1) (2) (3) (4) and $\alpha/\beta > 1$. In addition $\theta^\omega(x) = \pi^\omega(x) \geq c$ for a positive constant $c > 0$ in the case of CSRW. Let $h : (1,\infty) \to (0,\infty)$ be a function such that $h(t) \searrow 0$ as $t \to \infty$ and the function $\varphi(t) := t^{1/\beta}h(t)$ is increasing. If $h(t)$ satisfies Assumption 3.1.4 (3), then*

$$P_x^\omega\left(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t\right) = 1$$

for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$, or

$$P_x^\omega\left(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t\right) = 0$$

for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$, according as $\int_1^\infty \frac{1}{t}h(t)^{\alpha-\beta}dt < \infty$ or $= \infty$ respectively.

Note that the condition $\alpha/\beta > 1$ implies the transience of $\{Y_t^\omega\}_{t\geq 0}$.

Finally we discuss the constants c_1, c_2, c_3 in (3.1.9) and (3.1.10). When we consider a case of $G = \mathbb{Z}^d$, we can take c_1, c_2 as deterministic constants under some appropriate assumptions. To state this, we take the base point $x_0 = 0 \in \mathbb{Z}^d$ and we write shift operators as $\tau_x, (x \in \mathbb{Z}^d)$, where τ_x is given by

$$(\tau_x \omega)_{yz} = \omega_{x+y,x+z}. \tag{3.1.11}$$

We assume the following conditions.

Assumption 3.1.7. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfies the following conditions;*

(1) $\mathbb{P}$ *is ergodic with respect to the translation operators* τ_x*, namely* $\mathbb{P} \circ \tau_x = \mathbb{P}$ *and if* $\tau_x(A) = A$ *for all* $x \in \mathbb{Z}^d$ *and for all* $A \in \mathcal{F}$ *then* $\mathbb{P}(A) = 0$ *or* 1.

(2) For almost all environment ω, $V(G^\omega)$ *contains a unique infinite connected component.*

(3) (VSRW case) $\mathbb{E}\left[\dfrac{1}{\pi^\omega(0)}\right] \in (0, \infty)$.

Theorem 3.1.8. *Suppose that the same assumptions as in Theorem 3.1.5 are fulfilled and suppose in addition Assumption 3.1.7. Then we can take* c_1, c_2, c_3 *in* (3.1.9) *and* (3.1.10) *as deterministic constants (i.e. do not depend on* ω*).*

3.1.2 Example

In this subsection, we give some examples for which our results are applicable.

Example 3.1.9 (Bernoulli supercritical percolation cluster)**.** *Let* $G = (\mathbb{Z}^d, E_d)$ *be a graph, where* $E_d = \{\{x, y\} \mid x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$. *Put a Bernoulli random variable* ω_e *with* $\mathbb{P}(\eta_e = 1) = p$ *on each edge. This model is called bond percolation. We write* $p_c(d)$ *as the critical probability. It is known that there exists a unique infinite connected component when* $p > p_c(d)$. *See [34] for more details about the percolation.*

Barlow [4] proved that heat kernels of CSRWs on the super-critical percolation cluster (that is, when $p > p_c(d)$*) on* $\mathbb{Z}^d$, $d \geq 2$ *satisfy Assumption 3.1.2 (1) (2) (3) (4), Assumption 3.1.4 (1) (2) with* $\alpha = d$, $\beta = 2$ *and* $f_\epsilon(t) = c\exp(-c' t^\delta)$ *for some* $c, c', \delta > 0$. *Since the media is i.i.d. and there exists an unique infinite connected component, we can obtain Theorem 3.1.5 with deterministic constants by Theorem 3.1.8.*

In addition, we can easily check that $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ *for* $\kappa > 0$ *satisfy the conditions in Assumption 3.1.4 (3) and the assumptions of Theorem 3.1.6 in the case of* $d > 2$. *Thus* $P_x^\omega\left(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t\right) = 1, 0$ *according as* $\kappa > d-2, \leq d-2$ *respectively by Theorem 3.1.6.*

Note that (3.1.9) *for the supercritical percolation cluster was already obtained by [27, Theorem 1.1].*

Example 3.1.10 (Gaussian free fields and random interlacements)**.** *Gaussian free field on a graph* $G = (V, E)$ *is a family of centered Gaussian variables* $\{\varphi_x\}_{x\in G}$ *with covariance* $E[\varphi_x\varphi_y] = g(x, y)$, *where* $g(x, y)$ *is the Green function of a random walk on* G. *Here we are interested in the level sets of the Gaussian free field* $E_h = \{x \in V \mid \varphi_x \geq h\}$. *We can regard the level sets as one of the percolation models which has correlation among the vertices in* V. *See [67] for the details.*

The random interlacements concern geometries of random walk trajectories, e.g. how many random walk trajectories are needed to make the underlying graph disconnected? Sznitman [66] formulated the model of random interlacements. Although the model of random interlacements is defined through Poisson point process on a trajectory space, we can also regard this model as the percolation model with long range correlation. From the viewpoint of the RCM, we can regard the model of random interlacements as one of the RCM whose conductances take the value 0 *or* 1 *and the conductances are not independent. See [26] for the details.*

Sapozhnikov [63, Theorem 1.15] proved that for $\mathbb{Z}^d$, $d \geq 3$, *the CSRWs on (i) certain level sets of Gaussian free fields; (ii) random interlacements at level* $u > 0$*; (iii) vacant sets of random interlacements for suitable level sets, satisfy our Assumption 3.1.2 (1) (2) (3) with* $\alpha = d$, $\beta = 2$ *and the tail estimates of* $N_x(\omega)$ *as* $f_\epsilon(t) = c\exp(-c'(\log t)^{1+\delta})$ *for some* $c, c', \delta > 0$. *As the same reason with the case of Bernoulli supercritical percolation cluster, Assumption 3.1.2 (3) is also satisfied in these models. This subexponential tail estimate is sufficient for Assumption 3.1.4* (3) *with* $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ *for* $\kappa > 0$. *Since the media is ergodic and there is an unique infinite connected components (see [60], [66, Corollary 2.3] and [71, Theorem 1.1]), Theorem 3.1.5 holds with deterministic constants by Theorem 3.1.8, and Theorem 3.1.6 holds with* $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ *for* $\kappa \geq d-2$, $< d-2$ *respectively.*

Example 3.1.11 (Uniform elliptic case)**.** *Suppose that a graph* $G = (V, E)$ *is endowed with weight* 1 *on each edge and satisfies* (3.1.2) *and the scaled Poincaré inequalities. Take* c_1, c_2 *as positive constants and put random conductances on all edges so that* $c_1 \leq \omega(e) \leq c_2$ *for all* $e \in E$ *and for almost all* ω. *Delmotte [24] obtained*

Gaussian heat kernel estimates for CSRWs in this framework. Thus Assumption 3.1.2 (1) (2) (3) hold with $\beta = 2$ and $N_x \equiv 1$. Hence Theorem 3.1.5 holds.

In addition, this model satisfies Assumption 3.1.2 by [23, Corollary 11, 12]. (See also Proposition 3.1.3, note that the graph distance satisfies (3.1.8) *for CSRW case.) Thus Theorem 3.1.6 holds with $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ ($\kappa \geq d-2$, $< d-2$ respectively).*

Example 3.1.12 (Unbounded conductance bounded from below)**.** *Let $G = \mathbb{Z}^d$ ($d \geq 2$) and put random conductances $\omega = \{\omega_{xy}\}_{xy \in E}$ which take the value $[1, \infty)$. Barlow and Deuschel [12, Theorem 1.2] proved that the heat kernels of VSRW satisfy Assumption 3.1.2 (1) (2), Assumption 3.1.4 (1) (2) with $\alpha = d$, $\beta = 2$ and $f_\epsilon(t) = c_1 \exp(-c_2 t^\delta)$ for some $c_1, c_2, \delta > 0$. (Note that Assumption 3.1.2 (3) is trivial since $V^\omega(x,r) = \sharp B(x,r)$ for the VSRW.) Hence Theorem 3.1.5 holds.*

In addition, this model satisfies Assumption 3.1.2 (4) by either [12, Theorem 2.3, Theorem 4.3 (b)] or [29, Theorem 2.1, Theorem 2.2]. Thus Theorem 3.1.6 for the VSRW holds with $h(t) = \dfrac{1}{(\log t)^{\kappa/(d-2)}}$ ($\kappa \geq d-2$, $< d-2$ respectively).

Moreover, if the conductances $\{\omega_e\}_e$ satisfy Assumption 3.1.7 (3) then Theorem 3.1.5 holds with deterministic constants.

3.2 Consequences of Assumption 3.1.2

In this section we give some preliminary results of our assumptions.

3.2.1 Consequences of heat kernel estimates

In this subsection, we give preliminary results of Assumption 3.1.2 (1) (2) (3).

Recall the notations in (3.1.4).

Lemma 3.2.1. *Suppose Assumption 3.1.2 (1) (3). For all $\delta \in (0, c_{1.2} \wedge c_{1.4})$ there exist positive constants $c_1 = c_1(\delta), c_2 = c_2(\delta), c_3 = c_3(\delta)$ such that*

$$P_x^\omega\left(d(x, Y_t^\omega) \geq r\right) \leq c_1 \exp\left[-(c_{1.2} - \delta)\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right] + c_2 \exp\left(-c_3 t\right) \qquad (3.2.1)$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$, $r \geq N_x(\omega)$ and $t \geq N_x(\omega)$.

This lemma is standard except for the part of estimates of Poissonian regime (the bottom line of (3.1.4)). For the sake of completeness we give the proof here.

Proof. We first prepare some preliminary facts to estimate $P_x^\omega\left(d(x, Y_t^\omega) \geq r\right)$. Set $h_1(\eta, s) = \exp\left[-\eta s^{\beta/(\beta-1)}\right]$ and $h_2(\eta, s) = \exp\left[-\eta s\right]$. For $h_1(\eta, s)$, we can easily see that there exists a constant $\zeta_0 > 1$ such that

$$h_1(\eta, \zeta s) \leq h_1(\eta, 1) h_1(\eta, s) \tag{3.2.2}$$

for all $\zeta \geq \zeta_0$, $\eta > 0$ and $s \geq 1$. (We can take ζ_0 as the positive number which satisfies $\zeta_0^{\beta/(\beta-1)} - 1 = 1$.) For $h_2(\eta, s)$, we can easily see that

$$h_2(\eta, \zeta s) \leq h_2(\eta, 1) h_2(\eta, s) \tag{3.2.3}$$

for all $\zeta \geq 2$, $\eta > 0$ and $s \geq 1$. Next, we easily see that for all $\zeta > 1$ there exists $c_1 = c_1(\zeta)$ such that for almost all $\omega \in \Omega$

$$V^\omega(x, r\zeta) \leq c_1 V^\omega(x, r) \tag{3.2.4}$$

for all $x \in V(G)$ and for all $r \geq N_x(\omega)$. (Use (3.1.6) and take $c_1 = \frac{c_{3.2}\zeta^\alpha}{c_{3.1}}$.) Thirdly, it is also easy to see that for all $\delta \in (0, c_{1.2})$ there exists $c_2(\delta)$ such that

$$s^\alpha \exp\left[-c_{1.2} s^{\beta/(\beta-1)}\right] \leq c_2(\delta) \exp\left[-(c_{1.2} - \delta) s^{\beta/(\beta-1)}\right] \tag{3.2.5}$$

for all $s \geq 1$, where $c_{1.2}$ is the same constant as in (3.1.4). We can also see that for all $\delta \in (0, c_{1.4})$ there exists a positive constant $c_3 = c_3(\delta)$ such that

$$s^\alpha \exp\left[-c_{1.4} s\right] \leq c_3(\delta) \exp\left[-(c_{1.4} - \delta) s\right] \tag{3.2.6}$$

for all $s \geq 1$. Using (3.2.5), we can see that for $d(x, z) \geq s \geq t^{1/\beta}$ and $\delta \in (0, c_{1.2})$

$$\begin{aligned}
&\frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] = \frac{c_{1.1}}{d(x,z)^\alpha}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^\alpha \exp\left[-c_{1.2}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \\
&\leq \frac{c_4(\delta)}{d(x,z)^\alpha} \exp\left[-(c_{1.2} - \delta)\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \quad \text{(use (3.2.5))} \\
&\leq \frac{c_4(\delta)}{s^\alpha} \exp\left[-(c_{1.2} - \delta)\left(\frac{s}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right], \quad \text{(use } d(x,z) \geq s\text{)}\,.
\end{aligned} \tag{3.2.7}$$

Now we estimate $P_x^\omega(d(x, Y_t^\omega) \geq r)$. We first consider the case $r \leq t^{1/\beta}$. Since $s \mapsto h_1(\eta, s)$, $(\eta > 0)$ is non-increasing, we have

$$P_x^\omega(d(x, Y_t^\omega) \geq r) \leq 1 \leq \frac{h_1\left(c_{1.2}, \frac{r}{t^{1/\beta}}\right)}{h_1(c_{1.2}, 1)} = c_5 h_1\left(c_{1.2}, \frac{r}{t^{1/\beta}}\right), \tag{3.2.8}$$

where we set $c_5 = 1/h(c_{1.2}, 1)$. So we may and do assume $r \geq t^{1/\beta}$. Take $\zeta \geq \zeta_0 \vee 2$ so that (3.2.2), (3.2.3) and (3.2.4) hold. We divide $P_x^\omega(d(x, Y_t^\omega) \geq r)$ into

$$\begin{aligned}
&\left(\sum_{k=0}^{K-1} \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} + \sum_{z \in B^\omega(x, \lfloor t \rfloor) \setminus B^\omega(x, r\zeta^K)}\right) q_t^\omega(x,z)\theta^\omega(z), \\
&\left(\sum_{z \in B^\omega(x, r\zeta^{K+1}) \setminus B^\omega(x, \lfloor t \rfloor)} + \sum_{k=K+1}^{\infty} \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)}\right) q_t^\omega(x,z)\theta^\omega(z),
\end{aligned} \tag{3.2.9}$$

where K is the positive integer which satisfies $r\zeta^K \leq t < r\zeta^{K+1}$ and $\lfloor t \rfloor$ is the greatest integer which is less than or equal to t. We have for $t \geq N_x(\omega)$, $r \geq N_x(\omega)$ and using (3.1.4)

$$\begin{aligned}
&(\text{The first term of } (3.2.9)) \\
&\leq \sum_{k=0}^{K} \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2}\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \theta^\omega(z) \\
&\leq \sum_{k=0}^{K} \frac{c_6(\delta)}{(r\zeta^k)^\alpha} \exp\left[-(c_{1.2}-\delta)\left(\frac{r\zeta^k}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] (r\zeta^{k+1})^\alpha \qquad \text{(use (3.2.7) and (3.1.6))} \\
&\leq \sum_{k=0}^{K} c_7(\delta,\zeta) h_1\left(c_{1.2}-\delta, \frac{r\zeta^k}{t^{1/\beta}}\right) \\
&\leq c_7(\delta,\zeta) h_1\left(c_{1.2}-\delta, \frac{r}{t^{1/\beta}}\right) \sum_{k=0}^{K} h_1(c_{1.2}-\delta, 1)^k \qquad \text{(use (3.2.2))} \\
&\leq c_8(\delta,\zeta) \exp\left[-(c_{1.2}-\delta)\left(\frac{r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right], \qquad \text{(since } h_1(c_{1.2}-\delta, 1) < 1). \qquad (3.2.10)
\end{aligned}$$

For the second term of (3.2.9), using (3.1.4), $t \geq N_x(\omega)$ and $r \geq N_x(\omega)$ we have

$$
\begin{aligned}
&(\text{The second term of (3.2.9)}) \\
&\leq \sum_{k=K}^{\infty} \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} c_{1.3} \exp\left[-c_{1.4} d(x,z) \left(1 \vee \log \frac{d(x,z)}{t}\right)\right] \theta^\omega(z) \\
&\leq \sum_{k=K}^{\infty} \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} c_{1.3} \exp\left[-c_{1.4} d(x,z)\right] \theta^\omega(z) \quad \left(\text{since } 1 \vee \log \frac{d(x,z)}{t} \geq 1\right) \\
&\leq \sum_{k=K}^{\infty} c_9 \exp\left[-c_{1.4}(r\zeta^k)\right] (r\zeta^{k+1})^\alpha \qquad (\text{use (3.1.6)}) \\
&\leq c_{10}(\zeta, \delta) \sum_{k=K}^{\infty} \exp\left[-(c_{1.4} - \delta) r\zeta^k\right] \qquad (\text{use (3.2.6)}) \\
&= c_{10}(\zeta, \delta) \sum_{k=K}^{\infty} h_2\left(c_{1.4} - \delta, r\zeta^k\right) \\
&\leq c_{11}(\zeta, \delta) h_2(c_{1.4} - \delta, r\zeta^K) \sum_{k=0}^{\infty} h_2(c_{1.4} - \delta, 1)^k \qquad (\text{use (3.2.3)}) \\
&\leq c_{12}(\zeta, \delta) \exp\left[-c_{13}(\zeta, \delta) t\right], \qquad \left(\text{since } r\zeta^K \leq t < r\zeta^{K+1}\right). \qquad (3.2.11)
\end{aligned}
$$

Therefore, by (3.2.8), (3.2.10), (3.2.11) and adjusting the constants, we obtain (3.2.1). We thus complete the proof. □

Again recall the notations $c_{1.2}$ and $c_{1.4}$ in (3.1.4).

Lemma 3.2.2. *Suppose Assumption 3.1.2 (1) (3). For all $\delta \in (0, c_{1.2} \wedge c_{1.4})$ there exist positive constants $c_1 = c_1(\delta), c_2 = c_2(\delta), c_3 = c_3(\delta)$ such that*

$$
P_x^\omega\left(\sup_{0 \leq s \leq t} d(x, Y_s^\omega) \geq 2r\right) \leq c_1 \exp\left[-(c_{1.2} - \delta)\left(\frac{r}{(2t)^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2 \exp\left[-c_3 t\right] \qquad (3.2.12)
$$

$$
P_x^\omega\left(\sup_{0 \leq s \leq t} d(y, Y_s^\omega) \geq 4r\right) \leq c_1 \exp\left[-(c_{1.2} - \delta)\left(\frac{r}{(2t)^{1/\beta}}\right)^{\beta/(\beta-1)}\right] + c_2 \exp\left[-c_3 t\right] \qquad (3.2.13)
$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$, $t \geq 1$ and $r \geq 1$ with $d(x,y) \leq 2r$, $t \geq \max_{u \in B(x,2r)} N_u(\omega)$ and $r \geq \max_{u \in B(x,2r)} N_u(\omega)$.

Proof. This is standard (see the proof of [3, Lemma 3.9 (c)]), so we omit the proof. □

Lemma 3.2.3. *Suppose Assumption 3.1.2 (1) (2) (3). Then there exist positive constants* $\eta \geq 1, c_1, c_2 > 0$ *such that*

$$P_x^\omega\left(\sup_{0\leq s\leq t} d(x, Y_s^\omega) \leq 3\eta r\right) \geq c_1 \exp\left[-c_2\frac{t}{r^\beta}\right] \tag{3.2.14}$$

for almost all $\omega \in \Omega$, *all* $x \in V(G^\omega)$, $t \geq r \geq 1$ *with* $r^{1/\beta} \geq \max_{z\in B(y,3\eta r)} N_z(\omega)$.

Proof. The proof is quite similar to that of [51, Proposition 3.3], so we omit the proof. □

Let c_1, c_2 be as in Lemma 3.2.3. Note that we can assume that $c_1 < 1$ (and therefore $c_1 \exp[-c_2] \in (0,1)$). We define $\rho_1, a_k, b_k, \lambda_k, u_k, \sigma_k$ as

$$\begin{aligned} &\rho_1 = c_1\exp[-c_2], \quad a_k^\beta = e^{k^2}, \quad b_k^\beta = e^k, \\ &\lambda_k = \frac{2}{3|\log\rho_1|}\log(1+k), \quad u_k = \lambda_k a_k^\beta, \quad \sigma_k = \sum_{i=1}^{k-1} u_i. \end{aligned} \tag{3.2.15}$$

Corollary 3.2.4 (Corollary of Lemma 3.2.3)**.** *Let* $\eta \geq 1$ *be as in Lemma 3.2.3. Then under Assumption 3.1.2 (1) (2) (3) we have*

$$\inf_{z\in B(x,a_k)} P_z^\omega\left(\sup_{0\leq s\leq u_k} d(z, Y_s^\omega) \leq 3\eta a_k\right) \geq \rho_1^{\lambda_k} \tag{3.2.16}$$

for almost all $\omega \in \Omega$, *all* k *with* $\max_{z\in B(x,4\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$.

Proof. We can see from Lemma 3.2.3 that

$$P_z^\omega\left(\sup_{0\leq s\leq u_k} d(z, Y_s^\omega) \leq 3\eta a_k\right) \geq c_1\exp\left[-c_2\frac{u_k}{a_k^\beta}\right] \geq \rho_1^{\lambda_k}$$

for all $k \geq 1$ with $\max_{v\in B(z,3\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$. Hence (3.2.16) holds for k with $\max_{z\in B(x,a_k)} \max_{v\in B(z,3\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$.

□

Lemma 3.2.5. *Suppose Assumption 3.1.2 (1) (3). Then there exist positive constants c_1, c_2 such that*

$$P_x^\omega\left(\sup_{0\le s\le t} d(x, Y_s^\omega) \le r\right) \le c_1 \exp\left(-c_2 \frac{t}{r^\beta}\right)$$

for almost all environment $\omega \in \Omega$, all $x \in V(G^\omega)$, $t \ge 1$ and $r \ge 1$ with $\max_{y\in B(x,r)} N_y(\omega) \le 2r$.

Proof. The proof is quite similar to that of [51, Lemma 3.2], so we omit it. □

We will need the following version of 0-1 law.

Theorem 3.2.6 (0 − 1 law for tail events)**.** *For almost all environment $\omega \in \Omega$, the following holds; Let A^ω be a tail event, i.e. $A^\omega \in \bigcap_{t=0}^{\infty} \sigma\{Y_s^\omega : s \ge t\}$. Then either $P_x^\omega(A^\omega) = 0$ for all x or $P_x^\omega(A^\omega) = 1$ for all x.*

The proof of the above theorem is quite similar to that of [14, Proposition 2.3] (see also [4, Theorem 4]), so we omit the proof here.

3.2.2 Green function

In this subsection, we deduce the Green function estimates. We define the Green function as

$$g^\omega(x, y) = \int_0^\infty q_t^\omega(x, y)dt. \tag{3.2.17}$$

Recall that $\theta^\omega(x) = \pi^\omega(x)$ in the case of CSRW and $\theta^\omega(x) = 1$ in the case of VSRW.

Proposition 3.2.7. *Let $\alpha > \beta$ and suppose Assumption 3.1.2 (1) (2) (4). In addition we assume there exists a positive constant $c > 0$ such that $\theta^\omega(x) \ge c$ for all $x \in V(G^\omega)$ in the case of CSRW. Then there exist positive constants c_1, c_2 such that*

$$\frac{c_1}{d(x, y)^{\alpha-\beta}} \le g^\omega(x, y) \le \frac{c_2}{d(x, y)^{\alpha-\beta}} \tag{3.2.18}$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$ with $d(x, y) \ge N_x(\omega) \wedge N_y(\omega)$.

Proof. This proof is similar to [13, Proposition 6.2]. We first prove the upper bound of (3.2.18).

$$
\begin{aligned}
&g^{\omega}(x,y)\\
&= \int_0^{(c_{4.3}d(x,y))\wedge N_x(\omega)} q_t^{\omega}(x,y)dt + \int_{(c_{4.3}d(x,y))\wedge N_x(\omega)}^{N_x(\omega)} q_t^{\omega}(x,y)dt + \int_{N_x(\omega)}^{d(x,y)} q_t^{\omega}(x,y)dt\\
&\qquad + \int_{d(x,y)}^{\infty} q_t^{\omega}(x,y)dt\\
&:= J_1 + J_2 + J_3 + J_4. \qquad (3.2.19)
\end{aligned}
$$

We estimate J_1, J_2, J_3, J_4 as follows.

$$
\begin{aligned}
J_1 &\le \int_0^{(c_{4.3}d(x,y))\wedge N_x(\omega)} \frac{c_{4.4}}{\sqrt{\theta^{\omega}(x)\theta^{\omega}(y)}} \exp\left[-c_{4.5}d(x,y)\right] dt \qquad \text{(use (3.1.7))}\\
&\le c_1 d(x,y) \exp\left[-c_2 d(x,y)\right],\\
J_2 &\le \int_{(c_{4.3}d(x,y))\wedge N_x(\omega)}^{N_x(\omega)} \frac{c_{4.1}}{\sqrt{\theta^{\omega}(x)\theta^{\omega}(y)}} \exp\left[-c_{4.2}\frac{d(x,y)^2}{t}\right] dt \qquad \text{(use (3.1.7))}\\
&\le c_3 N_x(\omega) \exp\left[-c_4 \frac{d(x,y)^2}{N_x(\omega)}\right] \le c_3 d(x,y) \exp\left[-c_4 d(x,y)\right] \quad (\text{ use } d(x,y) \ge N_x(\omega)),\\
J_3 &\le \int_{N_x(\omega)}^{d(x,y)} c_{1.3} \exp\left[-c_{1.4}d(x,y)\right] dt \qquad \text{(use (3.1.4))}\\
&\le c_{1.3} d(x,y) \exp\left[-c_{1.4}d(x,y)\right],\\
J_4 &\le \int_{d(x,y)}^{\infty} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] dt \le \frac{c_5}{d(x,y)^{\alpha-\beta}}.
\end{aligned}
\qquad (3.2.20)
$$

By (3.2.19) and (3.2.20) we have $g^{\omega}(x,y) \le \frac{c_6}{d(x,y)^{\alpha-\beta}}$ for $d(x,y) \ge N_x(\omega)$. Note that $g^{\omega}(x,y) = g^{\omega}(y,x)$. Thus we complete the upper bound of (3.2.18).

Next we prove the lower bound of (3.2.18). We can obtain the lower bound in the following way.

$$
\begin{aligned}
g^{\omega}(x,y) &\ge \int_{d(x,y)^{\beta}}^{\infty} q_t^{\omega}(x,y)dt \ge \int_{d(x,y)^{\beta}}^{\infty} \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left[-c_{2.2}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] dt\\
&\ge \frac{c_7}{d(x,y)^{\alpha-\beta}}.
\end{aligned}
$$

We thus complete the proof. □

3.2.3 Consequences of the Green function and Assumption 3.1.2

In this subsection we give some preliminary results of Assumption 3.1.2 (1) (2) (3) (4) in the case of $\alpha > \beta$. This subsection is based on [64, Section 4.1]. In this subsection we assume the following conditions.

Assumption 3.2.8. *(1) $\alpha > \beta$,*

(2) (CSRW case) There exists a positive constant c such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$.

Recall that Proposition 3.2.7 holds under Assumption 3.1.2 (1) (2) (4) and Assumption 3.2.8.

We write $e_F^\omega(x) = P_x^\omega\left(\sigma_F^{+\omega} = \infty\right) 1_F(x)$ as the equilibrium measure of $F \subset V(G^\omega)$, and define $\mathrm{Cap}^\omega(F) = \sum_{x\in F} e_F^\omega(x)\theta^\omega(x)$ as the capacity of $F \subset V(G^\omega)$. Then we have

$$P_x^\omega\left(\sigma_F^{+\omega} < \infty\right) = \sum_{y\in F} g^\omega(x,y)e_F^\omega(y)\theta^\omega(y) \tag{3.2.21}$$

for any finite set F and for any $x \in V(G^\omega)$ since

$$\begin{aligned}
&P_x^\omega\left(\sigma_F^{+\omega} < \infty\right)\\
&= \int_0^\infty \sum_{y\in F} P_x^\omega\left(Y_t^\omega = y, Y_s^\omega \notin F \text{ for any } s > t\right) dt \quad \text{(last exit decomposition)}\\
&= \int_0^\infty \sum_{y\in F} q_t^\omega(x,y)\theta^\omega(y)P_y^\omega\left(\sigma_F^{+\omega} = \infty\right) dt \quad \text{(by the Markov property)}\\
&= \sum_{y\in F} g^\omega(x,y)e_F^\omega(y)\theta^\omega(y).
\end{aligned}$$

Lemma 3.2.9. *Under Assumption 3.1.2 (1) (2) (3) (4) and Assumption 3.2.8, there exists a positive constant c such that*

$$\mathrm{Cap}^\omega(B^\omega(x,2r)) \geq cr^{\alpha-\beta}$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$ and $r \geq 1$ with $r \geq \max_{v\in B(x,r)} N_v(\omega)$.

Proof. Recall the notations in (3.1.3).

$$
\begin{aligned}
1 &= \frac{1}{\theta^\omega(B(x,r))} \sum_{y\in B^\omega(x,r)} P^\omega_y \left(\sigma^{+\omega}_{B(x,2r)} < \infty\right) \theta^\omega(y) \\
&= \frac{1}{\theta^\omega(B(x,r))} \sum_{y\in B^\omega(x,r)} \sum_{\substack{z\in B^\omega(x,2r)\\ d(x,z)=2r}} g^\omega(y,z) e^\omega_{B^\omega(x,2r)}(z)\theta^\omega(z)\theta^\omega(y) \qquad \text{(we use (3.2.21))} \\
&\le \frac{c_1}{\theta^\omega(B(x,r))} \frac{1}{r^{\alpha-\beta}} \sum_{\substack{z\in B^\omega(x,2r)\\ d(x,z)=2r}} \sum_{y\in B^\omega(x,r)} e^\omega_{B^\omega(x,2r)}(z)\theta^\omega(z)\theta^\omega(y) \\
&\quad (\text{ since } d(y,z) \ge r \ge N_y(\omega) \text{ and Proposition 3.2.7 }) \\
&= \frac{c_1}{\theta^\omega(B(x,r))} \frac{\theta^\omega(B(x,r))}{r^{\alpha-\beta}} \sum_{\substack{z\in B^\omega(x,2r)\\ d(x,z)=2r}} e^\omega_{B^\omega(x,2r)}(z)\theta^\omega(z) \\
&= \frac{c_1}{r^{\alpha-\beta}} \operatorname{Cap}^\omega(B^\omega(x,2r)).
\end{aligned}
$$

We thus complete the proof. □

Recall the notations in (3.1.3) and set

$$
\begin{aligned}
\gamma^\omega_{x,F}(K_1) &= P^\omega_x \left(Y^\omega_{\sigma^+_F} \in K_1\right), \\
\pi^\omega_{x,F}(dt, K_2) &= P^\omega_x \left(Y^\omega_{\sigma^+_F} \in K_2, \sigma^+_F \in dt\right)
\end{aligned}
$$

for $F, K_1, K_2 \subset V(G^\omega)$. Note that $\int_0^\infty \pi^\omega_{x,F}(dt,K) = \gamma^\omega_{x,F}(K)$ and $\gamma^\omega_{x,F}(F) = P^\omega_x \left(\sigma^{\omega+}_F < \infty\right)$.

Lemma 3.2.10. *For almost all $\omega \in \Omega$,*

$$
g^\omega(x,y) = \sum_{v\in F^\omega} g^\omega(v,y)\gamma^\omega_{x,F^\omega}(v) \tag{3.2.22}
$$

for any finite set $F^\omega \subset V(G^\omega)$, $x \notin F^\omega$ and $y \in F^\omega$. In particular we have

$$
P^\omega_x \left(Y^\omega_t \in F^\omega \text{ for some } t > 0\right) \le \inf_{y\in F^\omega} \left(\frac{g^\omega(x,y)}{\inf_{z\in F^\omega} g^\omega(z,y)}\right). \tag{3.2.23}
$$

Proof. We write $F = F^\omega$ and $\sigma = \sigma^{\omega+}_{F^\omega} = \inf\{t > 0 \mid Y^\omega_t \in F\}$ for notational simplification. Then for any $x \notin F$, $y \in F$ we have

$$P^\omega_x\left(Y^\omega_t = y\right) = E^\omega_x\left[1_{\{\sigma\le t\}} P^\omega_{Y^\omega_\sigma}\left(Y^\omega_{t-\sigma} = y\right)\right] = \sum_{v\in F} E^\omega_x\left[1_{\{\sigma\le t\}} 1_{\{Y^\omega_\sigma = v\}} P^\omega_{Y_\sigma}\left(Y^\omega_{t-\sigma} = y\right)\right]$$

$$= \sum_{v\in F}\int_0^t P^\omega_v\left[Y^\omega_{t-s} = y\right]\pi^\omega_{x,F}(ds, v).$$

Hence we have

$$g^\omega(x,y) = \int_0^\infty \sum_{v\in F}\int_0^t q^\omega_{t-s}(v,y)\pi^\omega_{x,F}(ds,v)dt = \int_0^\infty \sum_{v\in F}\int_s^\infty q^\omega_{t-s}(v,y)dt\pi^\omega_{x,F}(ds,v)$$

$$= \int_0^\infty \sum_{v\in F} g^\omega(v,y)\pi^\omega_{x,F}(ds,v) = \sum_{v\in F} g^\omega(v,y)\gamma^\omega_{x,F}(v).$$

We thus complete the proof of (3.2.22). (3.2.23) is immediate from (3.2.22). □

Lemma 3.2.11. *Under Assumption 3.1.2 (1) (2) (3) (4) and Assumption 3.2.8 there exist positive constants c_1, c_2 such that for almost all $\omega \in \Omega$ the following hold.*

(1) $P^\omega_x\left(\sigma^{+\omega}_{B(x_0,2r)} < \infty\right) \le c_1 \dfrac{r^{\alpha-\beta}}{(d(x,x_0)-r)^{\alpha-\beta}}$ *for all* $x, x_0 \in V(G^\omega)$, $r \ge 1$ *with* $d(x,x_0) \ge 2r+1$ *and* $r \ge \max_{v\in B(x_0,r)} N_v(\omega)$.

(2) $P^\omega_x\left(\sigma^{+\omega}_{B(x_0,2r)} < \infty\right) \ge c_2 \dfrac{r^{\alpha-\beta}}{(d(x,x_0)+2r)^{\alpha-\beta}}$ *for all* $x, x_0 \in V(G^\omega)$, $r \ge 1$ *with* $d(x,x_0) \ge 2r$, $r \ge N_x(\omega)$ *and* $r \ge \max_{v\in B(x_0,r)} N_v(\omega)$.

Proof. We first prove (1) by using (3.2.23). Let $x, x_0 \in V(G^\omega)$ satisfy $d(x,x_0) \ge 2r+1$. For any $y \in B(x_0,r)$ we have

$$d(x,y) \ge d(x,x_0) - d(x_0,y) \ge d(x,x_0) - r \ge 2r - r = r.$$

By Proposition 3.2.7, for any $y \in B^\omega(x_0,r)$ and for any r with $r \ge \max_{y\in B(x_0,r)} N_y(\omega)$ we have

$$g^\omega(x,y) \le \frac{c_1}{d(x,y)^{\alpha-\beta}} \le \frac{c_1}{(d(x,x_0)-r)^{\alpha-\beta}}. \tag{3.2.24}$$

Next note that $B(x_0, 2r) \subset B(y, 3r)$ for any $y \in B(x_0, r)$. Since $g^\omega(\cdot, y)$ is a superharmonic function, using the minimum principle and Proposition 3.2.7 we have

$$\inf_{z \in B^\omega(x_0,2r)} g^\omega(z,y) \geq \inf_{z \in B^\omega(y,3r)} g^\omega(z,y) \geq \inf_{\substack{z \in B^\omega(y,3r+1) \\ d(y,z)=3r+1}} g^\omega(z,y) \geq \frac{c_2}{r^{\alpha-\beta}} \tag{3.2.25}$$

for all $r \geq 1$ and $y \in B^\omega(x_0, r)$ with $3r+1 \geq \max_{v \in B(x_0,r)} N_v(\omega)$. Hence by (3.2.23), (3.2.24) and (3.2.25) we have

$$P_x^\omega\left(\sigma^+_{B(x_0,2r)} < \infty\right) \leq \inf_{y \in B^\omega(x_0,r)} \left(\frac{g^\omega(x,y)}{\inf_{z \in B^\omega(x_0,2r)} g(z,y)}\right) \leq c_3 \frac{r^{\alpha-\beta}}{(d(x,x_0)-r)^{\alpha-\beta}}$$

for all r with $r \geq \max_{v \in B(x_0,r)} N_v(\omega)$. Thus we complete the proof of (1).

Next we prove (2). Note that

$$\begin{aligned}
P_x^\omega\left(\sigma^{+\omega}_{B(x_0,2r)} < \infty\right) &= \sum_{y \in B^\omega(x_0,2r)} g^\omega(x,y) e^\omega_{B^\omega(x_0,2r)}(y)\theta^\omega(y) \qquad \text{(use (3.2.21))} \\
&\geq \left(\inf_{y \in B^\omega(x_0,2r)} g^\omega(x,y)\right) \sum_{y \in B^\omega(x_0,2r)} e^\omega_{B(x_0,2r)}(y)\theta^\omega(y) \\
&= \left(\inf_{y \in B^\omega(x_0,2r)} g^\omega(x,y)\right) \mathrm{Cap}^\omega(B(x_0,2r)).
\end{aligned}$$

By $B(x_0, 2r) \subset B(x, d(x, x_0) + 2r)$, the minimum principle for superharmonic functions and our assumptions we have

$$\begin{aligned}
&\inf_{y \in B^\omega(x_0,2r)} g^\omega(x,y) \geq \inf_{y \in B^\omega(x,d(x,x_0)+2r)} g^\omega(x,y) \geq \inf_{\substack{y \in B^\omega(x,d(x,x_0)+2r+1) \\ d(y,x)=d(x,x_0)+2r+1}} g^\omega(x,y) \\
&\geq \frac{c_4}{(d(x,x_0)+2r)^{\alpha-\beta}}
\end{aligned}$$

for $r \geq N_x(\omega)$. By Lemma 3.2.9 $\mathrm{Cap}^\omega(B(x_0,r)) \geq c_5 r^{\alpha-\beta}$ for $r \geq \max_{v \in B(x_0,r)} N_v(\omega)$. Hence

$$P_x^\omega\left(\sigma^{+\omega}_{B(x_0,2r)} < \infty\right) \geq \frac{c_6 r^{\alpha-\beta}}{(d(x,x_0)+2r)^{\alpha-\beta}}$$

for $r \geq N_x(\omega)$ and $r \geq \max_{v \in B(x_0,r)} N_v(\omega)$. We thus complete the proof.

□

Lemma 3.2.12. *Under Assumption 3.1.2 (1) (2) (3) (4) and Assumption 3.2.8 there exist positive constants c_1 and T_0 such that*

$$P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t \right) \leq \frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}}$$

for almost all $\omega \in \Omega$, all $t \geq T_0$, $r \geq 1$ and $x, x_0 \in V(G^\omega)$ with $t^{1/\beta} \geq r$, $d(x, x_0) \leq r$ and $r \geq \max_{z \in B(x_0, r)} N_z(\omega)$.

Proof. First note that

$$\begin{aligned}
&P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t \right) \\
&= \sum_{y \in V(G^\omega)} P_x^\omega \left(Y_t^\omega = y \right) P_y^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0 \right) \\
&= \sum_{y; t^{1/\beta} < d(x_0, y) - r} P_x^\omega \left(Y_t^\omega = y \right) P_y^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0 \right) \\
&\quad + \sum_{y; r < d(x_0, y) - r \leq t^{1/\beta}} P_x^\omega \left(Y_t^\omega = y \right) P_y^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0 \right) \\
&\quad + \sum_{y; d(x_0, y) \leq 2r} P_x^\omega \left(Y_t^\omega = y \right) P_y^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0 \right) \\
&:= J_1 + J_2 + J_3.
\end{aligned}$$

We estimate J_1, J_2 and J_3 in the following way.

For $t, r \geq 1$ with $t \geq N_x(\omega)$ and $r \geq \max_{z \in B(x_0, r)} N_z$ (note that $t \geq N_x(\omega)$ follows from our assumptions), using (3.1.4), Lemma 3.2.11, (3.1.6) we have

$$
\begin{aligned}
J_1 \leq & \sum_{y; t^{1/\beta} < d(x_0,y)-r} \frac{c_1 r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \\
& \cdot \left\{ \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2} \left(\frac{d(x,y)}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right] + c_{1.3} \exp\left[-c_{1.4} d(x,y) \right] \right\} \theta^{\omega}(y) \\
& \text{(use (3.1.4) and Lemma 3.2.11)} \\
\leq & \sum_{\ell=1}^{\infty} \sum_{y; d(x_0,y) \in [\ell t^{1/\beta}+r, (\ell+1)t^{1/\beta}+r]} \\
& \frac{c_2 r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \frac{1}{t^{\alpha/\beta}} \exp\left[-c_{1.2} \left(\frac{d(y,x_0)-r}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right] \theta^{\omega}(y) \\
& + \sum_{\ell=1}^{\infty} \sum_{y; d(x_0,y) \in [\ell t^{1/\beta}+r, (\ell+1)t^{1/\beta}+r]} \frac{c_3 r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \exp\left[-c_{1.4} (d(y,x_0)-r) \right] \theta^{\omega}(y) \\
& \text{(since } d(x,y) \geq d(y,x_0) - d(x_0,x) \text{ and } d(x_0,x) \leq r) \\
\leq & \sum_{\ell=1}^{\infty} \frac{c_2 r^{\alpha-\beta}}{(\ell t^{1/\beta})^{\alpha-\beta}} \frac{1}{t^{\alpha/\beta}} \exp\left[-c_{1.2} \ell^{\frac{\beta}{\beta-1}} \right] \theta^{\omega}\left(B(x_0, (\ell+1)t^{1/\beta}+r) \right) \\
& + \sum_{\ell=1}^{\infty} \frac{c_3 r^{\alpha-\beta}}{(\ell t^{1/\beta})^{\alpha-\beta}} \exp\left[-c_{1.4} \ell t^{1/\beta} \right] \theta^{\omega}\left(B(x_0, (\ell+1)t^{1/\beta}+r) \right) \\
\leq & \frac{c_4 r^{\alpha-\beta}}{t^{\alpha/\beta-1}} \sum_{\ell=1}^{\infty} \ell^{\beta} \exp\left[-c_{1.2} \ell^{\beta/\beta-1} \right] + \frac{c_5 r^{\alpha-\beta}}{t^{\alpha/\beta-1}} t^{\alpha/\beta} \sum_{\ell=1}^{\infty} \ell^{\beta} \exp\left[-c_{1.4} \ell t^{1/\beta} \right] \\
& \text{(use } \theta^{\omega}(B(x_0, (\ell+1)t^{1/\beta}+r)) \leq c(\ell t^{1/\beta})^{\alpha} \text{ since } t^{1/\beta} \geq r \text{)} \\
\leq & \frac{c_6 r^{\alpha-\beta}}{t^{\alpha/\beta-1}}, \qquad \text{(since } t \mapsto t^{\alpha/\beta} \sum_{\ell=1}^{\infty} \ell^{\beta} \exp\left[-c_{1.4} \ell t^{1/\beta} \right] \text{ is bounded).}
\end{aligned}
$$

Next we see J_2. First, set $\phi_r(k) = (r+k)^{\beta} (k, r \geq 1)$. We can easily see that

$$\phi_r(k) \leq \frac{1}{2} \phi_r(4rk) \tag{3.2.26}$$

for all $k \geq 1$. Using this inequality we see that for $r \geq N_{x_0}(\omega)$

$$\begin{aligned}
&\sum_{y;r\leq d(x_0,y)-r\leq t^{1/\beta}} \frac{\theta^\omega(y)}{(d(y,x_0)-r)^{\alpha-\beta}} = \sum_{k\in[2r,r+t^{1/\beta}]} \frac{\theta^\omega(B(x_0,k)\setminus B(x_0,k-1))}{(k-r)^{\alpha-\beta}} \\
&\leq \sum_{\ell\in[0,(t^{1/\beta}-r)/(4r)+1]} \sum_{k\in[2r+4r\ell,2r+4r(\ell+1)]} \frac{\theta^\omega(B(x_0,k)\setminus B(x_0,k-1))}{(k-r)^{\alpha-\beta}} \\
&\leq c_7 \sum_{\ell\in[0,(t^{1/\beta}-r)/(4r)+1]} (r+4r\ell)^\beta \leq c_8 \left\{ r + 4r\left(\frac{t^{1/\beta}-r}{4r}+1\right)\right\}^\beta \qquad \text{(use (3.2.26))} \\
&\leq c_9 t, \qquad (\text{use } t^{1/\beta} \geq r). \qquad (3.2.27)
\end{aligned}$$

We go back to estimate J_2. Note that for y with $r \leq d(x_0,y) - r \leq t^{1/\beta}$ we see $d(x,y) \leq d(x,x_0) + d(x_0,y) \leq 3t^{1/\beta}$. For $r \geq 1$, $t \geq 1$ with $t \geq T_0 := 3^{\beta/(\beta-1)}$ (so that $3t^{1/\beta} \leq t$ for $t \geq T_0$) and $r \geq \max_{z\in B(x_0,r)} N_z(\omega)$ (in particular $t \geq N_x(\omega)$), using Lemma 3.2.11, (3.1.4) and (3.2.27) we have

$$\begin{aligned}
J_2 &\leq \sum_{y;r<d(x_0,y)-r\leq t^{1/\beta}} \frac{c_{10}r^{\alpha-\beta}}{(d(y,x_0)-r)^{\alpha-\beta}} \frac{\theta^\omega(y)}{t^{\alpha/\beta}} \\
&= \frac{c_{10}r^{\alpha-\beta}}{t^{\alpha/\beta}} \sum_{y;r\leq d(x_0,y)-r\leq t^{1/\beta}} \frac{\theta^\omega(y)}{(d(y,x_0)-r)^{\alpha-\beta}} \\
&\leq \frac{c_{11}r^{\alpha-\beta}t}{t^{\alpha/\beta}}, \qquad (\text{use (3.2.27)}).
\end{aligned}$$

Finally we see J_3. For $t \geq T_0 := 3^{\beta/(\beta-1)}$, $N_x(\omega) \leq t$ and $N_x(\omega) \leq r$, using (3.1.4) we have

$$\begin{aligned}
J_3 &\leq \sum_{y;d(y,x_0)\leq 2r} P_x^\omega\left(Y_t^\omega = y\right) = \sum_{y;d(y,x_0)\leq 2r} q_t^\omega(x,y)\theta^\omega(y) \\
&\leq \sum_{y;d(x,y)\leq 3r} q_t^\omega(x,y)\theta^\omega(y) \leq \frac{c_{12}r^\alpha}{t^{\alpha/\beta}} \leq \frac{c_{12}r^{\alpha-\beta}t}{t^{\alpha/\beta}}.
\end{aligned}$$

We thus complete the proof. □

Lemma 3.2.13. *Under Assumption 3.1.2 (1) (2) (3) (4) and Assumption 3.2.8 there exist constants $c_1 > 0, c_2, T_0 \geq 1$ such that*

$$P_x^\omega\left(d(x_0,Y_s^\omega) \leq 2r \text{ for some } s > t\ \right) \geq \frac{c_1 r^{\alpha-\beta}t}{t^{\alpha/\beta}}$$

for almost all $\omega \in \Omega$, *all* $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^\omega)$ *with* $d(x, x_0) \leq r$, $t \geq r^\beta$, $r \geq \max_{z \in B(x_0, c_2 t^{1/\beta})} N_z(\omega)$.

Proof. Take a constant c_2 such that $c_{3.1}c_2^\alpha - c_{3.2}2^\alpha > 0$. Note that by (3.1.6) we have $\theta^\omega(\{y \in V(G) \mid d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]\}) \geq (c_{3.1}c_2^\alpha - c_{3.2}2^\alpha)t^{\alpha/\beta}$, and for y and sufficiently large t (say $t \geq T_0$) with $d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]$ we have $d(x, y)^{1+\epsilon} \leq (d(x, x_0) + d(x_0, y))^{1+\epsilon} \leq \{(c_2+1)t^{1/\beta}\}^{1+\epsilon} \leq t$ since $1+\epsilon < \beta$ (see Assumption 3.1.2). Then by Lemma 3.2.11 (2), (3.1.5), (3.1.6), for t, r as in the statement above we have

$$
\begin{aligned}
&P_x^\omega\left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t\right) \\
&\quad = \sum_{y \in V(G^\omega)} q_t^\omega(x, y)\theta^\omega(y) P_y^\omega\left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0\right) \\
&\quad \geq \sum_{y: d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} q_t^\omega(x, y)\theta^\omega(y) P_y^\omega\left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0\right) \\
&\quad \geq \sum_{y: d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left[-c_{2.2}\left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \theta^\omega(y) \frac{c_3 r^{\alpha-\beta}}{(d(x_0, y) + 2r)^{\alpha-\beta}} \\
&\qquad \text{(use (3.1.5), Lemma 3.2.11 and } d(x, y)^{1+\epsilon} \leq t, \\
&\qquad \text{note that } t \geq N_x(\omega) \text{ follows from our assumptions)} \\
&\quad \geq \sum_{y: d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} \frac{c_4}{t^{\alpha/\beta}} \theta^\omega(y) \frac{r^{\alpha-\beta}}{(t^{1/\beta})^{\alpha-\beta}} \\
&\qquad \big(\text{use } d(x, y) \leq d(x, x_0) + d(x_0, y) \leq (c_2 + 1)t^{1/\beta} \text{ for } y \in B(x_0, c_2 t^{1/\beta})\big) \\
&\quad \geq \frac{c_5(c_{3.1}c_1^\alpha - c_{3.2}2^\alpha) r^{\alpha-\beta} t}{t^{\alpha/\beta}}.
\end{aligned}
$$

We thus complete the proof by taking $c_1 = c_5(c_{3.1}c_2^\alpha - c_{3.2}2^\alpha)$. □

Lemma 3.2.14. *Under Assumption 3.1.2 (1) (2) (3) (4) and Assumption 3.2.8 there exist positive constants* c_1, c_2, η_0, T_0 *such that for any* $\eta \geq \eta_0$ *the following holds;*

$$
P_x^\omega\left(d(x_0, Y_s^\omega) \leq 2r \textit{ for some } s \in (t, \eta t]\right) \geq \frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}}
$$

for almost all $\omega \in \Omega$, *all* $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^\omega)$ *with* $d(x, x_0) \leq r$, $t \geq r^\beta$, $r \geq \max_{z \in B(x_0, c_2 t^{1/\beta})} N_z(\omega)$.

Proof. By Lemma 3.2.12 and Lemma 3.2.13 there exist positive constants c_1, c_2, c_3, T_0 such that for almost all $\omega \in \Omega$

$$\frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}} \leq P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t \ \right) \leq \frac{c_2 r^{\alpha-\beta} t}{t^{\alpha/\beta}}$$

for $r \geq 1,\ t \geq T_0,\ x, x_0 \in V(G^\omega)$ with $d(x, x_0) \leq r,\ t \geq r^\beta,\ r \geq \max_{z \in B(x_0, c_3 t^{1/\beta})} N_z(\omega)$.

Take η_0 such that $c_2 - \dfrac{c_1}{\eta^{\alpha/\beta-1}} > \dfrac{c_2}{2}$ for all $\eta \geq \eta_0$. Then we have

$$\begin{aligned}
&P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s \in (t, \eta t] \ \right) \\
&\geq P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t \ \right) - P_x^\omega \left(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > \eta t \ \right) \\
&\geq c_2 \frac{r^{\alpha-\beta} t}{t^{\alpha/\beta}} - c_1 \frac{r^{\alpha-\beta}(\eta t)}{(\eta t)^{\alpha/\beta}} = \frac{r^{\alpha-\beta} t}{t^{\alpha/\beta}} \left(c_2 - \frac{c_1}{\eta^{\alpha/\beta-1}} \right).
\end{aligned}$$

We complete the proof by adjusting the constants. □

3.2.4 Consequences of Assumption 3.1.4

In this subsection, we give easy consequences of Assumption 3.1.4. We use $\varphi(q) = \varphi_C(q) = C q^{1/\beta} (\log \log q)^{1-1/\beta}$ in this subsection.

Lemma 3.2.15. *(1) Under Assumption 3.1.4 (1), for all $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$ there exists a positive number $L^{(1)}(\omega) = L^{(1)}_{x,\epsilon,\gamma_1,\gamma_2,q}(\omega)$ such that*

$$\gamma_1 q^{n/\beta} \geq \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y(\omega), \quad \gamma_1 \varphi(q^n) \geq \max_{y \in B(x, \gamma_2 \varphi(q^n))} N_y(\omega),$$

for all $n \geq L^{(1)}(\omega)$.

(2) Under Assumption 3.1.4 (2), for all $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$ there exists a positive number $L^{(2)}(\omega) = L^{(2)}_{x,\epsilon,\gamma_1,\gamma_2,q}(\omega)$ such that

$$\gamma_1 q^{n/\beta} \geq \max_{y \in B(x, \gamma_2 q^n)} N_y(\omega)$$

for all $n \geq L^{(2)}(\omega)$.

(3) Set $\psi(t) := t^{1/\beta} h(t)$, where $h(t)$ is non-increasing and $\psi(t)$ is increasing function. Under Assumption 3.1.4 (3), for all $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$ there exists a positive number $L^{(3)}(\omega) = L^{(3)}_{x,\epsilon,\gamma_1,\gamma_2,q}(\omega)$ such that

$$\gamma_1 \psi(q^n) \geq \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y(\omega)$$

for all $n \geq L^{(3)}(\omega)$.

Proof. We can prove (1) (2) (3) similarly, so we prove only the first inequality in (1). Since

$$\begin{aligned}\mathbb{P}\left(\gamma_1 q^{n/\beta} < \max_{y\in B(x,\gamma_2 q^{n/\beta})} N_y\right) &\le \sum_{y\in B(x,\gamma_2 q^{n/\beta})} \mathbb{P}\left(\gamma_1 q^{n/\beta} < N_y\right) \\ &\le c(\gamma_2 q^{n/\beta})^\alpha f(\gamma_1 q^{n/\beta}),\end{aligned}$$

where we use union bound in the first inequality and use (3.1.2) in the second inequality. The conclusion follows by the Borel-Cantelli lemma. □

3.3 Proof of Theorem 3.1.5

In this section we give the proof of Theorem 3.1.5.

3.3.1 Proof of the LIL

We follow the strategy as in [27].

Theorem 3.3.1. *Let* $\varphi(t) = \varphi_C(t) = Ct^{1/\beta}(\log\log t)^{1-1/\beta}$, *where* $C > 2^{1+1/\beta} c_{1.2}^{-(\beta-1)/\beta}$. *Then under Assumption 3.1.2 (1) (2) (3) and Assumption 3.1.4 (1) the following hold for almost all* $\omega \in \Omega$;

$$\limsup_{t\to\infty} \frac{\sup_{0\le s\le t} d(Y_0^\omega, Y_s^\omega)}{\varphi(t)} \le 1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega), \tag{3.3.1}$$

$$P_x^\omega\left(\sup_{0\le s\le t} d(x, Y_s^\omega) \le \varphi(t) \text{ for all sufficiently large } t\right) = 1, \qquad \text{for all } x \in V(G^\omega). \tag{3.3.2}$$

In particular, we have

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \le 1, \qquad P_x^\omega\text{-a.s. for all } x \in V(G^\omega),$$
$$P_x^\omega\left(d(x, Y_t^\omega) \le \varphi(t) \text{ for all sufficient large } t\right) = 1, \qquad \text{for all } x \in V(G^\omega).$$

Proof. Take $\eta > 0$ and $\delta \in (0, c_{1.2} \wedge c_{1.4})$ sufficiently small constants which satisfy $C > 2^{1/\beta}(1+\eta)^{1/\beta}\left(\frac{1}{c_{1.2}-\delta}\right)^{(\beta-1)/\beta}$. Set $t_n = (1+\eta)^n$.

First we estimate $P_x^\omega\left(\sup_{0\le s\le t_{n+1}} d(x,Y_s^\omega)\ge 2\varphi_C(t_n)\right)$. For all $\delta\in(0,c_{1.2}\wedge c_{1.4})$, using Lemma 3.2.2 we have

$$\begin{aligned}
&P_x^\omega\left(\sup_{0\le s\le t_{n+1}} d(x,Y_s^\omega)\ge 2\varphi(t_n)\right)\\
&\le c_1\exp\left[-(c_{1.2}-\delta)\left(\frac{\varphi(t_n)}{(2t_{n+1})^{1/\beta}}\right)^{\beta/(\beta-1)}\right]+c_2\exp\left[-c_3t_{n+1}\right]\\
&\le c_1\exp\left[-(c_{1.2}-\delta)\left(\frac{\varphi(t_n)}{(2(1+\eta)t_n)^{1/\beta}}\right)^{\beta/(\beta-1)}\right]+c_2\exp\left[-c_3t_{n+1}\right] \qquad (3.3.3)
\end{aligned}$$

for $\sup_{z\in B(x,2\varphi(t_n))} N_z(\omega)\le\varphi(t_n)\wedge t_{n+1}$. Note that $\sup_{z\in B(x,2\varphi(t_n))} N_z(\omega)\le\varphi(t_n)\wedge t_{n+1}$ for all n larger than a certain constant $L=L(\omega)$ by Lemma 3.2.15 (1).

(3.3.1) is immediate from (3.3.3) and the Borel-Cantelli Lemma.

We prove (3.3.2). Let $C>2^{1/\beta}(1+\eta)^{1/\beta}\left(\frac{1}{c_{1.2}-\delta}\right)^{(\beta-1)/\beta}$ be as above. Since

$$P_x^\omega\left(\sup_{0\le s\le t_n} d(x,Y_s^\omega)\ge 2\varphi(t)\right)\le P_x^\omega\left(\sup_{0\le s\le t_{n+1}} d(x,Y_s^\omega)\ge 2\varphi(t_n)\right)$$

for $t\in[t_n,t_{n+1}]$ and the last term of (3.3.3) is summable by the definition of η and δ. By the Borel-Cantelli lemma we have

$$P_x^\omega\left(\sup_{0\le s\le t} d(Y_0^\omega,Y_s^\omega)\le 2\varphi(t)\text{ for all sufficiently large }t\right)=1,\qquad\text{for all }x\in V(G^\omega). \qquad (3.3.4)$$

We thus complete the (3.3.2) by adjusting the constants. □

Theorem 3.3.2. *Let $\varphi(t)=\varphi_C(t)=Ct^{1/\beta}(\log\log t)^{1-1/\beta}$, where $0<C<\frac{1}{2^{1+1/\beta}}\left(\frac{c_{3.1}}{c_{3.2}}\right)^{1/\alpha}\left(\frac{1}{c_{2.2}}\right)^{(\beta-1)/\beta}$. Then under Assumption 3.1.2 (1) (2) (3) and Assumption 3.1.4 (1) the following holds;*

$$\limsup_{t\to\infty}\frac{d(Y_0^\omega,Y_t^\omega)}{\varphi(t)}\ge 1,\qquad P_x^\omega\text{-a.s. for all }x\in V(G^\omega).$$

In particular, we have

$$P_x^\omega\left(d(Y_0^\omega,Y_t^\omega)\ge\varphi(t)\text{ for sufficiently large }t\right)=1,\qquad\text{for all }x\in V(G^\omega),$$

$$\limsup_{t\to\infty}\frac{\sup_{0\le s\le t} d(Y_0^\omega,Y_s^\omega)}{\varphi(t)}\ge 1,\qquad P_x^\omega\text{-a.s. for all }x\in V(G^\omega).$$

Proof. Define $\Phi(q) = q^{1/\beta}(\log\log q)^{1-1/\beta}$ and let C be as above. Take $\eta > 0$ as a sufficiently small constant such that

$$C < \frac{1}{2^{1/\beta}} \left\{ \frac{1}{2} \left(\frac{c_{3.1}}{c_{3.2}} \right)^{1/\alpha} - \eta \right\} \left(\frac{1}{c_{2.2}} \right)^{(\beta-1)/\beta}.$$

Set $\frac{1}{\lambda} = \frac{1}{2} \left(\frac{c_{3.1}}{c_{3.2}} \right)^{1/\alpha} - \eta$. Note that $c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha > 0$ and $c_{2.2}(2^{1/\beta}C\lambda)^{\beta/(\beta-1)} < 1$.

We prove that

$$\sum_n P_x^\omega \left(A_n^\omega \mid \mathcal{F}_{2^n}^\omega \right) = \infty, \tag{3.3.5}$$

where $A_n^\omega = \left\{ d(Y_{2^n}^\omega, Y_{2^{n+1}}^\omega) \geq 2\varphi(2^{n+1}) \right\}$ and $\mathcal{F}_t^\omega = \sigma\left(Y_s^\omega \mid s \leq t \right)$. To prove (3.3.5), first note that by Theorem 3.3.1 there exists a sufficiently large constant C_1 such that for almost all $\omega \in \Omega$

$$d(x, Y_{2^n}^\omega) \leq C_1\Phi(2^n) \qquad \text{for sufficiently large } n \text{ (say } n \geq \tilde{N}_1), \qquad P_x^\omega\text{-a.s.}$$

Set $B_n^\omega = A_n^\omega \cap \{d(Y_0^\omega, Y_{2^n}^\omega) \leq C_1\Phi(2^n)\}$. Then we have

$$\begin{aligned}
&P_x^\omega \left(A_n^\omega \mid \mathcal{F}_{2^n}^\omega \right) \geq P_x^\omega \left(B_n^\omega \mid \mathcal{F}_{2^n}^\omega \right) \\
&= 1_{\{d(Y_0^\omega, Y_{2^n}^\omega) \leq C_1\Phi(2^n)\}} P_{Y_{2^n}^\omega}^\omega \left(d(Y_0^\omega, Y_{2^{n+1}-2^n}^\omega) \geq 2\varphi(2^{n+1}) \right) \\
&\geq \left(\inf_{u \in B^\omega(x, C_1\Phi(2^n))} P_u^\omega \left(d(Y_0^\omega, Y_{2^n}^\omega) \geq 2\varphi(2^{n+1}) \right) \right) \cdot 1_{\{d(Y_0^\omega, Y_{2^n}^\omega) \leq C_1\Phi(2^n)\}}, \qquad P_x^\omega\text{-a.s.}
\end{aligned} \tag{3.3.6}$$

We consider the first term of (3.3.6). Take $u \in B^\omega(x, C_1\Phi(2^n))$. Since $1 + \epsilon < \beta$, there exists a positive integer $\tilde{N}_2 = \tilde{N}_2(\lambda)$ (which does not depend on u, ω) such that $d(u, v)^{1+\epsilon} \leq 2^n$ for all $n \geq \tilde{N}_2$ and $v \in B^\omega(u, \lambda\varphi(2^{n+1}))$. So for all $n \geq \tilde{N}_2$ with

$2^n \wedge 2\varphi(2^{n+1}) \geq N_u(\omega)$, using (3.1.5) and (3.1.6) we have

$$\begin{aligned}
&P_u^\omega\left(d(Y_0^\omega, Y_{2^n}^\omega) \geq 2\varphi(2^{n+1})\right) \geq P_u^\omega\left(2\varphi(2^{n+1}) \leq d(Y_0^\omega, Y_{2^n}^\omega) \leq \lambda\varphi(2^{n+1})\right) \\
&= \sum_{\substack{v \in V(G^\omega) \\ 2\varphi(2^{n+1}) \leq d(u,v) \leq \lambda\varphi(2^{n+1})}} q_{2^n}^\omega(u,v)\theta^\omega(v) \\
&\geq \sum_{\substack{v \in V(G^\omega) \\ 2\varphi(2^{n+1}) \leq d(u,v) \leq \lambda\varphi(2^{n+1})}} \frac{c_{2.1}}{(2^n)^{\alpha/\beta}} \exp\left[-c_{2.2}\left(\frac{d(u,v)}{(2^n)^{1/\beta}}\right)^{\beta/(\beta-1)}\right]\theta^\omega(v) \\
&\geq \frac{c_{2.1}}{(2^n)^{\alpha/\beta}} \exp\left[-c_{2.2}\left(\frac{\lambda\varphi(2^{n+1})}{(2^n)^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \\
&\qquad \theta^\omega(\{v \in V(G^\omega) \mid 2\varphi(2^{n+1}) \leq d(u,v) \leq \lambda\varphi(2^{n+1})\}) \\
&\geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha\left(\frac{1}{(n+1)\log 2}\right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/(\beta-1)}} \left(\log\log 2^{n+1}\right)^{(\beta-1)\alpha/\beta}.
\end{aligned}$$

By the above estimate we have

$$\begin{aligned}
&\inf_{u \in B^\omega(x, C_1\Phi(2^n))} P_u^\omega\left(d(Y_0^\omega, Y_{2^n}^\omega) \geq 2\varphi(2^{n+1})\right) \\
&\geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha\left(\frac{1}{(n+1)\log 2}\right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/(\beta-1)}} \left(\log\log 2^{n+1}\right)^{(\beta-1)\alpha/\beta}
\end{aligned} \tag{3.3.7}$$

for $n \geq \tilde{N}_2$ with $\max_{u \in B(x, C_1\Phi(2^n))} N_u(\omega) \leq 2^n \wedge 2\varphi(2^{n+1})$. By Lemma 3.2.15 (1), $\max_{u \in B(x, C_1\Phi(2^n))} N_u(\omega) \leq 2^n \wedge 2\varphi(2^{n+1})$ holds for sufficiently large n (say $n \geq \tilde{N}_3 = \tilde{N}_3(\omega)$). Hence by (3.3.6) and (3.3.7) we have

$$\begin{aligned}
&P_x^\omega\left(A_n^\omega \mid \mathcal{F}_{2^n}^\omega\right) \\
&\qquad \geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha\left(\frac{1}{(n+1)\log 2}\right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/(\beta-1)}} \left(\log\log 2^{n+1}\right)^{(\beta-1)\alpha/\beta}
\end{aligned} \tag{3.3.8}$$

for $n \geq \tilde{N}_1 \vee \tilde{N}_2 \vee \tilde{N}_3$. We thus complete to show (3.3.5).

By (3.3.5) and the second Borel-Cantelli lemma, we have $d(Y_{2^n}^\omega, Y_{2^{n+1}}^\omega) \geq 2\varphi(2^{n+1})$ for infinitely many n. This implies $d(x, Y_{2^n}^\omega) \geq \varphi(2^n)$ or $d(x, Y_{2^{n+1}}^\omega) \geq \varphi(2^{n+1})$ for infinitely many n. Hence

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \geq 1.$$

We thus complete the proof. $\square$

By Theorem 3.3.1, 3.3.2 and 3.2.6 we obtain (3.1.10).

3.3.2 Another law of the iterated logarithm

The proof of Theorem 3.1.5 (2) is quite similar to that of [50, Theorem 4.1] by using Lemma 3.2.2, Corollary 3.2.4, Lemma 3.2.5, Theorem 3.2.6 and Lemma 3.2.15 (2). So we omit the proof.

3.4 Lower Rate Function

In this section we give the proof of Theorem 3.1.6. We follow the strategy as in [64, Section 4.1].

Theorem 3.4.1. *Suppose that Assumption 3.1.2 (1) (2) (3) (4). In addition suppose that there exists a positive constant c such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$ in the case of CSRW. Let $\alpha/\beta > 1$, $h : [0,\infty) \to (0,\infty)$ be a function such that $h(t) \searrow 0$ as $t \to \infty$, $\varphi(t) := t^{1/\beta}h(t)$ be increasing for all sufficiently large t and satisfy Assumption 3.1.4 (3). If the function $h(t)$ satisfies*

$$\int_1^\infty \frac{1}{t} h(t)^{\alpha-\beta} dt < \infty \tag{3.4.1}$$

then for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$ we have

$$P_x^\omega \left(d(x, Y_t^\omega) \geq t^{1/\beta} h(t) \text{ for all sufficiently large } t \right) = 1.$$

Proof. Set $\varphi(t) := t^{1/\beta}h(t)$, $t_n := 2^n$ and
$A_n^\omega := \{d(x, Y_s^\omega) \leq \varphi(s) \text{ for some } s \in (t_n, t_{n+1}]\}$. Note that there exists a constant c_1 such that $\varphi(s) \leq 2c_1\varphi(t_n)$ for all sufficiently large n (say $n \geq N_1$) and for all $s \in (t_n, t_{n+1}]$. Then by Lemma 3.2.12 we have

$$P_x^\omega (A_n^\omega) \leq P_x^\omega (d(x, Y_s^\omega) \leq 2c_1\varphi(t_n) \text{ for some } s > t_n) \leq \frac{c_2\varphi(t_n)^{\alpha-\beta}t_n}{t_n^{\alpha/\beta}}$$

for n with

$$\begin{aligned}&n \geq N_1, \quad 2^n \geq T_0, \text{where } T_0 \text{ is as in Lemma 3.2.12}, \quad t_n^{1/\beta} \geq c_1\varphi(t_n),\\ &c_1\varphi(t_n) \geq \max_{z\in B(x,c_1\varphi(t_n))} N_z(\omega).\end{aligned} \tag{3.4.2}$$

Note that (3.4.2) is satisfied for sufficiently large n (say $n \geq N_2 = N_2(\omega)$) by Assumption 3.1.4 (3) and Lemma 3.2.15 (3). Thus

$$\begin{aligned}\sum_{n\geq N_2(\omega)} P_x^\omega(A_n^\omega) &\leq \sum_{n\geq N_2(\omega)} \frac{c_2\varphi(t_n)^{\alpha-\beta}t_n}{t_n^{\alpha/\beta}} = \sum_{n\geq N_2(\omega)} \frac{c_2 h(t_n)^{\alpha-\beta}t_n}{t_n}\\ &\leq \sum_{n\geq N_2(\omega)} \frac{c_3 h(t_n)^{\alpha-\beta}(t_n - t_{n-1})}{t_n} \leq c_4\int_{t_{N_2-1}}^\infty \frac{h(s)^{\alpha-\beta}}{s}ds.\end{aligned}$$

Since the last expression above is integrable by (3.4.1), by the Borel-Cantelli lemma we have

$$P_x^\omega\left(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t\right) = 1.$$

We thus complete the proof. □

Theorem 3.4.2. *Under the same setting as in Theorem 3.4.1, if the function $h(t)$ satisfies*

$$\int_1^\infty \frac{1}{t}h(t)^{\alpha-\beta}dt = \infty \tag{3.4.3}$$

then for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$

$$P_x^\omega\left(d(x, Y_t^\omega) \geq \varphi(t) \text{ for all sufficiently large } t\right) = 0. \tag{3.4.4}$$

We cite the following form of the Borel-Cantelli Lemma (see [64, Lemma 4.15], [68, Lemma B], [20, Theorem 1]).

Lemma 3.4.3. *Let $\{A_k\}_{k\geq 1}$ be a family of event which satisfies the following conditions;*

(1) $\sum_k P(A_k) = \infty$,

(2) $P(\limsup A_k) = 0$ *or* 1,

(3) There exist two constants c_1, c_2 such that for each A_j there exist $A_{j_1}, \cdots, A_{j_s} \in \{A_k\}_{k\geq 1}$ such that

(a) $\displaystyle\sum_{i=1}^{s} P(A_j \cap A_{j_i}) \leq c_1 P(A_j)$,

(b) for any $k \in \{j+1, j+2, \cdots\} \setminus \{j_1, j_2, \cdots, j_s\}$ *we have* $P(A_j \cap A_k) \leq c_2 P(A_j)P(A_k)$.

Then infinitely many events $\{A_k\}_{k\geq 1}$ occur with probability 1.

Proof of Theorem 3.4.2. First we prepare preliminary facts. Since $h(t) \searrow 0$ as $t \to \infty$, there exists a positive constant T_1 such that $h(t) < 1$ for all $t \geq T_1$. So there exists a constant $\kappa \in (0,1)$ such that $\varphi(t) \leq (\kappa t)^{1/\beta}$ for $t \geq T_1$. Take $\eta > 1 \vee \eta_0$ (where η_0 is as in Lemma 3.2.14) with $1 - \frac{1}{\eta} \geq \kappa$ and $c_1 = c_1(\eta) \in (0,1)$ such that $2c_1(\eta^{n+1})^{1/\beta} \leq (\eta^n)^{1/\beta}$ for all n. Note that for all s with $\eta^{n+1} \leq s \leq \eta^{n+2}$ we have

$$\varphi(\eta^{n+1}) = (\eta^{n+1})^{1/\beta} h(\eta^{n+1}) \geq 2c_1(\eta^{n+2})^{1/\beta} h(s) \geq 2c_1\varphi(s), \tag{3.4.5}$$

and for all sufficiently large i, j with $i \geq j+2$ and $\eta^j \geq T_1$ (say $j \geq N_1$) we have

$$(2c_1\varphi(\eta^{i+1}))^\beta \overset{(3.4.5)}{\leq} \varphi(\eta^i)^\beta \leq \kappa\eta^i \overset{1-\frac{1}{\eta}\geq\kappa}{\leq} \eta^i - \eta^{i-1} \leq \eta^i - \eta^{j+1}. \tag{3.4.6}$$

Now we prove (3.4.4). Set $A_n^\omega := \{d(Y_0^\omega, Y_s^\omega) \leq 2c_1\varphi(\eta^{n+1}) \text{ for some } s \in (\eta^n, \eta^{n+1}]\}$. We use Lemma 3.4.3 to show that infinitely many A_n^ω occur with probability 1.

Note that $\eta^n \geq (c_1\varphi(\eta^{n+1}))^\beta$ for sufficiently large n (say $n \geq N_2 = N_2(\eta)$) by (3.4.6). By Lemma 3.2.14 we have

$$P_x^\omega(A_n^\omega) \geq c_2 \frac{(c_1\varphi(\eta^{n+1}))^{\alpha-\beta}\eta^n}{\eta^{n\alpha/\beta}}$$

for $\eta \geq \eta_0$ (where η_0 is as in Lemma 3.2.14) and $n \geq N_2$ with

$$\eta^n \geq T_0, \text{ where } T_0 \text{ is as in Lemma 3.2.14}, \qquad c_1\varphi(\eta^{n+1}) \geq \max_{z \in B(x, c_2\eta^{n/\beta})} N_z(\omega). \tag{3.4.7}$$

Note that (3.4.7) holds for sufficiently large n (say $n \geq N_3(\omega)$) by Assumption 3.1.4 (3) and Lemma 3.2.15 (3). Hence

$$\begin{aligned}\sum_{n\geq N_3} P_x^\omega(A_n^\omega) &\geq \sum_{n\geq N_3} \frac{c_2(c_1\varphi(\eta^{n+1}))^{\alpha-\beta}\eta^n}{\eta^{n\alpha/\beta}} = \sum_{n\geq N_3} c_2 c_1^{\alpha-\beta}\eta^{\alpha/\beta}\frac{h(\eta^{n+1})^{\alpha-\beta}}{\eta\cdot\eta^{n+1}}\eta^{n+1} \\ &= \sum_{n\geq N_3}\frac{c_2c_1^{\alpha-\beta}\eta^{\alpha/\beta}}{\eta\cdot(\eta-1)}\frac{h(\eta^{n+1})^{\alpha-\beta}}{\eta^{n+1}}(\eta^{n+2}-\eta^{n+1}) \geq \frac{c_2c_1^{\alpha-\beta}\eta^{\alpha/\beta}}{\eta(\eta-1)}\int_{\eta^{N_3+1}}^{\infty}\frac{h(s)^{\alpha-\beta}}{s}ds.\end{aligned}$$

Thus we have $\sum_n P_x^\omega(A_n^\omega) = \infty$ by (3.4.5).

The condition (2) in Lemma 3.4.3 is immediate from Theorem 3.2.6, since $\limsup_k A_k^\omega$ is a tail event.

Next we show the condition (3) in Lemma 3.4.3. Set $\sigma_n^\omega := \inf\{t \in (\eta^n, \eta^{n+1}] \mid d(Y_0^\omega, Y_t^\omega) \leq 2c_1\varphi(\eta^{n+1})\}$. Then for $i \geq j+2$ we have

$$\begin{aligned} &P_x^\omega(A_i^\omega \cap A_j^\omega) = P_x^\omega(\sigma_j \leq \eta^{j+1}, \sigma_i \leq \eta^{i+1}) \\ &= E_x^\omega\left[1_{\{\sigma_j \leq \eta^{j+1}\}} P_{Y_{\sigma_j}}^\omega\left(d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t \in (\eta^i - \sigma_j, \eta^{i+1} - \sigma_j]\right)\right] \\ &\leq E_x^\omega\left[1_{\{\sigma_j \leq \eta^{j+1}\}} P_{Y_{\sigma_j}}^\omega\left(d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}\right)\right] \\ &\leq \left(\sup_{z: d(x,z) \leq 2c_1\varphi(\eta^{j+1})} P_z^\omega\left(d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}\right)\right) \\ &\qquad \times P_x^\omega\left(\sigma_j \leq \eta^{j+1}\right). \end{aligned} \tag{3.4.8}$$

By Lemma 3.2.12, for any $i \geq j+2$ with

$$\eta^i - \eta^{j+1} \geq (c_1\varphi(\eta^{i+1}))^\beta, \quad 2c_1\varphi(\eta^{j+1}) \leq c_1\varphi(\eta^{i+1}), \quad \varphi(\eta^{i+1}) \geq \max_{z \in B(x, \varphi(\eta^{i+1}))} N_z(\omega) \tag{3.4.9}$$

we have

$$\begin{aligned} &\sup_{z: d(x,z) \leq 2c_1\varphi(\eta^{j+1})} P_z^\omega\left(d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}\right) \\ &\leq \frac{c_3\left(c_1\varphi(\eta^{i+1})\right)^{\alpha-\beta}(\eta^i - \eta^{j+1})}{(\eta^i - \eta^{j+1})^{\alpha/\beta}} \leq \frac{c_4\left(c_1\varphi(\eta^{i+1})\right)^{\alpha-\beta}\eta^i}{(\eta^i)^{\alpha/\beta}}. \end{aligned} \tag{3.4.10}$$

(3.4.9) holds for sufficiently large i, j with $i \geq j+2$ (say $j \geq N_4 = N_4(\omega)$) by (3.4.5), (3.4.6), Assumption 3.1.4 (3) and Lemma 3.2.15 (3). By Lemma 3.2.14, for any i with

$$\begin{aligned} &\eta^i \geq T_0, \text{ where } T_0 \text{ is as in Lemma 3.2.14,} \\ &\eta^i \geq (c_1\varphi(\eta^{i+1}))^\beta, \quad c_1\varphi(\eta^{i+1}) \geq \max_{v \in B(x, c_5\eta^{i/\beta})} N_v(\omega) \end{aligned} \tag{3.4.11}$$

we have

$$\begin{aligned} &\frac{\left(c_1\varphi(\eta^{i+1})\right)^{\alpha-\beta}\eta^i}{(\eta^i)^{\alpha/\beta}} \leq c_6 P_x^\omega\left(d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t \in (\eta^i, \eta^{i+1}]\right) \\ &= c_6 P_x^\omega\left(A_i^\omega\right). \end{aligned} \tag{3.4.12}$$

(3.4.11) holds for sufficiently large j (say $j \geq N_5 = N_5(\omega)$) by (3.4.5), Assumption 3.1.4 (3) and Lemma 3.2.15 (3). Hence by (3.4.8), (3.4.10) and (3.4.12) we have $P_x^\omega\left(A_i^\omega \cap A_j^\omega\right) \leq cP_x^\omega(A_i^\omega)P_x^\omega(A_j^\omega)$ for sufficiently large j ($j \geq N_6 := N_4 \vee N_5$) and $i \geq j+2$. In the case of $i = j+1$ we have $P_x^\omega\left(A_{j+1}^\omega \cap A_j^\omega\right) \leq P_x^\omega(A_j^\omega)$. Thus we obtain the condition (3) of Lemma 3.4.3 for $\{A_i^\omega\}_{i \geq N_6}$.

By Lemma 3.4.3, we thus complete the proof. □

By Theorem 3.4.1 and Theorem 3.4.2 we complete the proof of Theorem 3.1.6.

3.5 Ergodic media

In this section, we consider the case $G = (V, E) = \mathbb{Z}^d$ and obtain Theorem 3.1.8 under Assumption 3.1.7. We follow the strategy as in [27]

3.5.1 Ergodicity of the shift operator on $\Omega^{\mathbb{Z}}$

We consider Markov chains on the random environment, which is called the environment seen from the particle, according to Kipnis and Varadhan [47].

Let $\Omega = [0,\infty)^E$ and define $\mathscr{B}$ as the natural σ-algebra (generated by coordinate maps). We write $\mathcal{Y} = \Omega^{\mathbb{Z}}$, $\mathscr{Y} = \mathscr{B}^{\otimes\mathbb{Z}}$. If each conductance may take the value 0, we regard 0 as the base point and define $\mathcal{C}_0(\omega) = \{x \in \mathbb{Z}^d \mid 0 \overset{\omega}{\longleftrightarrow} x\} = V(G^\omega)$, where $0 \overset{\omega}{\longleftrightarrow} x$ means that there exists a path $\gamma = e_1e_2\cdots e_k$ from 0 to x such that $\omega(e_i) > 0$ for all $i = 1, 2, \cdots, k$. Define $\Omega_0 = \{\omega \in \Omega \mid \sharp\mathcal{C}_0(\omega) = \infty\}$ and $\mathbb{P}_0 = \mathbb{P}(\cdot \mid \Omega_0)$.

Next we consider the Markov chains seen from the particle. Recall that $\{X_n^\omega\}_{n\geq 0}$ is the discrete time random walk which is introduced in Section 3.1.1. Let $\omega_n(\cdot) = \omega(\cdot + X_n^\omega) = \tau_{X_n^\omega}\omega(\cdot) \in \Omega$. We can regard this Markov chain $\{\omega_n\}_{n\geq 0}$ as being defined on $\mathcal{Y} = \Omega^{\mathbb{Z}}$. We define a probability kernel $Q : \Omega_0 \times \mathscr{B} \to [0,1]$ as

$$Q(\omega, A) = \frac{1}{\sum_{e':|e'|=1}\omega_{e'}} \sum_{v:|v|=1} \omega_{0v} 1_{\{\tau_v\omega \in A\}}.$$

This is nothing but the transition probability of the Markov chain $\{\omega_n\}_{n\geq 0}$.

Next we define the probability measure on $(\mathcal{Y}, \mathscr{Y})$ as

$$\mu\left((\omega_{-n}, \cdots, \omega_n) \in B\right) = \int_B \mathbb{P}_0(d\omega_{-n})Q(\omega_{-n}, d\omega_{-n+1})\cdots Q(\omega_{n-1}, d\omega_n).$$

By the above definition, $\{\tau_{X_k^\omega}\omega\}_{k\geq 0}$ has the same law in $\mathbb{E}_0(P_0^\omega(\cdot))$ as $(\omega_0, \omega_1, \cdots)$ has in μ, that is,

$$\mathbb{E}_0\left[P_0^\omega(\{\tau_{X_k^\omega}\omega\}_{k\geq 0} \in B)\right] = \mu((\omega_0, \omega_1, \cdots) \in B) \tag{3.5.1}$$

for any $B \in \mathscr{Y}$.

We need the following Theorem. Let $T : \mathcal{Y} \to \mathcal{Y}$ be a shift operator of $\mathcal{Y}$, that is,

$$(T\omega)_n = \omega_{n+1}.$$

Theorem 3.5.1. *Under Assumption 3.1.7, T is ergodic with respect to μ.*

The proof is similar to [16, Proposition 3.5], so we omit it.

We also need the following Zero-One law (see Proposition 3.5.2). Let $a \geq 0$ and $A_1^\omega(a), A_2^\omega(a), A_3^\omega(a)$ be the events

$$A_1^\omega(a) = \left\{ \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} > a \right\},$$
$$A_2^\omega(a) = \left\{ \limsup_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}} > a \right\},$$
$$A_3^\omega(a) = \left\{ \liminf_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{-1/\beta}} > a \right\}.$$

Define

$$\tilde{A}_i(a) = \{\omega \in \Omega \mid \ A_i^\omega(a) \text{ holds for } P_x^\omega\text{-a.s. and for all } x \in \mathcal{C}_0(\omega)\}.$$

Proposition 3.5.2. *$\mathbb{P}_0(\tilde{A}_i(a))$ is either 0 or 1.*

Proof. See [50, Proposition 5.2]. □

3.5.2 Proof of Theorem 3.1.8

In this subsection we discuss the proof of Theorem 3.1.8. Recall $T_0^\omega = 0$, $T_{n+1}^\omega = \inf\{t > T_n^\omega \mid Y_t^\omega \neq Y_{T_n^\omega}^\omega\}$ and $X_n^\omega = Y_{T_n^\omega}^\omega$.

First we consider the CSRW. $\{T_{n+1}^\omega - T_n^\omega\}_{n\geq 0}$ is a family of i.i.d. random variables whose distributions are exponential with mean 1, so the law of large number gives us

$$\frac{T_n^\omega}{n} \to 1 \qquad P_0^\omega\text{-a.s.}$$

Thus

$$\limsup_{t\to\infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \limsup_{n\to\infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$
$$\limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \limsup_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$
$$\liminf_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log\log t)^{-1/\beta}} = \liminf_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log\log n)^{-1/\beta}}.$$

By Assumption 3.1.7, Proposition 3.5.2 and Theorem 3.1.5 we obtain Theorem 3.1.8.

Next we consider the VSRW. $\{T^{\omega}_{n+1} - T^{\omega}_n\}_{n\geq 0}$ are non-i.i.d., and the distribution of $T^{\omega}_{n+1} - T^{\omega}_n$ is exponential with mean $\dfrac{1}{\pi^{\omega}(X^{\omega}_n)}$. Write S^{ω}_x be a exponential random variable with parameter $\pi^{\omega}(x)$ and $\bar{S}_x(\bar{\omega}) := S^{\bar{\omega}_0}_x$, $(\bar{\omega} \in \mathcal{Y})$. Then by (3.5.1) and the ergodicity we have

$$\begin{aligned}
\frac{1}{n} T^{\omega}_n &= \frac{1}{n}\sum_{k=0}^{n-1} S^{\omega}_{X^{\omega}_k} \overset{d}{=} \frac{1}{n}\sum_{k=0}^{n-1} \bar{S}_0(T^k\bar{\omega}) \to \mathbb{E}^{\mu}\left[\bar{S}_0\right] \\
&= \mathbb{E}\left[E^{\omega}_0[S^{\omega}_0]\right] = \int_{\Omega}\int_0^{\infty} x\pi^{\omega}(0)\exp(-\pi^{\omega}(0)x)dxd\mathbb{P} = \mathbb{E}\left[\frac{1}{\pi^{\omega}(0)}\right].
\end{aligned}$$

Thus

$$\limsup_{t\to\infty} \frac{d(Y^{\omega}_0, Y^{\omega}_t)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \left(\frac{1}{\mathbb{E}\left[\frac{1}{\pi^{\omega}(0)}\right]}\right)^{1/\beta} \limsup_{n\to\infty} \frac{d(X^{\omega}_0, X^{\omega}_n)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$

$$\limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y^{\omega}_0, Y^{\omega}_s)}{t^{1/\beta}(\log\log t)^{1-1/\beta}} = \left(\frac{1}{\mathbb{E}\left[\frac{1}{\pi^{\omega}(0)}\right]}\right)^{1/\beta} \limsup_{n\to\infty} \frac{\sup_{0\leq k\leq n} d(X^{\omega}_0, X^{\omega}_k)}{n^{1/\beta}(\log\log n)^{1-1/\beta}},$$

$$\liminf_{t\to\infty} \frac{\sup_{0\leq s\leq t} d(Y^{\omega}_0, Y^{\omega}_s)}{t^{1/\beta}(\log\log t)^{-1/\beta}} = \left(\frac{1}{\mathbb{E}\left[\frac{1}{\pi^{\omega}(0)}\right]}\right)^{1/\beta} \liminf_{t\to\infty} \frac{\sup_{0\leq k\leq n} d(X^{\omega}_0, X^{\omega}_k)}{n^{1/\beta}(\log\log n)^{-1/\beta}}.$$

By Assumption 3.1.7, Proposition 3.5.2 and Theorem 3.1.5 we obtain Theorem 3.1.8.

Chapter 4

Cutoff for lamplighter chains on fractals

We show that the total-variation mixing time of the lamplighter random walk on fractal graphs exhibit sharp cutoff when the underlying graph is transient (namely of spectral dimension greater than two). In contrast, we show that such cutoff can not occur for strongly recurrent underlying graphs (i.e. of spectral dimension less than two).

4.1 Introduction

Markov chain mixing rate is an active subject of study in probability theory (see [52, 61] and the references therein). Mixing is usually measured in terms of total variation distance, which for probability measures μ, ν on a countable set H is

$$\|\mu - \nu\|_{\mathrm{TV}} := \sup_{A \subset H} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in H} |\mu(x) - \nu(x)| = \sum_{x \in H} [\mu(x) - \nu(x)]_+ \,.$$

Specifically, the (ϵ-)total variation mixing time of a Markov chain $Y = \{Y_t\}_{t \geq 0}$ on the set of vertices of a finite graph $G = (V, E)$, having the invariant distribution π, is

$$T_{\mathrm{mix}}(\epsilon; G) := \min \Big\{ t \geq 0 \;\Big|\; \max_{x \in V(G)} \|P_x(Y_t = \cdot) - \pi\|_{\mathrm{TV}} \leq \epsilon \Big\}.$$

One of the interesting topics in the study of Markov chains is the cutoff phenomena, mainly for the total variation mixing time (see e.g. [52, Chapter 18]). The study of cutoff phenomena for Markov chains was initiated by Aldous, Diaconis and their collaborators early in 80s, and there has been extensive work in the past several decades. Specifically, a sequence of Markov chains $\{Y^{(N)}\}_{N\geq 1}$ on the vertices of finite graphs $\{G^{(N)}\}_{N\geq 1}$ has cutoff with threshold $\{a_N\}_{N\geq 1}$ iff

$$\lim_{N\to\infty} a_N^{-1} T_{\text{mix}}(\epsilon; G^{(N)}) = 1, \qquad \forall \epsilon \in (0,1).$$

In the (switch-walk-switch) lamplighter Markov chains, each vertex of a locally connected, countable (or finite) graph $G = (V, E)$ is equipped with a lamp (from $\mathbb{Z}_2 = \{0,1\}$), and a move consists of three steps:

(a). The walker turns on/off the lamp at the vertex where he/she is, uniformly at random.

(b). The walker either stays at the same vertex, or moves to a randomly chosen nearest neighbor vertex.

(c). The walker turns on/off the lamp at the vertex where he/she is, uniformly at random.

Such a lamplighter chain on the graph G is precisely the random walk on the corresponding wreath product $G^* = \mathbb{Z}_2 \wr G$ (see Section 4.1.1 for the precise definitions), and the total variation mixing time of a lamplighter chain is closely related to the expected cover time of the underlying graph G, denoted hereafter by $T_{\text{cov}}(G)$. The study of cutoff for lamplighter chains goes back to Häggström and Jonasson [36] who showed that cutoff does not occur for the chain on one-dimensional tori, whereas for lamplighter chains on complete graphs, it occurs at the threshold $a_N = \frac{1}{2} T_{\text{cov}}(G^{(N)})$. Peres and Revelle [59] further explore the relation between the mixing time of lamplighter chain on $G^{(N)}$ and $T_{\text{cov}}(G^{(N)})$, showing that, under suitable assumptions,

$$\big(\frac{1}{2} + o(1)\big) T_{\text{cov}}(G^{(N)}) \leq T_{\text{mix}}(\mathbb{Z}_2 \wr G^{(N)}; \epsilon) \leq (1 + o(1)) T_{\text{cov}}(G^{(N)}). \tag{4.1.1}$$

The bounds of (4.1.1) cannot be improved in general, as the lower and the upper bounds are achieved for complete graphs, and two-dimensional tori, respectively. The same bounds apply for any Markov chain on $\mathfrak{X} \wr G^{(N)}$, where in steps (a) and (c) the walker independently chooses the element from the finite set $\mathfrak{X}$ according to some fixed strictly positive law. Indeed, for such chains total variation mixing time has mostly to do with the geometry of late points of G, namely those reached by the walker much later than most points. In particular, the LHS of (4.1.1) represents the need to visit all but $O(\sqrt{\sharp V(G)})$ points before mixing of the lamps can occur and the RHS reflects having the lamps at the invariant product measure once all vertices have been visited. Miller and Peres [54] provide a large class of graphs for which the LHS of (4.1.1) is sharp, with cutoff at $\frac{1}{2}T_{\text{cov}}(G^{(N)})$. Among those are lazy simple random walkers on d-dimensional tori, any $d \geq 3$, for which [55] further examines the total-variation distance between the law of late points and i.i.d. Bernoulli points (c.f. [55, Section 1] and the references therein). Finally, the analysis of effective resistance on $G^{(N)} = \mathbb{Z}_N^2 \times \mathbb{Z}_{[h \log N]}$ plays a key role in [25], where it is shown that the threshold $a(h)T_{\text{cov}}(G^{(N)})$ for mixing time cutoff of lamplighter chain on such graphs, continuously interpolates between $a(0) = 1$ and $a(\infty) = \frac{1}{2}$.

Another topic of much current interest is the long time asymptotic behavior of random walks $\{X_t\}$ on (infinite) fractal graphs (see [3, 45, 49] and the references therein). Such random walks are typically anomalous and sub-diffusive, so generically $E_x[d(X_0, X_t)] \asymp t^{1/d_w}$ and the *walk-dimension* d_w exceeds two for many fractal graphs, in contrast to the SRW on $\mathbb{Z}^d$ for which $d_w = 2$ (the notation $a_t \asymp b_t$ is used hereafter whenever $c^{-1}a_t \leq b_t \leq ca_t$ for some $c < \infty$). A related important parameter is the *volume growth exponent* d_f such that $\sharp B(x, r) \asymp r^{d_f}$, where $\sharp B(x, r)$ counts the number of vertices whose graph distance from x is at most r. The growth of the eigenvalues of the corresponding generator is then measured by the *spectral dimension* $d_s := 2d_f/d_w$, with the Markov chain $\{X_t\}$ strongly recurrent when $d_s < 2$ and transient when $d_s > 2$ (while $d_f = d_s = d$ for the SRW on $\mathbb{Z}^d$).

We study here the cutoff for total variation mixing time of the lamplighter chain when $G^{(N)}$ are increasing finite subsets of a fractal graph. While gaining important insights on the geometry of late points for the corresponding walks, our main result (see Theorem 4.1.4), is the following dichotomy:

- When $d_s < 2$ there is no cutoff for the corresponding lamplighter chain, whereas
- if $d_s > 2$, such cutoff occurs at the threshold $a_N = \frac{1}{2}T_{\text{cov}}(G^{(N)})$.

In contrast, in the critical case $d_s = 2$ (i.e. $d_f = d_w$), we expect the mixing time and the corresponding cutoff phenomena to depend also on some other properties of $\{G^{(N)}\}$.

4.1.1 Framework and main results

Given a countable, locally finite and connected graph $G = (V(G), E(G))$, denote by $d(\cdot, \cdot) = d_G(\cdot, \cdot)$ the graph distance (with $d(x, y)$ the length of the shortest path between x and y), and by $B(x, r) = B_G(x, r) := \{y \in V(G) \mid d(x, y) \leq r\}$ the corresponding ball of radius r centered at x. A weighted graph is a pair (G, μ)

with $\mu : V(G) \times V(G) \to [0, \infty)$ a conductance, namely a function $(x, y) \mapsto \mu_{xy}$ such that $\mu_{xy} = \mu_{yx}$ and $\mu_{xy} > 0$ if and only if $xy \in E(G)$. We use the notation $V(x, r) := \mu(B(x, r))$ and more generally $\mu(A) := \sum_{x \in A} \mu_x$ for $A \subset V(G)$, where

$$\mu_x := \sum_{y : xy \in E(G)} \mu_{xy}, \qquad \forall x \in V. \tag{4.1.2}$$

The discrete time random walk $X = \{X_t\}_{t \geq 0}$ associated with the weighted graph (G, μ) is the Markov chain on $V(G)$ having the transition probability

$$P(x, y) := \frac{\mu_{xy}}{\mu_x}.$$

Let $P_t(x, y) = P_t(x, y; G) := P_x(X_t = y)$ denote the distribution of X_t with the corresponding heat kernel

$$p_t(x, y) := \frac{P_t(x, y)}{\mu_y} \qquad \forall t \in \mathbb{N} \cup \{0\}$$

and Dirichlet form

$$\mathcal{E}(f, f) := \frac{1}{2} \sum_{x, y \in V(G)} (f(x) - f(y))^2 \mu_{xy} = -\langle f, (P - I) f \rangle_\mu, \quad \text{for } f : V(G) \to \mathbb{R}, \tag{4.1.3}$$

where $\langle f, g \rangle_\mu := \sum_x f(x) g(x) \mu(x)$. The corresponding effective resistance $R_{\text{eff}}(\cdot, \cdot)$ is given by

$$R_{\text{eff}}(A, B)^{-1} := \inf\{\mathcal{E}(f, f) \mid f|_A = 1, f|_B = 0\}, \quad \text{for } A, B \subset V(G).$$

We also consider the lazy random walk $\tilde{X} = \{\tilde{X}_t\}_{t \geq 0}$ on (G, μ), having the transition probability

$$\tilde{P}(x, y) := \begin{cases} \frac{1}{2} P(x, y), & \text{if } x \neq y, \\ \frac{1}{2}, & \text{if } x = y. \end{cases} \tag{4.1.4}$$

The Dirichlet form and heat kernel of $\tilde{X}$ are then, respectively $\tilde{\mathcal{E}}(f, f) = \frac{1}{2} \mathcal{E}(f, f)$ and

$$\tilde{p}_t(x, y) := \frac{\tilde{P}_t(x, y)}{\mu_x} \qquad \forall t \in \mathbb{N} \cup \{0\}.$$

We consider finite weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_{N\geq 1}$ with $\sharp V(G^{(N)}) \to \infty$. Using hereafter $\cdot^{(N)}$ for objects on $(G^{(N)}, \mu^{(N)})$ (e.g. denoting by $R_{\text{eff}}^{(N)}(\cdot,\cdot)$ the effective resistance on $(G^{(N)}, \mu^{(N)})$), we make the following assumptions, which are standard in the study of sub-Gaussian heat kernel estimates (SUB-GHKE) (c.f. [5, 49]).

Assumption 4.1.1. *For some $1 \leq d_f < \infty$, $c_{\text{e}}, c_{\text{v}} < \infty$, $p_0 > 0$ and all $N \geq 1$ we have*

(a) Uniform ellipticity: $\quad c_{\text{e}}^{-1} \leq \mu_{xy}^{(N)} \leq c_{\text{e}} \quad \forall xy \in E(G^{(N)})$.

(b) p_0-condition: $\quad \dfrac{\mu_{xy}^{(N)}}{\mu_x^{(N)}} \geq p_0 \quad \forall xy \in E(G^{(N)})$.

(c) d_f-set condition: $\quad c_{\text{v}}^{-1} r^{d_f} \leq V^{(N)}(x,r) \leq c_{\text{v}} r^{d_f} \quad \forall x \in V(G^{(N)})$, $1 \leq r \leq \operatorname{diam}\{G^{(N)}\} \to \infty$.

Assumption 4.1.2 (Uniform Parabolic Harnack Inequality)**.** *For some $2 \leq d_w < \infty$, $C_{\text{PHI}} < \infty$, $c_{\text{PHI}} \in (0,1]$ and all $N \geq 1$, whenever $u : [0,\infty) \times V(G^{(N)}) \to [0,\infty)$ satisfies*

$$u(t+1,x) - u(t,x) = (P^{(N)} - I)u(t,x), \qquad \forall t \in [0,4T], x \in B^{(N)}(x_0, 2R)\,, \quad (4.1.5)$$

for some $x_0 \in V(G^{(N)})$, $R \leq c_{\text{PHI}} \operatorname{diam}\{G^{(N)}\}$ and $T \geq 2R$, $T \asymp R^{d_w}$, one also has that

$$\max_{\substack{z \in B^{(N)}(x_0,R) \\ s \in [T,2T]}} \{u(s,z)\} \leq C_{\text{PHI}} \min_{\substack{z \in B^{(N)}(x_0,R) \\ s \in [3T,4T]}} \{u(s,z) + u(s+1,z)\}. \quad (4.1.6)$$

Remark 4.1.3. *Thanks to the p_0-condition we have that $1 \geq \deg_{G^{(N)}}(x) p_0$, so the graphs $\{G^{(N)}\}$ are of uniformly bounded degrees*

$$\sup_N \sup_{x \in V(G^{(N)})} \{\deg_{G^{(N)}}(x)\} < \infty.$$

Together with the uniform ellipticity, this implies that for some $\tilde{c} < \infty$

$$\tilde{c}^{-1} \leq \mu_x^{(N)} \leq \tilde{c}, \qquad \forall N \geq 1, \;\; x \in V(G^{(N)}),$$

and thereby

$$\tilde{c}^{-1}\, \sharp A \leq \mu^{(N)}(A) \leq \tilde{c}\, \sharp A, \qquad \forall N \geq 1, \;\; A \subset V(G^{(N)})\,. \quad (4.1.7)$$

To any finite underlying graph $G = (V, E)$ corresponds the wreath product $G^* = \mathbb{Z}_2 \wr G$ such that

$$V(G^*) = \mathbb{Z}_2^V \times V,$$
$$E(G^*) = \{\{(f,x),(g,y)\} \mid \; f = g \text{ and } xy \in E, \text{ or } x = y \text{ and } f(v) = g(v) \text{ for } v \neq x\}$$

and we adopt throughout the convention of using $\boldsymbol{y} = (f, y)$ for the vertices of $\mathbb{Z}_2 \wr G$. The lazy random walk $\tilde{X}$ on (G, μ) induces the switch-walk-switch lamplighter chain, namely the random walk $Y = \{Y_t = (f_t, \tilde{X}_t)\}_{t \geq 0}$ on $\mathbb{Z}_2 \wr G$ whose transition probability is

$$P^*((f,x),(g,y)) = \begin{cases} \frac{1}{4}\tilde{P}(x,y) = \frac{1}{8}, & \text{if } x = y \text{ and } f(v) = g(v) \text{ for any } v \neq x, \\ \frac{1}{4}\tilde{P}(x,y) = \frac{1}{8}P(x,y), & \text{if } x \neq y \text{ and } f(v) = g(v) \text{ for any } v \neq x, y, \\ 0, & \text{otherwise.} \end{cases}$$

One way to describe the moves of the Markov chain Y is as done before: first Y switches the lamp of the current position, then moves on G according to $\tilde{P}$, and finally switches the lamp on vertex on which it landed. We denote by $Y^{(N)} = \{Y_t^{(N)} = (f_t, \tilde{X}_t^{(N)})\}_{t \geq 0}$ the lamplighter chain on weighted graphs $(G^{(N)}, \mu^{(N)})$, using $P^*(\cdot, \cdot; G)$ whenever we wish to emphasize its underlying graph. The invariant (reversible) distribution of each $X^{(N)}$, and its lazy version $\tilde{X}^{(N)}$, is clearly

$$\pi^{(N)}(x) = \frac{\mu_x^{(N)}}{\mu^{(N)}(G^{(N)})}, \qquad \forall x \in V(G^{(N)}) \tag{4.1.8}$$

with the corresponding invariant distribution of $Y^{(N)}$ being

$$\pi^*(\boldsymbol{y}; G^{(N)}) = 2^{-\sharp V(G^{(N)})}\, \pi^{(N)}(y), \qquad \forall \boldsymbol{y} = (f, y) \in V(\mathbb{Z}_2 \wr G^{(N)}). \tag{4.1.9}$$

We next state our main result.

Theorem 4.1.4. *Consider lamplighter chains $Y^{(N)}$ whose underlying weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_{N \geq 1}$ satisfy Assumptions 4.1.1, 4.1.2.*

(a) If $d_f < d_w$, then there is no cutoff for the total variation mixing time of $Y^{(N)}$.

(b) If $d_f > d_w$, then the total variation mixing time for $Y^{(N)}$ admits cutoff at $a_N = \frac{1}{2}T_{\mathrm{cov}}(G^{(N)})$.

Note that for countable, infinite weighted graph (G,μ), having $d_f < d_w$ (resp. $d_f > d_w$), corresponds to a strongly recurrent (resp. transient), random walk $\tilde{X}$ in the sense of [10, Definition 1.2] (see [10, Theorem 1.3, Proposition 3.5 and Lemma 3.6]). In Section 4.2 we provide a host of fractal graphs satisfying Assumptions 4.1.1 and 4.1.2, with the Sierpinski gaskets and the two-dimensional Sierpinski carpets as typical examples of Theorem 4.1.4(a), while high-dimensional Sierpinski carpets with small holes serve as typical examples of Theorem 4.1.4(b).

In Section 4.3, we adapt to the setting of large finite weighted graphs, certain consequences of Assumptions 4.1.1 and 4.1.2 which are standard for infinite graphs. In case $d_f < d_w$, the relevant time scale for the cover time $\tau_{\rm cov}(G^{(N)})$ is shown there to be

$$T_N := (R_N)^{d_w} \qquad \text{where} \qquad R_N := \operatorname{diam}\{G^{(N)}\}\,. \tag{4.1.10}$$

Applying in Section 4.4 results from Section 4.3 that apply for $d_f < d_w$, we derive the following uniform exponential tail decay for $\tau_{\rm cov}(G^{(N)})/T_N$, which is of independent interest.

Proposition 4.1.5. *If Assumptions 4.1.1, 4.1.2 hold with $d_f < d_w$, then for some c_0 finite and all t, N,*

$$\sup_{z\in V(G^{(N)})} \{P_z(\tau_{\rm cov}(G^{(N)}) > t)\} \le c_0 e^{-t/(c_0 T_N)}\,. \tag{4.1.11}$$

Starting with all lamps off, namely at $Y_0 = \boldsymbol{x} := (\mathbf{0}, x)$, on the event $\{\sup_{0\le s\le t} d(\tilde{X}_0, \tilde{X}_s) \le \frac{1}{4}R_N\}$, all lamps outside $B^{(N)}(x, \frac{1}{4}R_N)$ are off at time t. Hence, then $\|P_t^*(\boldsymbol{x},\cdot;G^{(N)}) - \pi^*(\cdot;G^{(N)})\|_{\rm TV}$ is still far from 0. Using this observation, we prove in Section 4.5 the following uniform lower bound on the lamplighter chain distance from equilibrium at time $t \asymp T_N$.

Proposition 4.1.6. *If Assumptions 4.1.1, 4.1.2 hold, then for some finite c_1, N_1, any t and $N \ge N_1$,*

$$\max_{\boldsymbol{x}\in V(\mathbb{Z}_2 \wr G^{(N)})} \|P_t^*(\boldsymbol{x},\cdot;G^{(N)}) - \pi^*(\cdot;G^{(N)})\|_{\rm TV} \ge c_1^{-1} e^{-c_1 t/T_N} - \tilde{c}\, c_{\rm v} R_N^{-d_f}\,. \tag{4.1.12}$$

In Proposition 4.5.1 we bound the LHS of (4.1.12) by $\max_x P_x(\tau_{\rm cov}(G^{(N)}) > t)$ provided t/S_N is large (for S_N of (4.3.10)). Since $S_N \asymp T_N$ when $d_f < d_w$, contrasting Propositions 4.1.5 and 4.1.6 yields Theorem 4.1.4(a) (c.f. Remark 4.5.2 for information about $T_{\rm mix}(\epsilon; G^{(N)})/T_{\rm cov}(G^{(N)})$ and lack of concentration of $\tau_{\rm cov}(G^{(N)})/T_N$).

Propositions 4.1.6 and 4.5.1 apply also when $d_w < d_f$, but in that case $\tau_{\mathrm{cov}}(G^{(N)}) \geq \sharp V(G^{(N)}) \gg T_N$, and the proof of Theorem 4.1.4(b), provided in Section 4.5.2, amounts to verifying the sufficient conditions of [54, Theorem 1.5] for cutoff at $\frac{1}{2}T_{\mathrm{cov}}(G^{(N)})$. Indeed, the required uniform Harnack inequality follows from the UPHI of Assumption 4.1.2, which as we see in Section 4.2 is more amenable to analytic manipulations than the Harnack inequality.

4.2 Cutoff in fractal graphs

We provide here a few examples for which Theorem 4.1.4 applies, starting with the following.

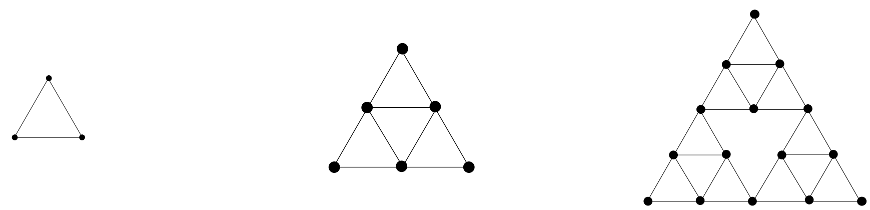

Figure 4.1: A sequence of the Sierpinski gasket graphs ($G^{(0)}, G^{(1)}, G^{(2)}$ respectively).

Example 4.2.1 (Sierpinski gasket graph in two dimension)**.**

Let $G^{(0)}$ denote the equilateral triangle of side length 1. That is,

$$V(G^{(0)}) = \Big\{x_0 = (0,0), x_1 = (1,0), x_2 = \big(\frac{1}{2}, \frac{\sqrt{3}}{2}\big)\Big\}, \qquad E(G^{(0)}) = \{x_0x_1, x_0x_2, x_1x_2\}.$$

Setting $\psi_i(x) := (x + x_i)/2$ for $i = 0, 1, 2$, we define the graphs $\{G^{(N)}\}_{N\geq 1}$ via

$$V(G^{(N+1)}) = 2 \cdot \Big(\bigcup_{i=1}^{3} \psi_i(V(G^{(N)}))\Big) \quad \text{and} \quad E(G^{(N+1)}) = 2 \cdot \Big(\bigcup_{i=1}^{3} \psi_i(E(G^{(N)}))\Big).$$

The limit graph $G = (V(G), E(G))$, where $V(G) = \cup_{N\geq 0} V(G^{(N)})$ and $E(G) = \cup_{N\geq 0} E(G^{(N)})$, is called the Sierpinski gasket graph. It is easy to confirm that if Assumption 4.1.1(a) holds for weight $\mu^{(N)}$ on $G^{(N)}$ then such $\mu^{(N)}$ satisfies also Assumption 4.1.1(b) and Assumption 4.1.1(c) for $d_f = \log 3/\log 2$.

We further prove in Section 4.2.2 the following.

Proposition 4.2.2. *The weighted graphs* $\{(G^{(N)}, \mu^{(N)})\}_{N\geq 0}$ *of Example 4.2.1 further satisfy Assumption 4.1.2 with* $d_w = \log 5/\log 2$.

In view of Proposition 4.2.2 and having $d_f < d_w$, we deduce from Theorem 4.1.4(a) that the total variation mixing time of the lamplighter chains of Example 4.2.1, admits no cutoff.

Remark 4.2.3. *For* $d \geq 3$, *the* d*-dimensional Sierpinski gasket graph is similarly defined, and by the same reasoning the corresponding lamplighter chains admit no mixing cutoff. In fact, one can deduce for a more general family of nested fractal graphs (see for instance [37, Section 2] for definition), that no cutoff applies.*

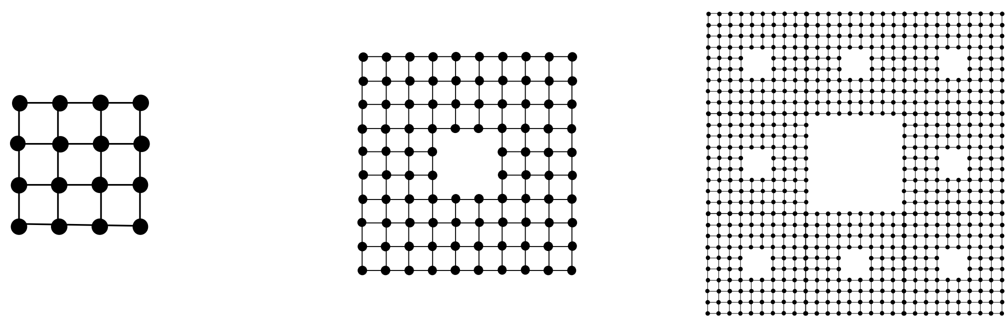

Figure 4.2: A sequence of the Sierpinski carpet graphs.

Example 4.2.4 (Sierpinski carpet graph)**.** *Fixing integers* $L \geq 2$ *and* $K \in [L, L^d]$, *partition the* d*-dimensional unit cube* $H_0 = [0,1]^d$ *into the collection* $Q := \{\prod_{i=1}^{d}[\frac{(k_i-1)}{L}, \frac{k_i}{L}] \mid 1 \leq k_i \leq L \text{ for all } i \in \{1,2,\ldots,d\}\}$ *of* L^d *sub-cubes. Then fixing* L*-similitudes* $\{\psi_i, 1 \leq i \leq K\}$ *of* H_0 *onto mutually distinct elements of* Q, *such that* $\psi_1(x) := L^{-1}x$, *there exists a unique non-empty compact* $F \subset H_0$ *such that* $F = \bigcup_{i=1}^{K} \psi_i(F)$. *We call* F *the generalized Sierpinski carpet if the following four conditions hold:*

(a) (Symmetry) $H_1 := \bigcup_{i=1}^{K} \psi_i(H_0)$ *is preserved by all isometries of* H_0.

(b) (Connectedness) $\mathrm{Int}(H_1)$ *is connected, and contains a path connecting the hyperplanes* $\{x_1 = 0\}$ *and* $\{x_1 = 1\}$.

(c) (Non-diagonality) If $\mathrm{Int}(H_1 \cap B)$ *is nonempty for some* d*-dimensional cube* $B \subset H_0$ *which is the union of* 2^d *distinct elements of* Q, *then* $\mathrm{Int}(H_1 \cap B)$ *is a connected set.*

(d) (Borders included) H_1 *contains the line segment* $\{(x_1, 0, \ldots, 0) \mid 0 \leq x_1 \leq 1\}$.

For a generalized Sierpinski carpet, let $V^{(0)}$ and $E^{(0)}$ denote the the 2^d corners of H_0 and $d2^{d-1}$ edges on the boundary of H_0 respectively, with

$$V(G^{(N)}) := \bigcup_{i_1,i_2,\dots,i_N=1}^{K} L^N \psi_{i_1,i_2,\dots,i_N}(V^{(0)}), \qquad V(G) := \bigcup_{N\geq 1} V(G^{(N)})\,.$$

$$E(G^{(N)}) := \bigcup_{i_1,i_2,\dots,i_N=1}^{K} L^N \psi_{i_1,i_2,\dots,i_N}(E^{(0)}), \qquad E(G) := \bigcup_{N\geq 1} E(G^{(N)})\,.$$

Once again, it is easy to check that if Assumption 4.1.1(a) holds for weight $\mu^{(N)}$ on $G^{(N)}$, then such $\mu^{(N)}$ satisfies also Assumptions 4.1.1(b) and 4.1.1(c) for $d_f = \log K/\log L$.

We prove in Section 4.2.2 the following.

Proposition 4.2.5. *For any generalized Sierpinski carpet, the weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_{N\geq 0}$ of Example 4.2.4 further satisfy Assumption 4.1.2 for some finite $d_w = \log(\rho K)/\log L$.*

Whereas directly verifying Assumption 4.1.2 is often difficult, as shown in Section 4.2.1, certain conditions from the research on sub-GHKE are equivalent to PHI and more robust. Indeed those equivalent conditions are key to our proof of Propositions 4.2.2 and 4.2.5.

In the context of Example 4.2.4, for carpets with central block of size b^d removed (so $K = L^d - b^d$), for some $1 \leq b \leq L-1$, one always have $\rho > 1$ when $d = 2$ (see [7, LHS of (5.9)]), hence by Theorem 4.1.4(a) no cutoff for the corresponding lamplighter chain. In contrast, from [7, RHS of (5.9)] we know that $\rho < 1$ for high-dimensional carpets of small central hole (specifically, whenever $b^{d-1} < L^{d-1} - L$), so by Theorem 4.1.4(b) the corresponding lamplighter chains then admit cutoff at $a_N = \frac{1}{2} T_{\text{cov}}(G^{(N)})$.

4.2.1 Stability of heat kernel estimates and parabolic Harnack inequality

We recall here various stability results for Heat Kernel Estimates (HKE) and Parabolic Harnack Inequalities (PHI), in case of a countably infinite weighted graph (G, μ). To this end, we assume

- *Uniform ellipticity:* $c_{\text{e}}^{-1} \leq \mu_{xy} \leq c_{\text{e}}$ for some $c_{\text{e}} < \infty$ and all $xy \in E(G)$,

- p_0-*condition:* $\frac{\mu_{xy}}{\mu_x} \geq p_0$ for some $p_0 > 0$ and all $xy \in E(G)$,

and recall few relevant properties of such (G, μ).

Definition 4.2.6. *Consider the following properties for $d_w \geq 2$ and $d_f \geq 1$:*

- *(VD) There exists $C_{\mathrm{D}} < \infty$ such that*

$$V(x, 2r) \leq C_{\mathrm{D}} V(x, r) \qquad \text{for all } x \in V(G) \text{ and } r \geq 1.$$

- *(V(d_f)) There exists $C_{\mathrm{V}} < \infty$ such that*

$$C_{\mathrm{V}}^{-1} r^{d_f} \leq V(x, r) \leq C_{\mathrm{V}} r^{d_f} \qquad \text{for all } x \in V(G) \text{ and } r \geq 1.$$

- *(CS(d_w)) There exist $\theta > 0$, $C_{\mathrm{CS}} < \infty$ and for each $z_0 \in V(G)$, $R \geq 1$ there exists a cut-off function $\psi = \psi_{z_0,R} : V(G) \to \mathbb{R}$ such that:*

  (a) $\psi(x) \geq 1$ when $d(x, z_0) \leq R/2$, while $\psi(x) \equiv 0$ when $d(x, z_0) > R$,

  (b) $|\psi(x) - \psi(y)| \leq C_{\mathrm{CS}} \left(d(x, y)/R\right)^{\theta}$,

  (c) for any $z \in V(G)$, $f : B(z, 2s) \to \mathbb{R}$ and $1 \leq s \leq R$

$$\sum_{x \in B(z,s)} f(x)^2 \sum_{y \in V(G)} |\psi(x) - \psi(y)|^2 \mu_{xy}$$
$$\leq C_{\mathrm{CS}}^2 \Big(\frac{s}{R}\Big)^{2\theta} \Big(\sum_{x,y \in B(z,2s)} |f(x) - f(y)|^2 \mu_{xy} + s^{-d_w} \sum_{y \in B(z,2s)} f(y)^2 \mu_y \Big).$$

- *(PI(d_w)) There exists $C_{\mathrm{PI}} < \infty$ such that*

$$\sum_{x \in B(z,R)} (f(x) - \bar{f}_{B(z,R)})^2 \mu_x \leq C_{\mathrm{PI}}\, R^{d_w} \sum_{x,y \in B(z,2R)} (f(x) - f(y))^2 \mu_{xy}$$

  for all $R \geq 1$, $x \in V(G)$ and $f : V(G) \to \mathbb{R}$, where $\bar{f}_{B(z,R)} = \frac{1}{V(z,R)} \sum_{x \in B(z,R)} f(x) \mu_x$.

- *(HKE(d_w)) There exists $C_{\mathrm{HK}} < \infty$ such that*

$$p_t(x, y) \leq \frac{C_{\mathrm{HK}}}{V(x, t^{1/d_w})} \exp\Big[- \frac{1}{C_{\mathrm{HK}}} \Big(\frac{d(x, y)^{d_w}}{t}\Big)^{1/(d_w - 1)} \Big]$$

  for all $x, y \in V(G)$ and $t \geq 0$, whereas

$$p_t(x, y) + p_{t+1}(x, y) \geq \frac{1}{C_{\mathrm{HK}} V(x, t^{1/d_w})} \exp\Big[- C_{\mathrm{HK}} \Big(\frac{d(x, y)^{d_w}}{t}\Big)^{1/(d_w - 1)} \Big]$$

  for all $x, y \in V(G)$ and $t \geq d(x, y)$.

- *(PHI(d_w)) There exists $C_{\mathrm{PHI}} < \infty$ such that if $u : [0,\infty) \times V(G) \to [0,\infty)$ satisfies*

$$u(t+1,x) - u(t,x) = (P-I)u(t,x), \qquad \forall\,(t,x) \in [0,4T] \times B(x_0,2R)$$

for some $x_0 \in V(G)$, $T \geq 2R$ with $T \asymp R^{d_w}$, then, for such x_0, T, R,

$$\max_{\substack{z\in B(x_0,R)\\ s\in[T,2T]}} u(s,z) \leq C_{\mathrm{PHI}} \min_{\substack{z\in B(x_0,R)\\ s\in[3T,4T]}} \{u(s,z) + u(s+1,z)\}.$$

Theorem 4.2.7 ([8, Theorems 1.2, 1.5])**.** *The following are equivalent for any uniformly elliptic, countably infinite (G,μ) satisfying the p_0-condition:*

(a) (VD), (PI(d_w)) and (CS(d_w)).

(b) (HKE(d_w)).

(c) (PHI(d_w)).

Note that in each implication of Theorem 4.2.7 the resulting values of $(C_{\mathrm{D}}, C_{\mathrm{PI}}, \theta, C_{\mathrm{CS}})$, C_{HK} and C_{PHI} depend only on p_0, c_{e}, d_w and the assumed constants. For example, in (PHI(d_w))$\Rightarrow$ (HKE(d_w)), the value of C_{HK} depends only on C_{PHI}, p_0, c_{e} and d_w.

The stability of such equivalence involves the following notion of rough isometry (see [37, Definition. 5.9]).

Definition 4.2.8. *Weighted graphs $(G^{(1)}, \mu^{(1)})$ and $(G^{(2)}, \mu^{(2)})$ are rough isometric if there exist $C_{\mathrm{QI}} < \infty$ and a map $T : V^{(1)} \to V^{(2)}$ such that*

$$\begin{aligned} C_{\mathrm{QI}}^{-1}\, d^{(1)}(x,y) - C_{\mathrm{QI}} \leq d^{(2)}(T(x),T(y)) &\leq C_{\mathrm{QI}}\, d^{(1)}(x,y) + C_{\mathrm{QI}}\,, &\qquad \forall x,y \in V^{(1)},\\ d^{(2)}(x', T(V^{(1)})) &\leq C_{\mathrm{QI}}, &\qquad \forall x' \in V^{(2)},\\ C_{\mathrm{QI}}^{-1}\, \mu^{(1)}_x \leq \mu^{(2)}_{T(x)} &\leq C_{\mathrm{QI}}\, \mu^{(1)}_x, &\qquad \forall x \in V^{(1)}, \end{aligned}$$

where $d^{(i)}(\cdot,\cdot)$ and $V^{(i)}$ denote the graph distance and vertex set of $G^{(i)}$, $i = 1,2$, respectively. Similarly, weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_N$ are uniformly rough isometric to a fixed, weighted graph (G,μ) if each $(G^{(N)}, \mu^{(N)})$ is rough isometric to (G,μ) for some $C_{\mathrm{QI}} < \infty$ which does not depend on N.

Recall [37, Lemma 5.10], that rough isometry is an equivalence relation. Further, (VD), (PI(d_w)) and (CS(d_w)) are stable under rough isometry. That is,

Theorem 4.2.9 ([37, Proposition 5.15])**.** *Suppose* $(G^{(1)}, \mu^{(1)})$ *and* $(G^{(2)}, \mu^{(2)})$ *have the* p_0*-condition and are rough isometric with constant* C_{QI}*. If* $(G^{(1)}, \mu^{(1)})$ *satisfies (VD), (PI(*d_w*)), (CS(*d_w*)) with constants* $(C_{\mathrm{D}}, C_{\mathrm{PI}}, \theta, C_{\mathrm{CS}})$*, then so does* $(G^{(2)}, \mu^{(2)})$ *with constants which depend only on* $(C_{\mathrm{D}}, C_{\mathrm{PI}}, \theta, C_{\mathrm{CS}})$*,* d_w*,* p_0 *and* C_{QI}*.*

Combining Theorems 4.2.7 and 4.2.9 we have the following useful corollary.

Corollary 4.2.10. *Suppose uniformly elliptic weighted graphs* $\{(G^{(N)}, \mu^{(N)})\}_N$ *satisfy the* p_0*-condition and are uniformly rough isometric to some countably infinite uniformly elliptic* (G, μ) *that also has the* p_0*-condition. If* (G, μ) *further satisfies (PHI(*d_w*)), then so do* $\{(G^{(N)}, \mu^{(N)})\}_N$ *with finite constant* C'_{PHI} *which is independent of* N*.*

4.2.2 Proof of Propositions 4.2.2 and 4.2.5

Proof of Proposition 4.2.2. Recall that for random walks on the Sierpinski gasket, namely $\mu_{xy} \equiv 1$ and the limit graph G of Example 4.2.1 (or its d-dimensional analog, $d \geq 3$), Jones [41, Theorems 17,18] established (HKE(d_w)), which by Theorem 4.2.7 implies that such (G, μ) must also satisfy (PHI(d_w)).

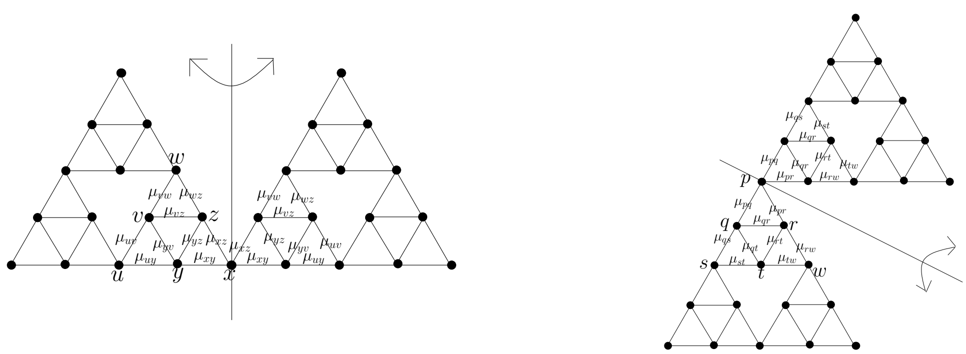

Figure 4.3: The construction of a weighted graph $(G^{(N+1)}, \mu^{(N+1)})$ for a given $(G^{(N)}, \mu^{(N)})$.

Proceeding to construct for each $N \geq 1$ a new weighted graph $(G, \mu'^{(N)})$, recall that $G^{(N+1)}$ consists of three copies $G^{(N,i)}$ of $G^{(N)}$, with $2^N x_i \in G^{(N,i)}$ for $i = 0, 1, 2$. Note that $G^{(N,0)} = G^{(N)}$ whereas each $G^{(N,i)}$, $i = 1, 2$ is the reflection of $G^{(N,0)}$ across a certain line $\ell^{N,i}$. Reflecting the weight $\mu^{(N,0)} := \mu^{(N)}$ on $G^{(N,0)}$, across $\ell^{N,i}$ yields weights $\mu^{(N,i)}$ on $G^{(N,i)}$, $i = 1, 2$ (see Figure 4.3). With $\{\mu^{(N,i)}, i = 0, 1, 2\}$ forming a new weight on $G^{(N+1)} \subset G$, we thus set

$$\mu'^{(N)}_{xy} := \begin{cases} \mu^{(N,i)}_{xy}, & \text{if } xy \in E(G^{(N+1)}), \\ 1, & \text{otherwise.} \end{cases} \tag{4.2.1}$$

Fixing a solution $u^{(N)} : [0,\infty) \times V(G^{(N)}) \to [0,\infty)$ of the heat equation (4.1.5) on the time-space cylinder of center $y_0 \in V(G^{(N)})$ and size $2R \leq T \asymp R^{d_w}$, $R \leq \frac{1}{4}R_N$, we extend $u^{(N)}(t,\cdot)$ to the non-negative function on $V(G)$

$$\tilde{u}^{(N)}(t,x) := \begin{cases} u^{(N)}(t,x), & \text{if } x \in V(G^{(N)}), \\ u^{(N)}(t,x'), & \text{if } x \notin V(G^{(N)}) \text{ and } x' \text{ are symmetric WRT } \ell^{N,1} \text{ or } \ell^{N,2}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.2}$$

Having $R \leq \frac{1}{4}R_N$ guarantees that $B_G(y_0, 2R) \subseteq G^{(N+1)}$, hence from our construction of $\mu'^{(N)}$ it follows that $\tilde{u}^{(N)}(t,x)$ satisfy the heat equation corresponding to $(G, \mu'^{(N)})$ on the time-space cylinder defined by (y_0, R, T). Since G has uniformly bounded degrees, the weighted graphs $\{(G, \mu'^{(N)})\}_N$ satisfy a p'_0-condition (for some $p'_0 > 0$ independent of N). Further, $\{(G, \mu'^{(N)})\}_N$ are uniformly rough isometric to (G, μ) (thanks to the uniform ellipticity of $\mu^{(N)}$). Hence, by Corollary 4.2.10, for some $C'_{\text{PHI}} < \infty$, which does not depend on N, nor on the specific choice of y_0, R and T,

$$\max_{\substack{t\in[T,2T] \\ y\in B_G(y_0,R)}} \tilde{u}^{(N)}(t,y) \leq C'_{\text{PHI}} \min_{\substack{t\in[3T,4T] \\ y\in B_G(y_0,R)}} \{\tilde{u}^{(N)}(t,y) + \tilde{u}^{(N)}(t+1,y)\}. \tag{4.2.3}$$

Since $\tilde{u}^{(N)}$ of (4.2.2) coincides with $u^{(N)}$ on $B^{(N)}(y_0,R) \subseteq B_G(y_0,R)$, replacing $\tilde{u}^{(N)}$ and $B_G(y_0,R)$ in (4.2.3) by $u^{(N)}$ and $B^{(N)}(y_0,R)$, respectively, may only decrease its LHS and increase its RHS. That is, (4.2.3) applies also for $u^{(N)}(\cdot,\cdot)$ and $B^{(N)}(y_0,R)$. This holds for all N and any of the preceding choices of y_0, R, T, yielding Assumption 4.1.2, as stated. □

Proof of Proposition 4.2.5. Consider the random walk, namely $\mu_{xy} \equiv 1$, on a limiting graph G that corresponds to a generalized Sierpinski carpet, as in Example 4.2.4. Clearly, (G,μ) is uniformly elliptic and of uniformly bounded degrees (so p_0-condition holds as well). Further, such random walk has properties (V(d_f)) and (HKE(d_w)), with $d_f = \log K / \log L \geq 1$ and $d_w = \log(\rho K)/\log L$ (see [6]). In particular, by Theorem 4.2.7 (G,μ) satisfies (PHI(d_w)). With $G^{(N+1)}$ consisting of K copies of $G^{(N)}$, we extend the given weight $\mu^{(N)}$ on $G^{(N)}$ to a weight $\mu'^{(N)}$ on G. Specifically, the weight on the edges of the reflected part of $G^{(N)}$, as in Figure 4.4, is $\mu_e'^{(N)} = K_e \mu_{e'}^{(N)}$, where $K_e \in [1, K]$ is the number of overlaps of e, and e' is the edge which moves to e by the reflection (so in Figure 4.4, we set $\mu_e'^{,(N)} = 2\mu_e^{(N)}$ for each edge e lying on a reflection axis). Taking $\mu_e'^{(N)} \equiv 1$ for all other $e \in E(G)$, the graphs $\{(G, \mu'^{(N)})\}_N$ are uniformly elliptic, satisfy a p_0'-condition (for some $p_0' > 0$ independent of N), and are uniformly rough isometric to (G, μ). Thus, by Corollary 4.2.10 the (PHI(d_w)) holds for $\{(G, \mu'^{(N)})\}_N$ with a constant C'_{PHI} which does not depend on N. Fixing center $y_0 \in V(G^{(N)})$ and size parameters $2R \leq T \asymp R^{d_w}$,

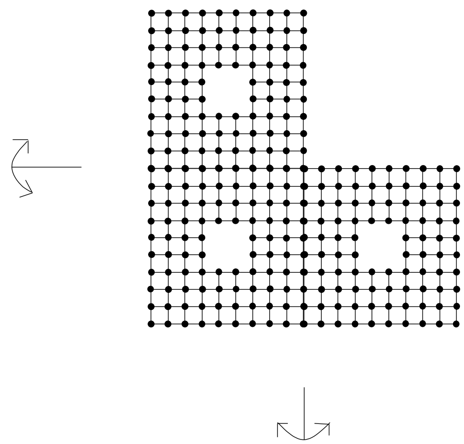

Figure 4.4: An example of the reflection

$R \leq \frac{1}{4} R_N$, we extend any given solution $u^{(N)} : [0, \infty) \times V(G^{(N)}) \to [0, \infty)$ of the heat equation (4.1.5) on the corresponding time-space cylinder, to the non-negative $\tilde{u}^{(N)} : [0, \infty) \times V(G) \to [0, \infty)$, symmetrically along reflections, analogously to (4.2.2). Since $R \leq \frac{1}{4} \operatorname{diam}\{G^{(N)}\}$ all edges of $B_G(y_0, 2R)$ not in $G^{(N)}$ are among those reflected to $G^{(N)}$, with our construction of $\mu'^{(N)}$ guaranteeing that $\tilde{u}^{(N)}(\cdot, \cdot)$ satisfy the heat equation on the corresponding time-space cylinder of $(G, \mu'^{(N)})$. Thanks to the (PHI(d_w)) for $\{(G, \mu'^{(N)})\}_N$, we have (4.2.3), and since $\tilde{u}^{(N)}$ coincides with $u^{(N)}$ on $B^{(N)}(y_0, R) \subseteq B_G(y_0, R)$, the same applies when replacing $\tilde{u}^{(N)}$ and $B_G(y_0, R)$ by $u^{(N)}$ and $B^{(N)}(y_0, R)$, respectively. As in our proof of Proposition 4.2.2, this holds for all relevant values of N, y_0, R and T, thereby establishing Assumption 4.1.2. $\square$

4.3 Random walk consequences of Assumptions 4.1.1 and 4.1.2

We summarize here those consequences of Assumptions 4.1.1 and 4.1.2 we need for Theorem 4.1.4, starting with sub-GHKE, an upper bound on the uniform mixing times and a covering statement which are applicable for all values of (d_f, d_w). Then, focusing in Section 4.3.1 on the case $d_f < d_w$, we control $R^{(N)}_{\text{eff}}(\cdot,\cdot)$ and relate it to T_N of (4.1.10), complemented in Section 4.3.2 by upper bounds on the Green functions, in case $d_f > d_w$. We provide only proof outlines since most of these results, and their proofs, are pretty standard.

Our first result is the uniform sub-GHKE one has on $\{(G^{(N)}, \mu^{(N)})\}_{N\ge 1}$, up to time of order T_N.

Proposition 4.3.1. *Under Assumptions 4.1.1 and 4.1.2, for any $\eta < \infty$, there exist $c_{\text{HK}} = c_{\text{HK}}(\eta) < \infty$, such that for all N, any $x, y \in V(G^{(N)})$ and $t \le \eta T_N$,*

$$p^{(N)}_t(x,y) \le \frac{c_{\text{HK}}}{t^{d_f/d_w}} \exp\Big[-\frac{1}{c_{\text{HK}}}\Big(\frac{d^{(N)}(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big]. \qquad (4.3.1)$$

Further, for all N, any $x, y \in V(G^{(N)})$ and $d^{(N)}(x,y) \le t \le \eta T_N$,

$$p^{(N)}_t(x,y) + p^{(N)}_{t+1}(x,y) \ge \frac{1}{c_{\text{HK}} t^{d_f/d_w}} \exp\Big[-c_{\text{HK}}\Big(\frac{d^{(N)}(x,y)^{d_w}}{t}\Big)^{1/(d_w-1)}\Big]. \qquad (4.3.2)$$

Proof. (Sketch:) This is a finite graph analogue of (PHI(d_w))⇒ (HKE(d_w)) of Theorem 4.2.7, which is standard for a countably infinite weighted graph (see [33, Theorem 3.1, (ii) ⇒ (i)]). Such implication holds also for metric measure space with a local regular Dirichlet form, as [11, Theorem 3.2 (c') ⇒ (a'')], and we sketch below how to adapt the latter proof, specifically [11, Sections 4.3 and 5], to the finite graph setting. First note that for $t \le \eta T_N$ the derivation of the (near-)diagonal upper-bound (4.3.1) (without the exponential term), follows as in the proof of [72, Proposition 7.1]. Setting $p^{(N,x,R)}_t$ for the heat kernel of the process killed upon exiting $B^{(N)}(x,R)$, upon adapting the arguments in [11, Section 4.3.4], one thereby establishes the corresponding (near)-diagonal lower bound, analogous to [11, (4.63)]. Namely, showing that for some $c'_{\text{PHI}} \in (0,1)$ and $c'_{\text{HK}} = c'_{\text{HK}}(\eta')$ finite, any $\eta' < \infty$, all $N \ge 1$, $x \in V(G^{(N)})$ and $R \le c'_{\text{PHI}} R_N$, if $c'_{\text{HK}} d^{(N)}(x,y) \le t^{1/d_w} \le \eta' R$, then

$$p^{(N,x,R)}_t(x,y) + p^{(N,x,R)}_{t+1}(x,y) \ge \frac{1}{c'_{\text{HK}} t^{d_f/d_w}}. \qquad (4.3.3)$$

Combining (4.3.3) and the (near-)diagonal upper bound, one then deduces (4.3.1) as done in [11, Sections 4.3.5-4.3.6]. Similarly, by adapting the proof of [11, Proposition 5.2(i) and (iii)], the near-diagonal lower bound (4.3.3) yields the full lower-bound of (4.3.2). Since all these arguments involve only η and the constants from Assumptions 4.1.1-4.1.2, we can indeed choose the constant $c_{\text{HK}}(\eta)$ in (4.3.1)-(4.3.2) independently of N. $\square$

Proposition 4.3.1 has the following immediate consequence.

Corollary 4.3.2. *Under Assumptions 4.1.1 and 4.1.2 there exist R_0 and c_2 finite, such that for any $N \geq 1$, $x \in V(G^{(N)})$ and $R_0 \leq r \leq R_N$*

$$P_x\big(\max_{0\leq j\leq t} d^{(N)}(x, X_j^{(N)}) \leq r\big) \geq c_2^{-1}\exp(-c_2 t/r^{d_w})\,.$$

Proof. Using the same arguments as in the proof of [50, Proposition 3.3], from (4.3.1) and (4.3.2) we get the finite graph analogs of [50, Lemma 3.1] and [50, Lemma 3.4], respectively. Combining these bounds and the Markov property, as done in [50, Lemma 3.5], results with the stated bound for $k[r^{d_w}] \leq t < (k+1)[r^{d_w}]$. All steps of the proof involve only our universal constants c_{e}, c_{v}, p_0, C_{PHI}, c_{PHI}, c_{HK} and with $X_j^{(N)}$ confined to certain balls, having our sub-HKE restricted to $t \leq \eta T_N$ is immaterial here. $\square$

Another consequence of (4.3.1) is the following upper bound on uniform mixing times.

Proposition 4.3.3. *Suppose Assumptions 4.1.1 and 4.1.2 hold. Then, for the invariant measures $\pi^{(N)}(\cdot)$ of* (4.1.8)*, some finite $c(\cdot)$, all $N \geq 1$ and $\epsilon > 0$,*

$$T^U_{\text{mix}}(\epsilon, G^{(N)}) := \min\Big\{t \geq 0 \,\Big|\, \max_{x,y\in V(G^{(N)})}\Big|\frac{P_x(\tilde{X}_t = y; G^{(N)})}{\pi^{(N)}(y)} - 1\Big| \leq \epsilon\Big\} \leq c(\epsilon) T_N\,. \tag{4.3.4}$$

For the proof of Proposition 4.3.3, consider the normalized Dirichlet forms of $\tilde{X}^{(N)}$ and $X^{(N)}$,

$$\mathcal{E}^{(N)}_{\text{norm}}(f,f) := -\langle f, (P^{(N)} - I)f\rangle_{\pi^{(N)}},$$
$$\tilde{\mathcal{E}}^{(N)}_{\text{norm}}(f,f) := -\langle f, (\tilde{P}^{(N)} - I)f\rangle_{\pi^{(N)}} = \frac{1}{2}\mathcal{E}^{(N)}_{\text{norm}}(f,f)\,.$$

Let $\mathcal{H}_0^+(S) := \{f : V(G^{(N)}) \to [0,\infty) \mid f \text{ not a constant function, } \mathrm{Supp}\{f\} \subseteq S\}$ for $S \subseteq V(G^{(N)})$ and define the spectral quantities

$$\lambda^{(N)}(S) := \inf \left\{ \frac{\mathcal{E}^{(N)}_{\mathrm{norm}}(f,f)}{\mathrm{Var}^{\pi^{(N)}}(f)} \;\middle|\; f \in \mathcal{H}_0^+(S) \right\},$$
$$T\tilde{\lambda}^{(N)}(S) := \inf \left\{ \frac{\tilde{\mathcal{E}}^{(N)}_{\mathrm{norm}}(f,f)}{\mathrm{Var}^{\pi^{(N)}}(f)} \;\middle|\; f \in \mathcal{H}_0^+(S) \right\} = \frac{1}{2}\lambda^{(N)}(S)\,.$$

Recall the following upper bound on uniform mixing times in terms of the corresponding spectral profile.

Lemma 4.3.4 ([30, Corollary 2.1])**.** *For $r \geq \pi_*^{(N)} := \inf_{x\in V(G^{(N)})}\{\pi^{(N)}(x)\}$, let*

$$\tilde{\Lambda}^{(N)}(r) := \inf \left\{ \tilde{\lambda}^{(N)}(S) \,|\, \pi^{(N)}(S) \leq r \right\}\,.$$

Then, for any $\epsilon > 0$ and all N,

$$T^U_{\mathrm{mix}}(\epsilon, G^{(N)}) \leq \int_{4\pi_*^{(N)}}^{4/\epsilon} \frac{4\,dr}{r\tilde{\Lambda}^{(N)}(r)}\,. \tag{4.3.5}$$

Our next lemma controls the spectral profiles on the LHS of (4.3.5) en-route to Proposition 4.3.3.

Lemma 4.3.5 (Faber-Krahn inequality)**.** *For any N and $S \subseteq V(G^{(N)})$ let*

$$\lambda_1^{(N)}(S) := \inf \left\{ \frac{\mathcal{E}^{(N)}(f,f)}{\|f\|_{L^2(\mu^{(N)})}} \;\middle|\; f \in \mathcal{H}_0(S) \right\}, \tag{4.3.6}$$

where $\mathcal{H}_0(S) := \{f : V(G^{(N)}) \to \mathbb{R} \mid \mathrm{Supp}\{f\} \subseteq S\}$. If Assumptions 4.1.1 and 4.1.2 hold, then for some $c_{\mathrm{FK}} > 0$ and all N,

$$\lambda_1^{(N)}(S) \geq c_{\mathrm{FK}}\, \mu^{(N)}(S)^{-d_w/d_f} \qquad \forall S \subseteq V(G^{(N)})\,. \tag{4.3.7}$$

Proof. (Sketch) For countably infinite (G,μ) satisfying the p_0-condition, such Faber-Krahn inequality is a standard consequence of $(\mathrm{V}(d_f))$ and the on-diagonal $(\mathrm{HKE}(d_w))$ upper bound. Indeed, its proof in [21, Theorem 5.4], while written for $d_w = 2$, is easily adapted to any $d_w > 0$, upon suitably adjusting various exponents (e.g. taking $\nu = d_w/d_f$ and $r = t^{1/d_w}$, c.f. the discussion in [32, Proposition 5.1]). To get (4.3.7) one instead relies on (4.3.1) at $y = x$, and on Assumption 4.1.1, noting that all steps of the proof involve only the universal d_f, d_w, p_0, c_{e}, c_{v} and c_{HK}. Further, following the proof of [21, Theorem 5.4] it now suffices to take only $r \leq R_N$, hence $t \leq \eta T_N$ for some fixed $\eta < \infty$. ◻

Proof of Proposition 4.3.3. Recall that $\mu^{(N)}(S) = \pi^{(N)}(S)\mu^{(N)}(V(G^{(N)}))$. By (4.3.6) we further have that $\lambda^{(N)}(S) \geq \lambda_1^{(N)}(S)$ for any choice of S and N, hence Lemma 4.3.5 results with

$$\tilde{\Lambda}^{(N)}(r) \geq \frac{c_{\mathrm{FK}}}{2}\Big[r\,\mu^{(N)}(V(G^{(N)}))\Big]^{-d_w/d_f}. \tag{4.3.8}$$

By the assumed d_f-set condition, $\mu^{(N)}(V(G^{(N)})) \leq c_{\mathrm{v}}\ \mathrm{diam}\{G^{(N)}\}^{d_f}$. Thus, combining (4.3.5) and (4.3.8) yields the bound

$$T^U_{\mathrm{mix}}(\epsilon, G^{(N)}) \leq \frac{8}{c_{\mathrm{FK}}}\frac{d_f}{d_w}\Big(\frac{4\,c_{\mathrm{v}}}{\epsilon}\Big)^{d_w/d_f} T_N\,,$$

as claimed. □

We conclude with a very useful covering property.

Proposition 4.3.6. *Assumption 4.1.1 implies that for any $\eta \in (0,1]$, there exist $L = L(\eta, d_f, \tilde{c}\,c_{\mathrm{v}}) < \infty$ such that each $G^{(N)}$ can be covered by L balls $\{B^{(N)}(x_i, \eta\,R_N)\}_{i=1}^L$ of $V(G^{(N)})$.*

Proof. Covering $V(G^{(N)})$ by a single ball of radius R_N, thanks to (4.1.7) and the assumed d_f-set condition $\sharp V(G^{(N)}) \leq \tilde{c}\,c_{\mathrm{v}}(R_N)^{d_f}$. Further, $G^{(N)}$ can be covered by L balls $B^{(N)}(x_i, \eta R_N)$ such that $\{B^{(N)}(x_i, \eta R_N/2)\}$ are disjoint (e.g. [5, Lemma 6.2(a)]). Consequently, $L(\tilde{c}\,c_{\mathrm{v}})^{-1}(\eta R_N/2)^{d_f} \leq \sharp V(G^{(N)})$ and we conclude that $L \leq (\tilde{c}\,c_{\mathrm{v}})^2(2/\eta)^{d_f}$ for all N, as claimed. □

4.3.1 Strongly recurrent case: $d_f < d_w$

A consequence of Assumptions 4.1.1, 4.1.2 for $d_f < d_w$ is the following relation between the resistance metric and the graph distance.

Proposition 4.3.7. *Suppose Assumptions 4.1.1, 4.1.2 and $d_f < d_w$. Then, for some c_{R} finite, all $N \geq 1$ and any $x, y \in V(G^{(N)})$,*

$$c_{\mathrm{R}}^{-1}\,d^{(N)}(x,y)^{d_w-d_f} \leq R^{(N)}_{\mathrm{eff}}(x,y) \leq c_{\mathrm{R}}\,d^{(N)}(x,y)^{d_w-d_f}\,. \tag{4.3.9}$$

Proof. (Sketch:) For a single infinite weighted graph this is a well known consequence of (HKE(d_w)), see for example [10, Theorem 1.3]. In our setting, the upper bound on $R^{(N)}_{\mathrm{eff}}$ is derived from Proposition 4.3.1 by going via (PI(d_w)), as done in the proof of [10, Lemma 2.3(ii), Proposition 4.2(1)]. The corresponding lower bound in (4.3.9) is proved as in [10, Proposition 4.2(2)], by showing instead the property (SRL(d_w)) (see remark at [10, bottom of Pg. 1650]). As in Proposition 4.3.1, all steps use only constants from Assumptions 4.1.1–4.1.2 and require our sub-HKE only at $t \leq \eta_0 T_N$. Hence, we end with finite c_{R} which is independent of N. □

The following corollary of Proposition 4.3.7 is immediate.

Corollary 4.3.8. *Suppose Assumptions 4.1.1 and 4.1.2 hold for some* $d_f < d_w$ *and let*

$$r(G^{(N)}) := \max_{x,y \in V(G^{(N)})} \{R_{\text{eff}}(x,y)\}, \qquad S_N := \mu^{(N)}(V(G^{(N)}))r(G^{(N)}). \tag{4.3.10}$$

Then, for some finite $c_\star$

$$c_\star^{-1} T_N \le S_N \le c_\star T_N, \qquad \forall N \ge 1\,. \tag{4.3.11}$$

Proof. By our d_f-set condition $\mu^{(N)}(G^{(N)}) \asymp (R_N)^{d_f}$, whereas $r(G^{(N)}) \asymp (R_N)^{d_w - d_f}$, thanks to Proposition 4.3.7. With $T_N := (R_N)^{d_w}$ we are thus done. □

4.3.2 Transient case: $d_f > d_w$

When $d_f > d_w$, Proposition 4.3.3 and (4.3.1) yield the following decay rate of the Green functions.

Proposition 4.3.9. *Suppose Assumptions 4.1.1, 4.1.2 and* $d_f > d_w$*. Then, for some* $c_{\text{g}}(\cdot)$ *finite, any* $\epsilon > 0$ *and finite* N*,*

$$\tilde{g}^{(N)}(x,y) := \sum_{t=0}^{T^U_{\text{mix}}(\epsilon, G^{(N)})} \tilde{p}^{(N)}_t(x,y) \le c_{\text{g}}(\epsilon)\, d^{(N)}(x,y)^{d_w - d_f}, \qquad \forall y \ne x \in V(G^{(N)})\,. \tag{4.3.12}$$

Proof. Clearly $\tilde{p}^{(N)}_t(x,y) = \sum_s q_t(s) p^{(N)}_s(x,y)$ with $q_t(s)$ the probability that a Binomial$(t, 1/2)$ equals s. Consequently, $\tilde{g}^{(N)}(x,y) \le 2g^{(N)}(x,y)$ (since $\sum_t q_t(s) = 2$). We further replace $T^U_{\text{mix}}(\epsilon, G^{(N)})$ in (4.3.12) by ηT_N, for $\eta := c(\epsilon)$ of Proposition 4.3.3. Hence, from (4.3.1) for some $c_{\text{HK}} = c_{\text{HK}}(\eta)$, all N and $x \ne y$,

$$\tilde{g}^{(N)}(x,y) \le 2c_{\text{HK}} \sum_{t=1}^{\infty} t^{-d_f/d_w} \exp\Big[- c_{\text{HK}}^{-1} \Big(\frac{d^{(N)}(x,y)^{d_w}}{t} \Big)^{1/(d_w - 1)} \Big]\,.$$

Since $d_f/d_w > 1$, the series on the RHS converges (even when $d^{(N)}(x,y) = 0$), and it is easy to further bound it by $c'_{\text{g}}\, d^{(N)}(x,y)^{d_w - d_f}$ for some $c'_{\text{g}} = c'_{\text{g}}(c_{\text{HK}})$ finite, as we claim in (4.3.12). □

4.4 Cover time: Proof of Proposition 4.1.5

We recall S_N, $r(G^{(N)})$ of (4.3.10) and use the following notations for $x, y \in V(G^{(N)})$, $r \in [0,1]$,

$$\widehat{R}_{\text{eff}}^{(N)}(x,y) := \frac{R_{\text{eff}}^{(N)}(x,y)}{r(G^{(N)})} \in [0,1], \qquad B_R^{(N)}(x,r) := \{y \in V(G^{(N)}) \mid \widehat{R}_{\text{eff}}^{(N)}(x,y) \le r\}. \tag{4.4.1}$$

We show in Lemma 4.4.1 that for some $\epsilon' > 0$, with positive probability, during its first S_N steps, a random walk on $G^{(N)}$ makes at least $\epsilon' \, r(G^{(N)})$ visits to the starting point. Combining this with the modulus of continuity of the relevant local times (of Lemma 4.4.2), we show in Proposition 4.4.3 and Corollary 4.4.4 that for some $\kappa > 0$, with positive probability, by time $4S_N$ a (small) ball $B_R^{(N)}(x,\kappa)$ is covered by the random walk trajectory. In view of Propositions 4.3.6 and 4.3.7, if in addition $d_f < d_w$, then for some $L = L(\kappa, c_{\text{R}})$ finite and all N, the set $V(G^{(N)})$ is covered by some $\{B_R^{(N)}(z_i,\kappa)\}_{i=1}^L$. Proposition 4.1.5 then follows by using this fact, the Markov property and having $S_N \asymp T_N$ (see Corollary 4.3.8).

We now implement the details of the preceding proof strategy.

Lemma 4.4.1. *Under Assumptions 4.1.1 and 4.1.2, there exists $\epsilon > 0$ such that*

$$\max_{N \ge 1} \max_{x \in V(G^{(N)})} P_x\left(\widehat{L}_{S_N}^{(N)}(x) \le 2\epsilon\right) \le \frac{1}{8}, \qquad \widehat{L}_t^{(N)}(x) := \frac{1}{r(G^{(N)})\mu_x^{(N)}} \sum_{s=0}^{t-1} 1_x(X_s^{(N)}). \tag{4.4.2}$$

Proof. Recall that the successive times in which the walk $X_t^{(N)}$ re-visits $x = X_0^{(N)}$, form a partial sum, whose i.i.d. $\mathbb{N}$-valued increments $\{\eta_x^{(N)}(i)\}_{i \ge 1}$ have mean

$$E_x\big[\eta_x^{(N)}\big] = \frac{1}{\pi^{(N)}(x)} = \frac{\mu^{(N)}(G^{(N)})}{\mu_x^{(N)}}.$$

Setting $m_x^{(N)} := [2\epsilon \, \mu_x^{(N)} \, r(G^{(N)})]$ we thus have by Markov's inequality that

$$P_x\left(\widehat{L}_{S_N}^{(N)}(x) \le 2\epsilon\right) = P_x\Big(\sum_{i=1}^{m_x^{(N)}} \eta_x^{(N)}(i) \ge S_N\Big) \le \frac{m_x^{(N)}}{S_N} E_x\big[\eta_x^{(N)}\big] \le 2\epsilon,$$

yielding (4.4.2) when $\epsilon \le 2^{-4}$. □

With our graphs having uniform volume growth, [22, Theorem 1.4] applies here, giving the following modulus of continuity result.

Lemma 4.4.2. *Suppose Assumptions 4.1.1 and 4.1.2. Then, for* $\varphi(\kappa) := \sqrt{\kappa(1+|\log\kappa|)}$ *we have that*

$$\Delta(\lambda) := \sup_{\kappa\in(0,1],N\geq 1}\ \sup_{z\in V(G^{(N)})} P_z\Big(\max_{t\leq S_N}\ \max_{\substack{x,y\in V(G^{(N)})\\ \widehat{R}^{(N)}_{\text{eff}}(x,y)\leq\kappa}} |\widehat{L}^{(N)}_t(x)-\widehat{L}^{(N)}_t(y)|\geq\lambda\varphi(\kappa)\Big)\to 0 \tag{4.4.3}$$

as $\lambda\to\infty$.

Combining Lemmas 4.4.1 and 4.4.2 yields the following uniform lower bound on the minimum over $y\in B^{(N)}_R(x,\kappa)$, of the normalized local time at y during the first $4S_N$ moves of the random walker.

Proposition 4.4.3. *Under Assumptions 4.1.1 and 4.1.2, for some positive* ϵ,κ

$$\inf_{N\geq 1}\ \inf_{x,z\in V(G^{(N)})} P_z\Big(\min_{y\in B^{(N)}_R(x,\kappa)}\{\widehat{L}^{(N)}_{4S_N}(y)\}\geq\epsilon\Big)\geq\frac{1}{2}\,. \tag{4.4.4}$$

Proof. Step 1. Taking $\epsilon>0$ as in Lemma 4.4.1, we first show that for some $\kappa>0$,

$$\inf_{N\geq 1}\ \inf_{x\in V(G^{(N)})} P_x\Big(\min_{y\in B^{(N)}_R(x,\kappa)}\Big\{\widehat{L}^{(N)}_{S_N}(y)\Big\}\geq\epsilon\ \Big)\geq\frac{3}{4}\,.$$

To this end considering Lemma 4.4.2 for $\lambda<\infty$ such that $\Delta(\lambda)<2^{-3}$ and $\kappa>0$ such that $\lambda\varphi(\kappa)\leq\epsilon$, we obtain that, for all N and any $z\in V(G^{(N)})$,

$$P_z\Big(\max_{x,y\in B^{(N)}_R(x,\kappa)}\Big\{|\widehat{L}^{(N)}_{S_N}(x)-\widehat{L}^{(N)}_{S_N}(y)|\Big\}\geq\epsilon\Big)\leq\frac{1}{8}\,.$$

Consequently, by Lemma 4.4.1,

$$\begin{aligned}
\frac{7}{8}&\leq P_x\Big(\widehat{L}^{(N)}_{S_N}(x)\geq 2\epsilon\Big)\\
&\leq\frac{1}{8}+P_x\Big(\widehat{L}^{(N)}_{S_N}(x)\geq 2\epsilon,\ \max_{y\in B^{(N)}_R(x,\kappa)}\Big\{|\widehat{L}^{(N)}_{S_N}(x)-\widehat{L}^{(N)}_{S_N}(y)|\Big\}\leq\epsilon\Big)\\
&\leq\frac{1}{8}+P_x\Big(\min_{y\in B^{(N)}_R(x,\kappa)}\Big\{\widehat{L}^{(N)}_{S_N}(y)\Big\}\geq\epsilon\Big)\,,
\end{aligned}$$

thereby completing Step 1.

Step 2. Turning to prove (4.4.4) when $z \neq x$, let $\tau_x^{(N)} := \inf\{t \geq 0 \mid X_t^{(N)} = x\}$ denote the first hitting time of $x \in V(G^{(N)})$ by the random walk. Recall the commute time identity (see [52, Proposition 10.6]), that for any N and $x \neq z$ in $V(G^{(N)})$,

$$E_x\big[\tau_z^{(N)}\big] + E_z\big[\tau_x^{(N)}\big] = R_{\text{eff}}^{(N)}(z,x)\mu^{(N)}(G^{(N)})\,. \tag{4.4.5}$$

Hence,

$$P_z\left(\tau_x^{(N)} \geq 3S_N\right) \leq \frac{1}{3\,S_N} E_z\big[\tau_x^{(N)}\big] \leq \frac{1}{3} \tag{4.4.6}$$

so by the strong Markov property at $\tau_x^{(N)}$, we see that for any $z \in V(G^{(N)})$,

$$\begin{aligned}
&P_z\Big(\min_{y\in B_R^{(N)}(x,\kappa)} \{\widehat{L}_{4S_N}^{(N)}(y)\} \geq \epsilon\Big) \\
&\geq \sum_{t=0}^{3S_N} P_z\Big(\min_{y\in B_R^{(N)}(x,\kappa)} \{\widehat{L}_{4S_N}^{(N)}(y) - \widehat{L}_t^{(N)}(y)\} \geq \epsilon\,,\ \tau_x^{(N)} = t\Big) \\
&= \sum_{t=0}^{3S_N} P_z\big(\tau_x^{(N)} = t\big)P_x\Big(\min_{y\in B_R^{(N)}(x,\kappa)} \Big\{\widehat{L}_{4S_N-t}^{(N)}(y)\Big\} \geq \epsilon\Big) \\
&\geq P_z\left(\tau_x^{(N)} \leq 3S_N\right) P_x\Big(\min_{y\in B_R^{(N)}(x,\kappa)} \Big\{\widehat{L}_{S_N}^{(N)}(y)\Big\} \geq \epsilon\Big) \geq \frac{1}{2}\,,
\end{aligned}$$

by combining Step 1 and (4.4.6). □

Denoting the range of the random walk by $\text{Range}_t^{(N)} := \{X_0^{(N)}, X_1^{(N)}, \ldots, X_{t-1}^{(N)}\}$, we have the following consequence of Proposition 4.4.3.

Corollary 4.4.4. *If Assumptions 4.1.1 and 4.1.2 hold, then for some $\kappa > 0$ and any t,*

$$\sup_{N\geq 1}\sup_{x,z\in V(G^{(N)})} P_z\left(\mathit{Range}_t^{(N)} \not\supseteq B_R^{(N)}(x,\kappa)\right) \leq 2^{1-t/(4S_N)}\,.$$

Proof. Taking $\kappa > 0$ as in Proposition 4.4.3, we have that for all N and $x, z \in V(G^{(N)})$,

$$P_z\left(\text{Range}_{4S_N}^{(N)} \supseteq B_R^{(N)}(x,\kappa)\right) \geq \frac{1}{2}\,.$$

Applying the Markov property at times $\{4iS_N\}$ for $i = 1, \dots, k-1$, it follows that

$$P_z\left(\text{Range}^{(N)}_{4kS_N} \not\supseteq B^{(N)}_R(x,\kappa)\right) \le 2^{-k}$$

and we are done, since $t \mapsto \text{Range}^{(N)}_t$ is non-decreasing. $\square$

Proof of Proposition 4.1.5. From Proposition 4.3.7, if $c_{\mathrm{R}}^2\, \eta^{d_w - d_f} \le \kappa$, then for any N and $x \in V(G^{(N)})$,

$$B^{(N)}(x, \eta\, R_N) \subseteq B^{(N)}_R(x,\kappa)\,.$$

Setting such $\eta = \eta(c_{\mathrm{R}}, \kappa) > 0$ we deduce from Proposition 4.3.6 that for any $\kappa > 0$ there exist $L = L(\kappa)$ finite and $x_1, \dots, x_L \in V(G^{(N)})$, such that for all N,

$$V(G^{(N)}) = \bigcup_{i=1}^{L} B^{(N)}_R(x_i,\kappa)\,.$$

We embed the walk $X^{(N)}_s$ within the sample path $s \mapsto \tilde{X}^{(N)}_s$ of its lazy counterpart, such that the number of steps M_t made by the lazy walk during the first t steps of $\{X^{(N)}_s\}$ is the sum of t i.i.d. Geometric(1/2) variables, which are further independent of $\{X^{(N)}_s\}$. Since the range of the lazy random walk at time M_t is then $\text{Range}^{(N)}_t$, we have for any t, N and $z \in V(G^{(N)})$

$$P_z\left(\tau_{\text{cov}}(G^{(N)}) > 3t\right) \le P(M_t > 3t) + \sum_{i=1}^{L} P_z\left(\text{Range}^{(N)}_t \not\supseteq B^{(N)}_R(x_i,\kappa)\right)\,.$$

By Cramer-Chernoff bound, the first term on the RHS is at most θ^t for some $\theta < 1$. With $L = L(\kappa)$ independent of N, z, and $S_N \le c_\star T_N$ (see Corollary 4.3.8), we thus reach (4.1.11) upon choosing $\kappa > 0$ as in Corollary 4.4.4 and $c_0 \ge 2L(\kappa) + 1$ such that $e^{-3/c_0} \ge \max(\theta, 2^{-1/(4c_\star)})$. $\square$

4.5 Lamplighter mixing: Theorem 4.1.4 and Proposition 4.1.6

Proof of Proposition 4.1.6. WLOG we may and do assume that $\boldsymbol{x_0} = (\mathbf{0}, x_0)$ for some $x_0 \in V(G^{(N)})$. Let

$$A^*_N := \left\{(f,x) \in V(\mathbb{Z}_2 \wr G^{(N)}) \;\middle|\; \exists y \in V(G^{(N)}) \text{ such that } f(b) \equiv 0, \quad \forall b \in B^{(N)}(y, r_N)\right\},$$

where taking $r_N := \lceil (2d_f \tilde{c}\, c_{\mathrm{v}} \log_2 R_N)^{1/d_f} \rceil$ we have thanks to (4.1.7) and the d_f-set condition, that

$$\sharp B^{(N)}(y, r_N) \geq \tilde{c}^{-1} V^{(N)}(y, r_N) \geq (\tilde{c}\, c_{\mathrm{v}})^{-1} (r_N)^{d_f} \geq 2d_f \log_2 R_N \,.$$

By the same reasoning $\sharp V(G^{(N)}) \leq \tilde{c}\, c_{\mathrm{v}} (R_N)^{d_f}$, so for the invariant distribution $\pi^*(\cdot; G^{(N)})$ of the lamplighter chain $Y^{(N)}$ on $\mathbb{Z}_2 \wr G^{(N)}$

$$\pi^*(A_N^*; G^{(N)}) \leq \sum_{y \in V(G^{(N)})} 2^{-\sharp B^{(N)}(y, r_N)} \leq \tilde{c}\, c_{\mathrm{v}} (R_N)^{-d_f} \,. \tag{4.5.1}$$

Part of our d_f-set condition is having $R_N \to \infty$, so there exists N_1 finite such that $R_0 \leq r_N \leq \frac{1}{4} R_N$ for R_0 of Corollary 4.3.2 and any $N \geq N_1$. Since $\max_y \{d^{(N)}(x_0, y)\} \geq \frac{1}{2} R_N$ for any $x_0 \in V(G^{(N)})$, whenever $N \geq N_1$ the event

$$\tilde{\Gamma}_t^{(N)} := \Big\{ \max_{0 \leq s \leq t} d(\tilde{X}_0^{(N)}, \tilde{X}_s^{(N)}) \leq \frac{1}{4} R_N \Big\}$$

implies that $\{Y_t^{(N)} \in A_N^*\}$. Consequently, for any such N we have by (4.5.1) that

$$\begin{aligned} \max_{\boldsymbol{x} \in V(\mathbb{Z}_2 \wr G^{(N)})} \|P_t^*(\boldsymbol{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\mathrm{TV}} &\geq P_{\boldsymbol{x}_0}^*(Y_t^{(N)} \in A_N^*; G^{(N)}) - \pi^*(A_N^*; G^{(N)}) \\ &\geq P_{x_0}(\tilde{\Gamma}_t^{(N)}; G^{(N)}) - \tilde{c}\, c_{\mathrm{v}} (R_N)^{-d_f} \,. \end{aligned} \tag{4.5.2}$$

Let $c_1 := 4^{d_w} c_2$ for $c_2 < \infty$ of Corollary 4.3.2. Then, by Corollary 4.3.2 at $r = \frac{1}{4} R_N$, we have for all $N \geq N_1$

$$P_{x_0}\big(\tilde{\Gamma}_t^{(N)}; G^{(N)}\big) \geq P_x\Big(\max_{0 \leq s \leq t} d^{(N)}(X_0^{(N)}, X_s^{(N)}) \leq \frac{1}{4} R_N \Big) \geq c_1^{-1} e^{-c_1 t / T_N} \,, \tag{4.5.3}$$

which together with (4.5.2) completes the proof. □

As shown next, at $t \gg S_N$ the lazy walk is near equilibrium (in total variation), and the total variation distance of $P_t^*(\boldsymbol{x}, \cdot; G^{(N)})$ from its equilibrium law is then controlled by the tail probabilities of $\tau_{\mathrm{cov}}(G^{(N)})$.

Proposition 4.5.1. *For any t, weighted graphs $(G^{(N)}, \mu^{(N)})$ and $\boldsymbol{x} \in V(\mathbb{Z}_2 \wr G^{(N)})$,*

$$\begin{aligned} \|P_t^*(\boldsymbol{x}, \cdot; G^{(N)}) - \pi^*(\cdot; G^{(N)})\|_{\mathrm{TV}} &\leq P_x(\tau_{\mathrm{cov}}(G^{(N)}) > t) + \|\tilde{P}_t(x, \cdot; G^{(N)}) - \pi(\cdot; G^{(N)})\|_{\mathrm{TV}} \\ &\leq P_x(\tau_{\mathrm{cov}}(G^{(N)}) > t) + \frac{\sqrt{S_N}}{2\sqrt{t}} \,. \end{aligned} \tag{4.5.4}$$

Proof. Using the uniform (invariant) distribution of lamp configurations at $t \geq \tau_{\text{cov}}(G^{(N)})$, yields

$$\begin{aligned}
&\|P_t^*(\boldsymbol{x},\cdot;G^{(N)}) - \pi^*(\cdot;G^{(N)})\|_{\text{TV}} \\
&\leq \sum_{\boldsymbol{y}\in V(\mathbb{Z}_2\wr G^{(N)})} P^*_{\boldsymbol{x}}(Y_t^{(N)} = \boldsymbol{y}, \tau_{\text{cov}}(G^{(N)}) > t) \\
&\quad + \sum_{\boldsymbol{y}\in V(\mathbb{Z}_2\wr G^{(N)})} [P^*_{\boldsymbol{x}}(Y_t^{(N)} = \boldsymbol{y}, t \geq \tau_{\text{cov}}(G^{(N)})) - \pi^*(\boldsymbol{y};G^{(N)})]_+ \\
&\leq P_x(\tau_{\text{cov}}(G^{(N)}) > t) + \sum_{y\in V(G^{(N)})} [P_x(\tilde{X}_t^{(N)} = y) - \pi^{(N)}(y)]_+ \,.
\end{aligned}$$

Applying the definition of total variation distance for $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$ yields the first inequality in (4.5.4). Next, let $\tilde{\tau}_x^{(N)} := \min\{t \geq 0 \mid \tilde{X}_t^{(N)} = x\}$. By the embedding of $X^{(N)}$ within $\tilde{X}^{(N)}$ (as in the proof of Proposition 4.1.5), and the commute time identity (see (4.4.5)), we have that for all N and $x, z \in V(G^{(N)})$

$$E_z\big[\tilde{\tau}_x^{(N)}\big] = 2E_z\big[\tau_x^{(N)}\big] \leq 2S_N \,. \tag{4.5.5}$$

While proving [56, Lemma 4.1], it shown that for all N, t and $x \in V(G^{(N)})$,

$$\Big(\|\tilde{P}_t(x,\cdot;G^{(N)}) - \pi(\cdot;G^{(N)})\|_{\text{TV}}\Big)^2 \leq \frac{1}{8t} \max_{z\in V(G^{(N)})} \big\{E_z[\tilde{\tau}_x^{(N)}]\big\} \tag{4.5.6}$$

and we get the second inequality in (4.5.4) by combining (4.5.5) and (4.5.6). □

4.5.1 The strongly recurrent case: $d_f < d_w$

For $d_f < d_w$ we get Theorem 4.1.4(a) by combining the lower bounds of Proposition 4.1.6 with the upper bounds of Propositions 4.1.5 and 4.5.1.

Proof of Theorem 4.1.4(a). Since $R_N \to \infty$, we deduce from Proposition 4.1.6 that for any $\epsilon \in (0, 1)$,

$$\liminf_{N\to\infty} \left\{\frac{T_{\text{mix}}(\epsilon;G^{(N)})}{T_N}\right\} \geq -c_1^{-1}\log(c_1\epsilon)\,. \tag{4.5.7}$$

In contrast, with $S_N \leq c_\star T_N$ and $\gamma = \gamma(\epsilon)$ denoting the unique solution of

$$\epsilon = c_0 e^{-\gamma/c_0} + \frac{\sqrt{c_\star}}{2\sqrt{\gamma}}\,, \tag{4.5.8}$$

we get from Propositions 4.1.5 and 4.5.1 that

$$\limsup_{N\to\infty}\left\{\frac{T_{\rm mix}(\epsilon; G^{(N)})}{T_N}\right\} \le \gamma(\epsilon)\,. \tag{4.5.9}$$

The RHS of (4.5.7) blows up as $\epsilon \to 0$, while the RHS of (4.5.9) is uniformly bounded above for $\epsilon \in [\frac{1}{2}, 1]$. Hence, there can be no cutoff for these lamplighter chains. □

Remark 4.5.2. *In view of Proposition 4.1.5, here $T_{\rm mix}(\epsilon; G^{(N)})/T_{\rm cov}(G^{(N)}) \gg 1$ for small ϵ. From Section 4.4 we also learn that, when $d_f < d_w$, the lamplighter chains have no mixing cutoff mainly because the laws of $\tau_{\rm cov}(G^{(N)})/T_{\rm cov}(G^{(N)})$ do not concentrate as $N \to \infty$ (unlike the transient case of $d_f > d_w$).*

4.5.2 The transient case: $d_f > d_w$

As mentioned before, in case $d_f > d_w$, we establish the cutoff for total-variation mixing time of the lamplighter chains by verifying that our weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_{N\ge1}$ satisfy the sufficient conditions from [54, Theorem 1.5]. To this end, recall the uniform mixing times $T^U_{\rm mix}(G^{(N)})$ and Green functions $\tilde g^{(N)}(\cdot,\cdot)$ that correspond to $\epsilon = \frac{1}{4}$ in (4.3.4) and (4.3.12), respectively. In [54], uniformly elliptic, finite weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_{N\ge1}$ are called *uniformly locally transient* if for all N,

$$g(x, A; G^{(N)}) := \sum_{y\in A} \tilde g^{(N)}(x, y) \le \rho(d^{(N)}(x, A), \mathrm{diam}\{A\})$$

for all $x \in V(G^{(N)}), A \subseteq V(G^{(N)})$, where $\rho : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\rho(r, s) \downarrow 0$ as $r \to \infty$, for each fixed s. Further setting

$$\bar\Delta(G) := \max_{x\in V}\{\mu_x\}, \quad \underline{\Delta}(G) := \min_{x\in V}\{\mu_x\}, \quad \Delta(G) := \frac{\bar\Delta(G)}{\underline{\Delta}(G)}\,,$$

the following two assumptions are made in [54].

Assumption 4.5.3 (Transience). *The finite weighted graphs $\{(G^{(N)}, \mu^{(N)})\}_{N\ge1}$ are such that for any fixed $r < \infty$, as $N \to \infty$,*

(a) $\mu^{(N)}(G^{(N)}) \to \infty$.

(b) $\sup_N\{\Delta(G^{(N)})\} < \infty$.

(c) $\sup_x\{\log V^{(N)}(x, r)\} = o(\log \mu^{(N)}(G^{(N)}))$.

(d) $T^U_{\rm mix}(G^{(N)})(\bar{\Delta}(G^{(N)}))^r = o(\mu^{(N)}(G^{(N)}))$.

Assumption 4.5.4 (Uniform Harnack inequalities)**.** *For some $C(\alpha) < \infty$ and all $N, r \geq 1$, $\alpha > 1$, $x \in V(G^{(N)})$, if $h(\cdot)$ is a positive $\mu^{(N)}$-harmonic on $B^{(N)}(x, \alpha r)$, then*

$$\max_{y \in B^{(N)}(x,r)} \{h(y)\} \leq C(\alpha) \min_{y \in B^{(N)}(x,r)} \{h(y)\} .$$

We next prove Theorem 4.1.4(b) by relying on the following restatement of [54, Theorem 1.5].

Theorem 4.5.5. *If uniformly locally transient $\{(G^{(N)}, \mu^{(N)})\}_{N\geq 1}$ satisfy Assumptions 4.5.3 and 4.5.4, then the lamplighter chains $\{Y^{(N)}\}_{N\geq 1}$ have cutoff at the threshold $\frac{1}{2}T_{\rm cov}(G^{(N)})$.*

Remark 4.5.6. *The derivation of [54, Theorem 1.5] is limited to lazy* SRW *on graphs $G^{(N)}$, namely with $\mu_{xy} \equiv 1$ for all $xy \in E(G)$. However, up to the obvious modifications we made in Assumptions 4.5.3 and 4.5.4, the same argument applies for uniformly elliptic weighted graphs, as re-stated in Theorem 4.5.5.*

Proof of Theorem 4.1.4(b). Thanks to Proposition 4.3.9 and (4.1.7) we confirm that $(G^{(N)}, \mu^{(N)})$ are uniformly locally transient for $\rho(r,s) = c_{\rm g}\, \tilde{c}\, c_{\rm v} r^{d_w - d_f} s^{d_f}$. Having $\mu^{(N)}(G^{(N)}) \geq c_{\rm v}^{-1}(R_N)^{d_f} \to \infty$ and $G^{(N)}$ of uniformly bounded degrees (see Remark 4.1.3), conditions (a)-(c) of Assumption 4.5.3 also hold here. Further, with $d_w < d_f$, the bound $T_{\rm mix}(G^{(N)}) \leq c(R_N)^{d_w}$ of Proposition 4.3.3 yields Assumption 4.5.3(d). Considering Assumption 4.1.2 for $u(t, \cdot) = h(\cdot)$ results with the lazy version $\tilde{P}^{(N)}$ satisfying the uniform Harnack inequality of Assumption 4.5.4 for any $\alpha > \max(2, 1/c_{\rm PHI})$. By our p_0-condition this is equivalent to the full Assumption 4.5.4 (see [73, Proposition 3.5]), and we complete the proof by applying Theorem 4.5.5. □

Bibliography

[1] S. Andres, M. T. Barlow, J. D. Deuschel and B. M. Hambly, Invariance principle for the random conductance model, Probab. Theory Related Fields **156** (2013), no. 3-4, 535–580.

[2] S. Andres, J-D. Deuschel, and M. Slowik, Harnack inequalities on weighted graphs and some applications for the random conductance model, Probab. Theory Related Fields, **164** (2016), no. 3, 931–977.

[3] M.T. Barlow, Diffusions on fractals, Lect. Notes in Math. **1690**, École d'Été de Probabilités de Saint-Flour XXV–1995, Springer, New York, (1998).

[4] M.T. Barlow, Random walks on supercritical percolation clusters, Ann. Probab. **32** (2004), no. 4, 3024–3084.

[5] M. T. Barlow, Random walks and heat kernels on graphs. Cambridge University Press, (2017).

[6] M. T. Barlow and R. F. Bass, Random walks on graphical Sierpinski carpets. Random walks and discrete potential theory (Cortona, 1997), 26-55, Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999.

[7] M. T. Barlow and R. F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets. Canad. J. Math. **51** (1999), no. 4, 673–744.

[8] M. T. Barlow and R. F. Bass, Stability of parabolic Harnack inequalities. Trans. Amer. Math. Soc. **356** (2003), no. 4, 1501–1533.

[9] M. T. Barlow, R. F. Bass and T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces. J. Math. Soc. Japan, **58** (2006), no. 2, 485–519.

[10] M. T. Barlow, T. Coulhon and T. Kumagai, Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs. Comm. Pure Appl. Math. **58** (2005), no. 12, 1642–1677.

[11] M.T. Barlow, A. Grigor'yan and T. Kumagai, On the equivalence of parabolic Harnack inequalities and heat kernel estimates. J. Math. Soc. Japan, **64** (2012), no. 4, 1091–1146.

[12] M. T. Barlow and J. D. Deuschel, Invariance principle for the random conductance model with unbounded conductances, Ann. Probab. **38** (2010), no. 1, 234–276.

[13] M. T. Barlow and B. M. Hambly, Parabolic Harnack inequality and local limit theorem for percolation clusters, Electron. J. Probab. **14** (2009), no. 1, 1–27.

[14] R.F. Bass and T. Kumagai, Laws of the iterated logarithm for some symmetric diffusion processes, Osaka J. Math. **37** (2000), no. 3, 625–650.

[15] O. Boukhadra, T. Kumagai, and P. Mathieu, Harnack inequalities and local central limit theorem for the polynomial lower tail random conductance model, J. Math. Soc. Japan **67** (2015), no. 4, 1413–1448.

[16] N. Berger and M. Biskup, Quenched invariance principle for simple random walk on percolation clusters, Probab. Theory Related Fields, **137** (2007), no. 1-2, 83–120.

[17] M. Biskup, Recent progress on the random conductance model, Probab. Surv. **8** (2011), 294–373.

[18] M. Biskup, personal communication 2016.

[19] T. K. Carne, A transmutation formula for Markov chains, Bull. Sci. Math. (2) **109** (1985), no. 4, 399–405.

[20] K. L. Chung and P. Erdős, On the application of the Borel-Cantelli lemma, Trans. Amer. Math. Soc. **72**, (1952), 179–186.

[21] T. Coulhon and A. Grigor'yan, A. Random walks on graphs with regular volume growth. Geom. Funct. Anal. **8** (1998), no. 4, 656–701.

[22] D. A. Croydon, Moduli of continuity of local times of random walks on graphs in terms of the resistance metric. Trans. London Math. Soc. **2** (2015), no. 1, 57–79.

[23] E. B. Davies, Large deviations for heat kernels on graphs, J. London Math. Soc. (2) **47** (1993), no. 1, 65–72.

[24] T. Delmotte, Parabolic Harnack inequality and estimates of Markov chains on graphs. Rev. Mat. Iberoamericana **15** (1999), no. 1, 181–232.

[25] A. Dembo, J. Ding, J. Miller and Y. Peres, Cut-off for lamplighter chains on tori: Dimension interpolation and phase transition. Preprint, available at `arXiv:1312.4522`.

[26] A. Drewitz, B. Ráth and A. Sapozhnikov, An introduction to random interlacements, Springer Briefs in Mathematics, Springer, Cham, 2014.

[27] H. Duminil-Copin, Law of the iterated logarithm for the random walk on the infinite percolation cluster, preprint 2008, available at `arXiv:0809.4380`.

[28] A. Erschler, Isoperimetry for wreath products of Markov chains and multiplicity of selfintersections of random walks. Probab. Theory Related Fields **136** (2006), no. 4, 560–586.

[29] M. Folz, Gaussian upper bounds for heat kernels of continuous time simple random walks, Electron. J. Probab. **16** (2011), no. 62, 1693–1722.

[30] S. Goel, R. Montenegro and P. Tetali, Mixing time bounds via the spectral profile. Electron. J. Probab. **11** (2006), no. 1, 1–26.

[31] A. Grigor'yan, Escape rate of Brownian motion on Riemannian manifolds, Appl. Anal. **71** (1999), no. 1-4, 63–89.

[32] A. Grigor'yan and A. Telcs, Sub-Gaussian estimates of heat kernels on infinite graphs. Duke Math. J. **109** (2001), no. 3, 451–510.

[33] A. Grigor'yan and A. Telcs, Harnack inequalities and sub-Gaussian estimates for random walks. Math. Ann. **324** (2002), no. 3, 521–556.

[34] G. Grimmett, Percolation, Second edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **321**, Springer-Verlag, Berlin, 1999.

[35] A. Grigor'yan and M. Kelbert, Range of fluctuation of Brownian motion on a complete Riemannian manifold, Ann. Probab. **26** (1998), no. 1, 78–111.

[36] O. Häggström and J. Jonasson, Rates of convergence for lamplighter processes. Stochastic Process. Appl. **67** (1997), no. 2, 227–249.

[37] B. Hambly, and T. Kumagai, Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries, Proc. of Symposia in Pure Math. **72**, Part 2, pp. 233–260, Amer. Math. Soc., Providence, (2004).

[38] W. J. Hendricks, Lower envelopes near zero and infinity for processes with stable components, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **16** (1970), 261–278.

[39] X. Huang, Escape rate of Markov chains on infinite graphs, J. Theoret. Probab. **27** (2014), no. 2, 634–682.

[40] X. Huang and Y. Shiozawa, Upper escape rate of Markov chains on weighted graphs, Stochastic Process. Appl. **124** (2014), no. 1, 317–347.

[41] D. O. Jones, Transition probabilities for the simple random walk on the Sierpiński graph. Stochastic Process. Appl. **61** (1996), no. 1, 45–69.

[42] A. Khinchin, Über einen Satz der Wahrscheinlichkeitcrechnung, Funcamenta Mathematica, **6** (1924), 9–20.

[43] A. Khinchin, Zwei Sätze über stochastische Prozesse mit stabilen Vereilungen, Rec. Math. [Mat.Sbornik] N.S. **3** (1938), 577–584.

[44] D. Khoshnevisan, Escape rates for Lévy processes, Studia Sci. Math. Hungar. **33** (1997), 177–183.

[45] J. Kigami, Analysis on fractals. Cambridge Tracts in Mathematics, **143**. Cambridge University Press, Cambridge, 2001.

[46] P. Kim, T. Kumagai, Takashi, J. Wang, Laws of the iterated logarithm for symmetric jump processes. Bernoulli **23** (2017), no. 4A, 2330–2379.

[47] C. Kipnis and S. Varadhan, A central limit theorem for additive functionals of reversible markov processes and applications to simple exclusions, Commun. Math. Phys. **104** (1986), no. 1, 1–19.

[48] N. Kubota, The law of the iterated logarithm for a class of transient random walk in random environment, Journal of Research Institute of Science and Technology, College Science and Technology, Nihon University, **127** (2012), 29–32, available at `arXiv:1004.5015`.

[49] T. Kumagai, Random walks on disordered media and their scaling limits, Lect. Notes in Math. **2101**, École d'Été de Probabilités de Saint-Flour XL–2010, Springer, New York, (2014).

[50] T. Kumagai and C. Nakamura, Laws of the iterated logarithm for random walks on random conductance models. Stochastic analysis on large scale interacting systems, 141–156, RIMS Kôkyûroku Bessatsu, **B59**, Res. Inst. Math. Sci. (RIMS), Kyoto, 2016.

[51] T. Kumagai, and C. Nakamura, Lamplighter random walks on Fractals, To appear in J. Theoret. Probab.

[52] D. A. Levin, Y. Peres and E. L. Wilmer, *Markov chains and mixing times.* American Mathematical Society, Providence, RI, 2009.

[53] P. Mathieu and A. Piatnitski, Quenched invariance principles for random walks on percolation clusters, Proc. Roy. Soc. A. **463** (2007), 2287–2307.

[54] J. Miller and Y. Peres, Uniformity of the uncovered set of random walk and cutoff for lamplighter chains. Ann. Probab. **40** (2012), no. 2, 535–577.

[55] J. Miller and P. Sousi, Uniformity of the late points of random walk on Z_n^d for d $\geq$3. Probab. Theory Related Fields **167** (2017), no. 3-4, 10011056.

[56] A. Nachmias and Y. Peres, Critical random graphs: diameter and mixing time. Ann. Probab. **36** (2008), no. 4, 1267–1286.

[57] C. Nakamura, Rate functions for random walks on random conductance models and related topics. Kodai Math. J. **40** (2017), no. 2, 289–321.

[58] C. Pittet and L. Saloff-Coste, Amenable groups, isoperimetric profiles and random walks. Geometric group theory down under (Canberra, 1996), 293-316, de Gruyter, Berlin, 1999.

[59] Y. Peres and D. Revelle, Mixing times for random walks on finite lamplighter groups. Electron. J. Probab. **9** (2004), no. 26, 825–845.

[60] P. Rodriguez and A. Sznitman, Phase transition and level-set percolation for the Gaussian free field, Comm. Math. Phys. **320** (2013), no. 2, 571–601.

[61] L. Saloff-Coste, Lectures on finite Markov chains. Lectures on probability theory and statistics (Saint-Flour, 1996), 301–413, Lecture Notes in Math., 1665, Springer, Berlin, 1997.

[62] L. Saloff-Coste and T. Zheng, Random walks and isoperimetric profiles under moment conditions. Ann. Probab. **44** (2016), no. 6, 4133–4183.

[63] A. Sapozhnikov, Random walks on infinite percolation clusters in models with long-range correlations. Ann. Probab. **45** (2017), no. 3, 1842–1898.

[64] Y. Shiozawa and J. Wang, Rate functions for symmetric Markov processes via heat kernel. Potential Anal. **46** (2017), no. 1, 23–53.

[65] V. Sidoravicius and A. S. Sznitman, Quenched invariance principles for walks on clusters of percolation or among random conductances, Probab. Theory Relat. Fields. **129** (2004), 219–244.

[66] A. S. Sznitman, Vacant set of random interlacements and percolation, Ann. of Math. (2) **171** (2010), no. 3, 2039–2087.

[67] A. S. Sznitman, Topics in occupation times and Gaussian free fields, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zűrich, 2012.

[68] J. Takeuchi, A local asymptotic law for the transient stable process, Proc. Japan Acad. **40** (1964), 141–144.

[69] J. Takeuchi, On the sample paths of the symmetric stable processes in spaces. J. Math. Soc. Japan **16** (1964), 109-127.

[70] J. Takeuchi, S. Watanabe, Spitzer's test for the Cauchy process on the line, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **3** (1964), 204–210.

[71] A. Teixeira, On the uniqueness of the infinite cluster of the vacant set of random interlacements, Ann. Appl. Probab. **19** (2009), no. 1, 454–466.

[72] A. Telcs, Local sub-Gaussian estimates on graphs: the strongly recurrent case. Electron. J. Probab. **6** (2001), no. 22, 33 pp.

[73] A. Telcs, The art of random walks. Lecture Notes in Mathematics, **1885**. Springer-Verlag, Berlin, 2006.

[74] N. Th. Varopoulos, *Random walks on soluble groups.* Bull. Sci. Math. **107** (1983), 337-344.

[75] N. Th. Varopoulos, Long range estimates for Markov chains. Bull. Sci. Math. (2) **109** (1985), no. 3, 225–252.