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Abstract

In this paper, we study cooperative games with the players whose preferences depend on all players' allocations, which we refer to as the social preferences. The social preferences we study in this paper are represented by the utility functions proposed by Fehr and Schmidt (1999) or the utility functions proposed by Charness and Rabin (2002). First, we define and characterize the cores, which are the same as the standard core except that the utility functions are the Fehr-Schmidt or the Charness-Rabin type. We show that the Fehr-Schmidt type core becomes smaller if the players become more envious and that it may become larger or smaller if the players become more compassionate. We also show that the Charness-Rabin type core becomes smaller if the players pay more attention to care about the minimal allocation and that it may become larger or smaller if the players pay more attention to care about the social welfare. Moreover, we analyze the alpha-core and the beta-core of the cooperative games consisting of players with these types of social preferences, as well as a new core concept that takes account of networks among the players. We show that the Fehr-Schmidt type core is the smallest among these cores and that the alpha-core coincides with the beta-core under the Fehr-Schmidt utility functions.

JEL Codes: C71, D63, D91

Keywords: Social preference, Inequality-aversion, Cooperative game, Core, Network.

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1 Introduction

There are many countries which seem to accept large income disparity. Meanwhile too large income disparity should decrease people's utilities and ought to cause social instability. Attitudes of each person toward income disparity are explained by inequality-averse social preferences, which depend on not only their own income but also the income distribution of the society. Many societies would consist of people who have such inequality-averse social preferences and would not be maintained if income disparity in the society is large enough. This is because such income disparity gives disutility even to the rich. In other words, realized income distributions in a society reflect the level of the income disparity which can be accepted by the individuals' preferences in the society. The main purpose of this paper is to study delicate relationships between the shape of social preferences and acceptable inequality in the society.

To study the relationships above, this paper analyzes solutions of cooperative games where players' preferences are inequality-averse social ones. The solutions of the game we study are the core which can be understood as a set of realized income distributions with acceptable disparity. We can apply any social preferences to cooperative games but to bring clear-cut results, we consider the two well-known models of social preferences that are proposed by Fehr and Schmidt (1999) and Charness and Rabin (2002), respectively.

The utility functions proposed by Fehr and Schmidt (1999) (the F-S utility functions hereafter) consist of the two parts, that is, the self-interest part and the part that decreases in proportion to the differences between his income and those of the others. This utility function represents inequality-averse social preferences in the sense that if we fix one's own income, his/her utility is maximized when all people get the same income. The utility functions proposed by Charness and Rabin (2002) (the C-R utility functions hereafter) depend on the self-interest part and the "social welfare" part characterized by the minimum income and the total income. This utility function represents inequality-averse social preferences in the sense that if we fix one's own income and the total income, his/her utility increases when the minimum income increases.

When we apply the ideas of social preferences to the core of cooperative games, we need to specify the set of players whose incomes affects blocking coalition's welfare. We examine three frameworks. In the first framework, players care about those who are in the coalition they belong to. In the second framework, players care about all the players in the game. The third framework is an intermediate case between the first and second frameworks. In other words, players have intimate players like family or friends and care about not only the members of their coalition but also the intimate players who do not belong to the coalition.

To motivate our analysis, consider the following example of a symmetric game.

There are three players, A, B, and C. If all players cooperate, they get 12. If two players cooperate, they get 5. If they work apart, everyone gets nothing. Assume all players are selfish. Namely, they are concerned with only their own income. The vector $(6, 6, 0)$ which means that A and B get 6, and C gets 0 is in the core because no coalition can block it. Similarly, the vector $(4, 4, 4)$ is also in the core.

Now suppose these players' preferences are represented by a strictly inequality-averse F-S utility function. Values of this utility function are one's income level when all players have the equal incomes and decrease when other players' income becomes higher or lower. We will show $(6, 6, 0)$ is not in the core under any framework. Notice that the utility of player C is negative. In the first framework where players care about those who are in the coalition they belong to, player C can get 0 by the coalition of only him/her and can block it, so the vector $(6, 6, 0)$ is not in the core. In the second framework where players care about all the players in the game, the vector $(6, 6, 0)$ is not in the core because if player C blocks the imputation, A's income and B's income become lower, and the differences between C's income and the others' respective incomes become lower. Then, C's utility becomes larger. In the third framework where players care about not only the members of their coalition but also an intimate player who does not belong to the coalition, the vector $(6, 6, 0)$ is not in the core. Assume C's intimate player is B[1]. If player C blocks the imputation, B's income becomes lower, and C's utility becomes larger. In contrast, $(4, 4, 4)$ is in the core in any framework because the utility of all players is 4, and it is readily seen that a two-player coalition can give at most 2.5 to either of the players.

This example shows that a vector of income levels which is in the core with self-interested players is not necessarily in the core with inequality-averse players. Moreover, it seems to suggest that if players are averse to income disparity, distributions with large variation tend to be blocked, and the core becomes smaller in general. So it is natural to ask if and how the core becomes smaller toward the equal income distribution as players get more averse to income disparity. Interestingly enough, it turns out that such monotonicity holds for a parameter for envy of the F-S utility functions but does not hold for the other parameter for compassion.

Our contributions can be summarized as follows. We define and characterize the F-S core and the C-R core which are the same as the standard core except that the utility functions are the Fehr-Schmidt or the Charness-Rabin type, using the first framework where players care about those who are in the coalition they belong to. More precisely, although the F-S and C-R cores are the cores of an NTU game, they are characterized by inequalities which resemble coalitional rationality inequalities of a TU game.

[1] If we replace B with A, we get the same result because A and B are symmetric. In addition, if C cares about both players, we also get the same result by the discussion of the second framework.

These inequalities enable us to carry out various comparative statics exercises. We show that the F-S core becomes smaller if the players become more envious, that is, if a parameter of the F-S utility functions for envy increases. However, the F-S core may become larger or smaller if the players become more compassionate, that is, if a parameter of the F-S utility functions for compassion increases.

The intuitive reason for such non-monotonicity is as follows. Since all players have the F-S utility functions, a player A who has a uniquely smallest income wants to decrease the number of players who have a larger income than his/her own. In other words, even if A's income becomes lower, A's utility can become larger when the number of the players who have a larger income decreases. This property of the F-S utility functions allows us to expect that there is a blocking coalition of two players which includes A such that A's income decreases, and the other player's income increases. This is because if A's disutility from envy decreases as the number of people whom A envies decreases, A's utility can increase.

In fact, such a blocking coalition exists when the other player is not compassionate enough. Nevertheless, if the player is compassionate enough to dislike inequality of the blocking coalition, the player's utility decreases even when his/her income increases, and such a blocking coalition does not exist. Therefore, there exist imputations which can be blocked under the F-S utility function with low compassion but not under the function with high compassion. So the F-S core may become larger or smaller if the players become more compassionate.

Similarly, we show that the C-R core may become larger or smaller if the players get more concerned with the social-welfare part than the self-interest part. In contrast, the C-R core becomes smaller if the players get more concerned with the minimum-income part than the total-income part of the social-welfare part.

Additionally, we analyze the alpha-core and the beta-core in the second framework where players care about all the players in the game. In this framework, when some players block an imputation, the players still care about the other players who remain on the original coalition, and the players' utilities depend on the other players' incomes. To analyze this framework, the alpha-core and the beta-core are adequate concepts. We show that the two cores coincide in both the F-S and C-R cases. In general, the alpha-core includes the beta-core, but in this framework, they coincide. In addition, these cores with the F-S utility functions include the F-S core we discussed above.

Moreover, we analyze the third framework where players care about not only the members of their coalition but also intimate players who do not belong to the coalition. In this framework, players are not likely to punish intimate players even when the intimate players try to block an imputation. To analyze this framework, we define and analyze a new core concept which comes from the alpha-core and the Equal Division Core proposed by Selten (1972). Furthermore, we compare the F-S core, the alpha-core, and the new core and conclude the F-S core is the smallest among these cores under the F-S utility functions.

The relation between egalitarianism and the cooperative games is studied by many papers, for example, Arin and Inarra (2001), Arin et al. (2008), van den Brink et al. (2013), Bhattacharya (2004), and Dutta and Ray (1989). These papers analyze egalitarianism, the core and/or the Shapley value. In particular, Dutta and Ray (1989) propose egalitarian allocations. The egalitarian allocation is the feasible allocation that is not Lorenz blocked by any sub-coalitions and is undominated by each other. Moreover, they prove there is at most one egalitarian allocation, and the allocation is in the core. Their paper is close to ours because both the Fehr-Schmidt and the Charness-Rabin preferences can be interpreted as some form of egalitarianism. The differences are that (i) Dutta and Ray (1989) use Lorenz domination but we use ordinary concepts of domination, and (ii) they use the usual transferable utility but we use the social preferences with non-transferable utilities.[2]

This paper is structured as follows. In Section 2, we will state our model. In Section 3, we will characterize the core with the Fehr-Schmidt preferences and examine how the F-S core changes according to parameters specifying the degree of inequity aversion. In Section 4, we will characterize the core with the Charness-Rabin preferences and examine how the C-R core changes according to the weights on the self-interest part and the social-welfare part, and the weights on the minimum-income part and the total-income part within the social-welfare part. In Section 5, we will analyze the alpha-core and the beta-core, based on the second framework, and we will propose a new core concept and analyze it, based on the third framework in Section 6. We will conclude in Section 7.

2 The Model

Let $N = \{1, 2, \ldots, n\}$ be the set of *n* players and $v : 2^N \setminus \{\emptyset\} \to \mathbb{R}_{\geq 0}$ be a characteristic function.

Definition 1. For any $S \subset N$, an imputation of S is defined as $x = (x_i)_{i \in S} \in \mathbb{R}^{|S|}$ such that

1. $\sum_{i \in S} x_i = v(S)$, and
2. for all $i \in S, x_i \geq 0$.

We call x_i the share of player i in an imputation x.

[2] In fact, Dutta and Ray (1989) also mention non-transferable utilities but their approach is to maximize the social welfare function, being different from our paper which studies the social preferences.

Let X_S denote the set of all imputations of S. We interpret $v(S)$ as the total wealth for the members in S and an imputation as an income distribution. We call an imputation $(\frac{v(S)}{|S|}, \dots, \frac{v(S)}{|S|}) \in X_S$ the equal imputation of S and call imputations which are not the equal imputation unequal imputations.

Each player has a social preference relation about income distributions. We write the utility of $x \in X_S$ with $S \subset N$ for player $i \in S$ as $u_i(x, S)$. In interpretation, $u_i(x, S)$ is the utility for i when i receives x_i and cares about the income distribution in S.

Definition 2. An imputation $x \in X_N$ is dominated by $y \in X_S$ through S if and only if for all $i \in S, u_i(y, S) > u_i(x, N)$.

Thus, we implicitly assume that each member of S cares about the income distribution solely in S. Next, we consider utility functions of a specific form.

Definition 3. The F-S utility function u_i^F is defined by

$$u_i^F(x, S) = x_i - \frac{\alpha}{n-1} \sum_{j \in S \setminus \{i\}} \max[x_j - x_i, 0] - \frac{\beta}{n-1} \sum_{j \in S \setminus \{i\}} \max[x_i - x_j, 0],$$

where $\alpha \geq \beta \geq 0$ and $\beta < 1$.

This class of utility functions is studied in Fehr and Schmidt (1999). Since $\alpha \geq 0$ and $\beta \geq 0$, the player with an F-S utility function (weakly) feels sympathy for the players whose income levels are less than his and (weakly) envies the players whose income levels are more than his. $\alpha \geq \beta \geq 0$ means that envy is not weaker than sympathy. $\beta < 1$ is a natural requirement for preferences for equality, since in the case that $\beta \geq 1$, even if the players who have a uniquely largest share threw their income to decrease disparity, their utilities would not decrease.

Definition 4. The F-S core, denoted by $C_{\alpha,\beta}$, is the set of the elements in X_N that are undominated by any elements in X_S through any $S \subset N$ when the utility functions are the F-S utility functions.

We consider another class of utility functions.

Definition 5. The C-R utility function u_i^C is defined by

$$u_i^C(x, S) = (1 - \gamma)x_i + \gamma \left[\delta \min_{j \in S} x_j + (1 - \delta) \sum_{j \in S} x_j \right],$$

where $\gamma \in [0, 1)$ and $\delta \in (0, 1)$.

This class of utility functions is studied in Charness and Rabin (2002). This function can be decomposed into two factors. One is a factor of their own shares. The other is a social-welfare factor:

$$W(x_1, \ldots, x_{|S|}) = \delta \min_{j \in S} x_j + (1 - \delta) \sum_{j \in S} x_j,$$

where $\delta \in (0, 1)$ is a parameter measuring the degree of concern for the minimum share. Namely, larger δ means that players become more compassionate. A C-R utility function is therefore a convex combination of this social-welfare factor and their own share.

Definition 6. The C-R core, denoted by $C_{\gamma,\delta}$, is the set of the elements in X_N that are undominated by any elements in X_S through any $S \subset N$ when the utility functions are the C-R utility functions.

By Definitions 3 and 5, it is clear that these functions are not linear in income. Therefore, players' utilities are not transferable, although the terminology comes from TU games.

3 Core with Fehr-Schmidt preferences

3.1 The Characterization of the F-S Core

In this section, we will consider what kinds of imputations are included in the core with the Fehr-Schmidt preferences. The following lemma states that for any imputation of a coalition, the shares and the utilities of players are aligned in the same way.

Lemma 1. For any $S \subset N$, any $i, j \in S$, and any $y \in X_S$, $u_i^F(y, S) \geq u_j^F(y, S)$ if and only if $y_i \geq y_j$.

Proof. First, we will show that if there is no $k \in S$ such that $y_i > y_k > y_j$, then $y_i \geq y_j$ if and only if $u_i^F(y, S) \geq u_j^F(y, S)$. The latter inequality is equivalent to

$$\begin{aligned} & 0 \geq \left(1 + \frac{l}{n-1}\alpha - \frac{|S| - l}{n-1}\beta\right)(y_j - y_i) \\ \Leftrightarrow \quad & y_i \geq y_j, \end{aligned}$$

where l denotes the number of players whose share is greater than y_j. By this inequality, we can prove the lemma inductively. □

The next lemma asserts that if some imputation of $S \subset N$ dominates $x \in X_N$, there exists $y \in X_S$ which dominates x such that the shares in x and y are aligned in the same way.

Lemma 2. Consider $x = (x_1, x_2, \ldots, x_n) \in X_N$ and $S \subset N$. Without loss of generality, assume $S = \{i_1, i_2, \ldots, i_{|S|}\}$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$. If $y \in X_S$ dominates x, then we can obtain $y' = (y'_{i_1}, y'_{i_2}, \ldots, y'_{i_{|S|}}) \in X_S$ which dominates x with $y'_{i_1} \geq y'_{i_2} \geq \cdots \geq y'_{i_{|S|}}$.

Proof. If $k < l$ and $y_{i_k} < y_{i_l}$, $u^F_{i_l}(x, N) \leq u^F_{i_k}(x, N) < u^F_{i_k}(y, S) \leq u^F_{i_l}(y, S)$ because of Lemma 1. Therefore, if the share of i_k is exchanged for that of i_l, the new imputation also dominates x. Repeating this manipulation, we can obtain $y' = (y'_{i_1}, y'_{i_2}, \ldots, y'_{i_{|S|}}) \in X_S$ which dominates x with $y'_{i_1} \geq y'_{i_2} \geq \cdots \geq y'_{i_{|S|}}$. □

The following theorem asserts that when β is large enough for the player who has the largest share to want to give his/her share and try to realize the equal imputation, the F-S core only includes the equal imputation or is empty. Additionally, we completely characterize the two cases.

Theorem 1. Assume $\beta > \frac{n-1}{n}$. If $\frac{v(N)}{n} \geq \frac{v(S)}{|S|}$ for all $S \subset N$, the F-S core includes only the equal imputation $(\frac{v(N)}{n}, \frac{v(N)}{n}, \ldots, \frac{v(N)}{n})$ and otherwise, the F-S core is empty.

Proof. First, we will show the F-S core does not include an unequal imputation. Consider $x \in X_N$. We assume $x_1 \geq x_2 \geq \cdots \geq x_n$ without loss of generality and assume that $x_1 > \frac{v(N)}{n}$. The utility of player 1 is

$$x_1 - \frac{\beta}{n-1} \sum_{i=1}^{n} (x_1 - x_i) = \left(1 - \frac{n}{n-1}\beta\right) x_1 + \frac{\beta}{n-1} v(N).$$

By $x_1 > \frac{v(N)}{n}$ and $\beta > \frac{n-1}{n}$,

$$\left(1 - \frac{n}{n-1}\beta\right) x_1 + \frac{\beta}{n-1} v(N) < \left(1 - \frac{n}{n-1}\beta\right) \frac{v(N)}{n} + \frac{\beta}{n-1} v(N) = \frac{v(N)}{n}$$

holds. Thus, $\frac{v(N)}{n} > u^F_1(x, N) \geq u^F_2(x, N) \geq \cdots \geq u^F_n(x, N)$ by $x_1 \geq x_2 \geq \cdots \geq x_n$ and Lemma 1. Therefore, $(\frac{v(N)}{n}, \frac{v(N)}{n}, \ldots, \frac{v(N)}{n})$ dominates all unequal imputations of N.

Next, we will prove that the F-S core includes the equal imputation if $\frac{v(N)}{n} \geq \frac{v(S)}{|S|}$ for any $S \subset N$. By the definition of the F-S utility functions, for any $S \subset N$ and any $y \in X_S$, there exists $i \in S$ such that

$$\frac{v(N)}{n} \geq \frac{v(S)}{|S|} \geq y_i \geq u^F_i(y, S)$$

because of $\sum_{i\in S} y_i = v(S)$. Thus, the F-S core includes the equal imputation. If $\frac{v(N)}{n} < \frac{v(S)}{|S|}$, the imputation $(\frac{v(S)}{|S|}, \ldots, \frac{v(S)}{|S|}) \in X_S$ clearly dominates $(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}) \in X_N$, and therefore the F-S core is empty. □

By this theorem, we can obtain a necessary and sufficient condition whether the F-S core is empty or not when $\beta > \frac{n-1}{n}$.

The proof of Theorem 1 reveals that all unequal imputations in X_N are dominated by the equal imputation through the grand coalition when β is sufficiently large. Note that this property never holds under ordinary TU games.

Next, we consider the case of $\beta \leq \frac{n-1}{n}$. The following proposition states that for any imputation of a coalition, $v(S)$ can be expressed by a weighted sum of utilities, and the weights are derived from α and β.

Proposition 1. Assume $\beta \leq \frac{n-1}{n}$. Fix $S \subset N$ and $y \in X_S$, so that $S = \{i_1, \ldots, i_{|S|}\}$ with $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_{|S|}}$. Then,

$$\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_j A_{j-1}} u^F_{i_j}(y, S) = v(S), \qquad \text{(FSTGR)}$$

where $A_j = 1 + \frac{j}{n-1}\alpha - \frac{|S|-j}{n-1}\beta$. The convention is that if $S = N$ and $\beta = \frac{n-1}{n}$ (hence $A_0 = 0$), $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1} A_j} u^F_{i_j}(y, N) = |S| u^F_{i_1}(y, N)$.

Proof. See Appendix A.1. □

Moreover, we can show the following important formula.

$$\frac{A_0 A_{|S|}}{A_j A_{j-1}} = |S| \frac{\frac{1}{A_{j-1}} - \frac{1}{A_j}}{\frac{1}{A_0} - \frac{1}{A_{|S|}}}.$$

Thus,

$$\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_j A_{j-1}} = |S|$$

holds. In particular, if $\alpha = \beta = 0$, $A_j = 1$ holds for all $j = 0, \ldots, |S|$, and the identity (FSTGR) is written as $\sum_{j=1}^{|S|} y_{i_j} = v(S)$ which expresses total group rationality in TU games. Then, (FSTGR) when $\alpha = \beta = 0$ corresponds to the identity of total group rationality which holds under the existing model of TU games.

We will show an intuitive meaning of $\frac{A_0 A_{|S|}}{A_j A_{j-1}}$. Fix $j \in S$ arbitrarily. Consider $a \in \mathbb{R}^{|S|}$ which satisfies $u^F_{i_k}(a) = 1$ for all $k = 1, \ldots, j$ and $u^F_{i_k}(a) = 0$ for all $k = j+1, \ldots, |S|$. By Lemma 1, we can assume that the shares on a of i_k for all

$k = 1, \dots, j$ are a_h, and the shares on a of i_k for all $k = j+1, \dots, |S|$ are a_l. By assumption,

$$\begin{cases} a_h - \frac{(|S|-j)\beta}{n-1}(a_h - a_l) = 1 \\ a_l - \frac{(|S|-j)\alpha}{n-1}(a_h - a_l) = 0 \end{cases}$$

holds, and then, $a_h - a_l = \frac{1}{A_j}$. Therefore,

$$\begin{cases} a_h = a_l + \frac{1}{A_j} = (1 + \frac{j\alpha}{n-1})\frac{1}{A_j} \\ a_l = \frac{j\alpha}{n-1}\frac{1}{A_j}. \end{cases}$$

The sum of the shares denoted by T_j is

$$ja_h + (|S| - j)a_l = |S|a_l + j(a_h - a_l) = \frac{j}{A_j}\left(1 + \frac{\alpha|S|}{n-1}\right) = \frac{j}{A_j}A_{|S|}.$$

Then,

$$T_j - T_{j-1} = \left(\frac{j}{A_j} - \frac{j-1}{A_{j-1}}\right)A_{|S|} = \frac{jA_{j-1} - (j-1)A_j}{A_jA_{j-1}}A_{|S|} = \frac{A_0A_{|S|}}{A_jA_{j-1}}.$$

The convention is that $T_0 = 0$. The last equality comes from $jA_{j-1} - (j-1)A_j = A_0$.

As a result, we can interpret $\frac{A_0A_{|S|}}{A_jA_{j-1}}$ as follows. Since T_{j-1} is the sum of the shares of a vector which gives 1 of utilities to i_k for all $k = 1, \dots, j-1$ and 0 to i_k for all $k = j, \dots, |S|$, $\frac{A_0A_{|S|}}{A_jA_{j-1}}$ is the marginal total shares when only i_j's utility increases by 1, and the other players' utilities are unchanged.

Under this interpretation, we can understand Proposition 1 as follows. For any $j = 1, \dots, |S|$, to increase the utility of only i_j by $u^F_{i_j}(y, S)$, the sum of the shares must increase by $\frac{A_0A_{|S|}}{A_jA_{j-1}}u^F_{i_j}(y, S)$. Then, if $\sum_{j=1}^{|S|}\frac{A_0A_{|S|}}{A_jA_{j-1}}u^F_{i_j}(y, S) = v(S)$, by dividing $v(S)$ properly, i_j gets the utility $u^F_{i_j}(y, S)$ for all $j = 1, \dots, |S|$.

By Proposition 1, we can get the next proposition, which states that any imputation of the grand coalition is not dominated through the grand coalition.

Proposition 2. Assume $\beta \le \frac{n-1}{n}$. Consider any $x \in X_N$. There is no $y \in X_N$ that dominates x.

Proof. Fix any $x \in X_N$ with $x_{i_1} \ge x_{i_2} \ge \dots \ge x_{i_n}$, where $\{i_1, \dots, i_n\} = N$.

Assume $\beta < \frac{n-1}{n}$. Recall that

$$\sum_{j=1}^{n}\frac{A_0A_{|S|}}{A_jA_{j-1}}u^F_{i_j}(x, N) = v(N)$$

for all $x \in X_N$ by Proposition 1. Then, for all $x, y \in X_N$,

$$\sum_{j=1}^{n} \frac{A_0 A_{|S|}}{A_j A_{j-1}} u_{i_j}^F(x, N) = \sum_{j=1}^{n} \frac{A_0 A_{|S|}}{A_j A_{j-1}} u_{i_j}^F(y, N)$$
$$\Leftrightarrow \sum_{j=1}^{n} \frac{A_0 A_{|S|}}{A_j A_{j-1}} (u_{i_j}^F(y, N) - u_{i_j}^F(x, N)) = 0.$$

Since $\frac{A_0 A_{|S|}}{A_j A_{j-1}} > 0$ for all $j = 1, \ldots, n$ because of $\beta < \frac{n-1}{n}$, y does not dominate x.

If $\beta = \frac{n-1}{n}$, by Proposition 1,

$$n u_{i_1}^F(x, N) = v(N)$$
$$\Leftrightarrow u_{i_1}^F(x, N) = \frac{v(N)}{n}$$

for all $x \in X_N$. Then, there exists no $y \in X_N$ that dominates x. □

Proposition 2 states that if $\beta \leq \frac{n-1}{n}$, domination through the grand coalition never occurs unlike the case of Theorem 1.

By these propositions, we can characterize the F-S core completely.

Theorem 2. Assume $\beta \leq \frac{n-1}{n}$. Consider $x \in X_N$. x is in the F-S core if and only if for all $S = \{i_1, \ldots, i_{|S|}\} \subsetneq N$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$, $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1} A_j} u_{i_j}^F(x, N) \geq v(S)$ holds, where $A_j = 1 + \frac{j}{n-1}\alpha - \frac{|S|-j}{n-1}\beta$.

Proof. See Appendix A.2. □

If $\alpha = \beta = 0$ (hence $A_j = 1$ for all $j = 0, \ldots, |S|$), the inequality in the theorem is written as $\sum_{j=1}^{|S|} x_{i_j} < v(S)$. Then, we generalize the inequality that is the necessary and sufficient condition that an imputation of the grand coalition is blocked through S. Namely, S does not block an imputation x of the grand coalition if and only if $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1} A_j} u_{i_j}^F(x, N) \geq v(S)$ holds. That is, these inequalities resemble the definition of the core with a super-additive characteristic function in a TU game. So we call this inequality the condition of "coalitional rationality", corresponding to the terminology in TU games.

3.2 The Comparative statics of the F-S core about α and β

In this section, we will see how α and β change the shape of the F-S core. The following proposition shows that the standard core with traditional preferences ($\alpha = \beta = 0$) includes the F-S core for any α and any β.

Proposition 3. For any α and any β, $C_{\alpha,\beta} \subset C_{0,0}$.

Proof. Fix any $x \notin C_{0,0}$. There exists $S \subset N$ that satisfies $\sum_{i \in S} x_i < v(S)$. Let $\epsilon = v(S) - \sum_{i \in S} x_i$ and $x'_i = x_i + \frac{\epsilon}{|S|}$ for any $i \in S$. Then, x' is an imputation of S. For any α and any β,

$$u_i^F(x', S) > x_i - \frac{\alpha}{n-1} \sum_{j \in S \setminus \{i\}} max[x_j - x_i, 0] - \frac{\beta}{n-1} \sum_{j \in S \setminus \{i\}} max[x_i - x_j, 0] \geq u_i^F(x, N)$$

holds for any $i \in S$. Therefore, there exists an imputation that dominates x, and we proved the proposition. □

According to this proposition, we can insist two things. One is that the condition for non-emptiness of the core is stronger compared to the case of the original core. The other is that unequal imputations can be excluded. This is because if $C_{0,0}$ includes the equal imputation, $C_{\alpha,\beta}$ also includes the equal imputation.

The following theorem is an answer to how the F-S core changes if α increases.

Theorem 3. Fix α and β arbitrarily. If $\alpha' > \alpha$, $C_{\alpha',\beta} \subset C_{\alpha,\beta}$.

Proof. See Appendix A.3. □

Next, how does the F-S core change if β increases? Does the F-S core always become smaller like the case of α? The answer is no, as the following examples show.

Assume $N = \{1, 2, 3\}, v(N) = 25, v(\{2, 3\}) = 14, v(\{1, 3\}) = v(\{1, 2\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \alpha = 0.8$, and $\beta = 0$. If $x = (10, 10, 5)$, the utility of player 2 is 10, and the utility of player 3 is

$$5 - \frac{\alpha}{2}\{(10 - 5) + (10 - 5)\} = 1.$$

Let $y = (10.1, 3.9) \in X_{\{2,3\}}$. Then, the utility of player 2 under y is 10.1, and the utility of player 3 under y is

$$3.9 - \frac{\alpha}{2}(10.1 - 3.9) = 1.42.$$

Therefore, y dominates x. This is because player 3 wants to decrease the number of players who have more shares than he/she has. If this effect is stronger than the effect that his/her share decreases, the coalition $\{2, 3\}$ will block x.

Next, assume $N = \{1, 2, 3\}, v(N) = 25, v(\{2, 3\}) = 14, v(\{1, 3\}) = v(\{1, 2\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \alpha = 0.8$, and $\beta = 0.5$. If $x = (10, 10, 5)$, the utility of player 2 under x is

$$10 - \frac{\beta}{2}\{(10 - 10) + (10 - 5)\} = \frac{35}{4},$$

and the utility of player 3 under x is 1. We can describe any $y' \in X_{\{2,3\}}$ as $(y'_2, 14-y'_2)$. Suppose y' dominates x. Then, $y'_2 \geq u_2(y', \{2,3\}) > 35/4 > 7$. The utility of player 2 under y' is

$$y'_2 - \frac{\beta}{2}\{y'_2 - (14 - y'_2)\} = \frac{y'_2}{2} + \frac{7}{2},$$

and the utility of player 3 under y' is

$$14 - y'_2 - \frac{\alpha}{2}\{y'_2 - (14 - y'_2)\} = 19.6 - 1.8y'_2.$$

Therefore, y' dominates x if and only if $\frac{21}{2} < y'_2$ and $y'_2 < \frac{93}{9}$. However, there is no such y'_2, and no $S \subset N$ has $y \in X_S$ which dominates x through S.

If $\beta > \frac{2}{3}$, x is not in the F-S core because of Theorem 1. Therefore, there is an imputation which belongs to the F-S core if β is at an intermediate level but does not belong to the F-S core if β is too large or too small.

4 Core with Charness-Rabin preferences

4.1 The Characterization of the C-R Core

In this section, we will propose the core with the C-R preferences. For any $S \subset N$, let m_S be the smallest integer that satisfies $\gamma\delta(|S|-m_S) < m_S(1-\gamma)$. The number of m_S can be understood as follows. Given any imputation where $m_S - 1$ or less players have the minimum share in the coalition S, a transfer among the players which increases the minimum share weakly increases all players' utilities in S. Thus, we call m_S the Maximal Number of the Minimal-Share Players of S (the MNMSP hereafter).

The next lemma implies the property of this number. In particular, when we confirm whether an imputation is dominated through a given coalition $S \subset N$ or not, we only have to check whether some imputation of S which has at least m_S minimal shares dominate the imputation or not.

Lemma 3. Consider $x = (x_1, x_2, \ldots, x_n) \in X_N$ and $S \subset N$. Without loss of generality, assume $S = \{i_1, i_2, \ldots, i_{|S|}\}$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$. If $y \in X_S$ dominates x, then we can obtain $y' = (y'_{i_1}, y'_{i_2}, \ldots, y'_{i_{|S|}}) \in X_S$ which dominates x with $y'_{i_1} \geq y'_{i_2} \geq \cdots \geq y'_{i_{|S|-m_S+1}} = \cdots = y'_{i_{|S|}}$.

Proof. If $k < l$ and $y_{i_k} < y_{i_l}$, $u^C_{i_l}(x, N) \leq u^C_{i_k}(x, N) < u^C_{i_k}(y, S) \leq u^C_{i_l}(y, S)$ by the definition of the C-R utility functions. Therefore, if the share of i_k is exchanged for that of i_l, the new imputation also dominates x. Repeating this manipulation, we can obtain $y'' = (y''_{i_1}, y''_{i_2}, \dots, y''_{i_{|S|}}) \in X_S$ which dominates x with $y''_{i_1} \geq y''_{i_2} \geq \dots \geq y''_{i_{|S|}}$.

Let $m' = |\operatorname*{arg\,min}_{k=1,\dots,|S|} y''_{i_k}|$. Then, if $m' \geq m_S$, define $y' = y''$, and we got the proof. Assume $m' \leq m_S - 1$. Define y' as follows:

$$y'_{i_j} = \begin{cases} y''_{i_j} - \frac{m'}{|S|-m'}\epsilon \ \ if\ j \leq |S| - m' \\ y''_{i_j} + \epsilon \ \ otherwise, \end{cases}$$

where $\epsilon \in \mathbb{R}_{\geq 0}$ satisfies that $y'_{i_{|S|-m'}} = y'_{i_{|S|-m'+1}}$. By the definition of m', $y''_{i_{|S|-m'}} > y''_{i_{|S|-m'+1}}$ holds, and therefore $\epsilon > 0$ and $|\operatorname*{arg\,min}_{k=1,\dots,|S|} y'_{i_k}| > m'$ hold. Moreover, $\sum_{j=1}^{|S|} y'_{i_j} = \sum_{j=1}^{|S|} y''_{i_j}$ holds, and y' is an imputation of S. We will show $u^C_{i_j}(y', S) \geq u^C_{i_j}(y'', S)$ holds for any $j = 1, \dots, |S|$. For any $j \notin \operatorname*{arg\,min}_{k=1,\dots,|S|} y''_{i_k}$,

$$\begin{aligned} u^C_{i_j}(y', S) - u^C_{i_j}(y'', S) &= (1-\gamma)(y'_{i_j} - y''_{i_j}) + \gamma \left[\delta(y'_{i_{|S|}} - y''_{i_{|S|}}) + (1-\delta)\left(\sum_{j=1}^{|S|} y'_{i_j} - \sum_{j=1}^{|S|} y''_{i_j} \right) \right] \\ &= (1-\gamma)(y'_{i_j} - y''_{i_j}) + \gamma\delta(y'_{i_{|S|}} - y''_{i_{|S|}}) \\ &= -(1-\gamma)\frac{m'}{|S| - m'}\epsilon + \gamma\delta\epsilon \\ &= \frac{-m'(1-\gamma) + \gamma\delta(|S| - m')}{|S| - m'}\epsilon \geq 0 \end{aligned}$$

holds because $m' \leq m_S - 1$ holds. If $j \in \operatorname*{arg\,min}_{k=1,\dots,|S|} y''_{i_k}$, $u^C_{i_j}(y', S) \geq u^C_{i_j}(y'', S)$ holds obviously. Then, y' dominates x because y'' dominates x. By repeating this manipulation, we got the imputation which dominates x and satisfies that the number of players who have the minimum share is not smaller than m_S. As a result, we complete the proof. □

The following proposition corresponds to Proposition 1 in the sense that for any imputation of a coalition, $v(S)$ can be expressed by a weighted sum of utilities, and the weights are derived from γ and δ.

Proposition 4. Fix any $S \subset N$ and any $y \in X_S$, so that $S = \{i_1, \dots, i_{|S|}\}$ with $y_{i_1} \geq y_{i_2} \geq \dots \geq y_{i_{|S|-m_S+1}} = \dots = y_{i_{|S|}}$. Then,

$$\frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S} \sum_{j=1}^{|S|-m_S} u^C_{i_j}(y, S) + \frac{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|}{(1-\gamma)\Gamma_S} u^C_{i_{|S|-m_S+1}}(y, S) = v(S) \tag{CRTGR}$$

holds, where $\Gamma_S = 1 - \gamma + \gamma\delta + \gamma(1-\delta)|S|$.

Proof. By $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_{|S|-m_S+1}} = \cdots = y_{i_{|S|}}$,

$$u^C_{i_j}(y,S) = (1-\gamma)y_{i_j} + \gamma\left[\delta y_{i_{|S|-m_S+1}} + (1-\delta)\sum_{j=1}^{|S|} y_{i_j}\right]$$

holds for any $j = 1, \ldots, |S|$. Then,

$$\begin{aligned}
& \frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S}\sum_{j=1}^{|S|-m_S} u^C_{i_j}(y,S) + \frac{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|}{(1-\gamma)\Gamma_S}u^C_{i_{|S|-m_S+1}}(y,S) \\
= \;& \frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S}\sum_{j=1}^{|S|-m_S}\{u^C_{i_j}(y,S) - u^C_{i_{|S|-m_S+1}}(y,S)\} + \frac{|S|}{\Gamma_S}u^C_{i_{|S|-m_S+1}}(y,S) \\
= \;& \frac{1-\gamma+\gamma\delta}{\Gamma_S}\sum_{j=1}^{|S|}\{y_{i_j} - y_{i_{|S|-m_S+1}}\} \\
& + \frac{|S|}{\Gamma_S}\left\{(1-\gamma)y_{i_{|S|-m_S+1}} + \gamma\left[\delta y_{i_{|S|-m_S+1}} + (1-\delta)\sum_{j=1}^{|S|} y_{i_j}\right]\right\} \\
= \;& \frac{1-\gamma+\gamma\delta+\gamma(1-\delta)|S|}{\Gamma_S}\sum_{j=1}^{|S|} y_{i_j} + \frac{|S|}{\Gamma_S}\{-(1-\gamma)-\gamma\delta+(1-\gamma)+\gamma\delta\}y_{i_{|S|-m_S+1}} \\
= \;& v(S)
\end{aligned}$$

holds. □

By definition, if all players' shares rise by 1 unit, all players' utilities rise by Γ_S unit. As in the case of the F-S core,

$$(|S| - m_S)\frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S} + \frac{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|}{(1-\gamma)\Gamma_S} = \frac{|S|}{\Gamma_S}$$

holds. Moreover, if $\gamma = 0$ (hence $\Gamma_S = 1$ and $m_S = 1$), (CRTGR) is written as $\sum_{j=1}^{|S|} y_{i_j} = v(S)$. Then, (CRTGR) when $\gamma = 0$ corresponds to the identity of total group rationality which holds under the existing model of TU games.

By the above proposition, we can prove the next theorem which completely characterizes the C-R core.

Theorem 4. Consider $x \in X_N$. x is in the C-R core if and only if for all $S = \{i_1, i_2, \ldots, i_{|S|}\} \subset N$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$,

$$\frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S}\sum_{j=1}^{|S|-m_S} u^C_{i_j}(x,N) + \frac{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|}{(1-\gamma)\Gamma_S}u^C_{i_{|S|-m_S+1}}(x,N) \geq v(S) \tag{1}$$

holds, where $\Gamma_S = 1 - \gamma + \gamma\delta + \gamma(1-\delta)|S|$.

Proof. See Appendix A.4. □

This theorem characterizes the C-R core. If $\gamma = 0$, the inequality in the theorem is written as $\sum_{j=1}^{|S|} x_{i_j} \geq v(S)$. Then, we generalize the inequality that is the necessary and sufficient condition that an imputation of the grand coalition is blocked through S. Similarly to Section 3, we call the inequality (1) the condition of "coalitional rationality".

4.2 The Comparative statics of the C-R core about γ and δ

In this section, we will see how γ and δ change the shape of the C-R core. In Section 3.2, we prove that the F-S core is in the original core, but this statement does not hold about the C-R core. This fact can be explained in the following way. The F-S preferences have a property where their utilities do not decrease if the number of players they take care of decreases. However, the C-R preferences do not have such a property. Rather, if the number of players they take care of decreases, the total share does not increase, and the utilities do not increase.

Moreover, the monotonicity of the size of the core does not hold for γ, as the following example shows. Assume $N = \{1,2,3\}, v(N) = 21, v(\{1,2\}) = 18, v(\{1,3\}) = v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \delta = 0.5$, and $\gamma = 0$. In this case, the C-R utility function is the same as the traditional utility which is constructed only of the self-interest part. Let $x = (13,4,4)$. This imputation is not in the C-R core. (Consider $y = (13.5, 4.5) \in X_{\{1,2\}}$.)

Next, assume $N = \{1,2,3\}, v(N) = 21, v(\{1,2\}) = 18, v(\{1,3\}) = v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \delta = 0.5$, and $\gamma = 0.5$. If $x = (13,4,4)$, the utility of player 1 under x is

$$0.5 \times 13 + 0.5(0.5 \times 4 + 0.5 \times 21) = 12.75,$$

and the utility of player 2 under x is

$$0.5 \times 4 + 0.5(0.5 \times 4 + 0.5 \times 21) = 8.25.$$

We can describe any $y' \in X_{\{1,2\}}$ as $(y'_1, 18 - y'_1)$. Suppose y' dominates x. If $0 \leq y'_1 \leq 9$, the utility of player 1 under y' is

$$0.5 \times y'_1 + 0.5(0.5 \times y'_1 + 0.5 \times 18) = 0.75 \times y'_1 + 4.5.$$

Since y' dominates x, $y'_1 > 11$ but this is a contradiction. If $9 \leq y'_1 \leq 18$, the utility of player 1 under y' is

$$0.5 \times y'_1 + 0.5(0.5 \times (18 - y'_1) + 0.5 \times 18) = 0.25 \times y'_1 + 9,$$

and the utility of player 2 under y' is

$$0.5 \times (18 - y'_1) + 0.5(0.5 \times (18 - y'_1) + 0.5 \times 18) = -0.75 \times y'_1 + 18.$$

Therefore, y' dominates x if and only if $15 < y'_1$ and $y'_1 < 13$. However, there is no such y'_1, and no $S \subset N$ has $y \in X_S$ which dominates x. By the definition of v, any imputation of any coalition $\{i\}$ for any player $i \in N$ does not dominate x. We consider domination through the grand coalition N. The MNMSP of N, m_N, is 2. By Proposition 4,

$$u_1^C(y'', N) + u_2^C(y'', N) = 21$$

holds, where $y'' = (y''_1, y''_2, y''_3) \in X_N$ satisfies $y''_1 \geq y''_2 = y''_3$. Moreover, $u_1^C(x, N) + u_2^C(x, N) = 21$ holds, and

$$\{u_1^C(y, N) - u_1^C(x, N)\} + \{u_2^C(y, N) - u_2^C(x, N)\} = 0$$

holds. By Lemma 3, there is no $y'' \in X_N$ that dominates x.

We can interpret this case as follows. If $\gamma = 0$, the C-R utility of players has only a self-interest part and does not depend on the social welfare. So they can block x by an imputation which gives more to player 1. If $\gamma = 0.5$, they care about the social welfare. In this case, player 1 faces a trade-off between his share and the social welfare. Then, his utility is not improved unless he gets more share compared to the case of $\gamma = 0$. Nevertheless, if he gets enough, player 2 can not get enough, and the utility of x for player 2 can not be improved.

Finally, assume $N = \{1, 2, 3\}, v(N) = 21, v(\{1, 2\}) = 18, v(\{1, 3\}) = v(\{2, 3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \delta = 0.5$, and $\gamma = 0.9$. In this case, the MNMSP of the coalition N is 3, so we only have to check whether $y''' = (7, 7, 7) \in X_N$ dominates $x = (13, 4, 4)$ or not. The utility of player 1 under y''' is the same as the utility of player 2 under y''' and that of player 3. The value is

$$0.1 \times 7 + 0.9(0.5 \times 7 + 0.5 \times 21) = 13.3.$$

The utility of player 1 under x is

$$0.1 \times 13 + 0.9(0.5 \times 4 + 0.5 \times 21) = 12.55.$$

Since the utility of player 2 under x is the same as that of player 3 which is not larger than that of player 1, y''' dominates x. Intuitively, the social welfare in the blocking coalition increases compared to the case of $\gamma = 0.5$, and that allows them to block x when $\gamma = 0.9$.

However, such monotonicity holds for δ. The next theorem shows the fact.

Theorem 5. Fix γ and δ arbitrarily. For any $\delta' > \delta$, $C_{\gamma,\delta'} \subset C_{\gamma,\delta}$.

Proof. See Appendix A.5. □

The intuitive reason of Theorem 5 is as follows. When δ increases, the players get more concerned with the minimum share but less concerned with the total shares. Then, imputations of a blocking coalition which have larger minimum income come to dominate an imputation of the grand coalition even though the total shares of the imputation of the blocking coalition get smaller. Therefore, the C-R core becomes smaller.

5 The alpha-core and the beta-core

In the discussion so far, when the grand coalition is blocked, players in the blocking coalition do not care about the remaining players by assumption. In this section, we consider the case where the members of the blocking coalition care about all players and study the alpha-core and the beta-core[3]. If players care about all other players regardless of their coalition, some players who do not belong to the coalition interrupt the coalition by raising or lowering their own incomes. To consider the implication of these possibilities, the alpha-core and the beta-core are adequate.

Definition 7. $S \subset N$ can alpha-improve upon $x \in X_N$ through S if and only if there exists $y_S \in X_S$ such that for all $z_{N \setminus S} \in X_{N \setminus S}$, $u_i((y_S, z_{N \setminus S}), N) > u_i(x, N)$ holds.

Definition 8. The alpha-core, denoted by C^A, is the set of all elements in X_N which can not be alpha-improved through any $S \subset N$.

Definition 9. $S \subset N$ can beta-improve upon $x \in X_N$ through S if and only if for all $z_{N \setminus S} \in X_{N \setminus S}$, there exists $y_S \in X_S$ such that $u_i((y_S, z_{N \setminus S}), N) > u_i(x, N)$ holds.

Definition 10. The beta-core, denoted by C^B, is the set of all elements in X_N which can not be beta-improved through any $S \subset N$.

Let $C^A_{\alpha,\beta}$ denote the alpha-core with the F-S preferences (the F-S alpha-core hereafter) and $C^A_{\gamma,\delta}$ denote the alpha-core with the C-R preferences (the C-R alpha-core hereafter). Similarly, let $C^B_{\alpha,\beta}$ denote the beta-core with the F-S preferences (the F-S beta-core hereafter) and $C^B_{\gamma,\delta}$ denote the beta-core with the C-R preferences (the C-R beta-core hereafter). By definition, the alpha-core includes the beta-core under both the utility functions, i.e., $C^B_{\alpha,\beta} \subset C^A_{\alpha,\beta}$ and $C^B_{\gamma,\delta} \subset C^A_{\gamma,\delta}$. In general, the inclusions are strict, but consider the following kind of imputations:

[3]The terms, the alpha-core and the beta-core, are written as the α-core and the β-core in general. To avoid confusion among the parameters (α and β) and these terms, we write these terms by alphabets.

Definition 11. For any $S \subset N$, $z^*_{N\setminus S} \in X_{N\setminus S}$ is a dominant punishment imputation of S if and only if for any $y_S \in X_S$ and any $z_{N\setminus S} \in X_{N\setminus S}$, $u_i((y_S, z_{N\setminus S}), N) \geq u_i((y_S, z^*_{N\setminus S}), N)$ holds for any $i \in S$.

For any imputation of a blocking coalition, the dominant punishment imputation is the worst imputation for the blocking coalition. Moreover, a dominant punishment imputation corresponds to a dominant punishment strategy proposed by Nakayama (1998). He proposes this strategy in the context of coalitional strategic games and shows that when there is a dominant punishment strategy for any coalition, the alpha-core coincides with the beta-core. It can be readily verified that if a dominant punishment imputation exists in our model, the alpha-core coincides with the beta-core under any utility function. The difference between the alpha-core and the beta-core is only orders between punishment by a remaining coalition and deciding an imputation of a blocking coalition, so these cores do not depend on the orders if there exists a dominant punishment imputation.

5.1 The alpha- and beta-core with the F-S preferences

The next theorem states that the F-S alpha-core coincides with the F-S beta-core.

Theorem 6. $C^A_{\alpha,\beta} = C^B_{\alpha,\beta}$ holds.

Proof. We only have to prove there is a dominant punishment imputation for any coalition. Fix any $S \subset N$. We will prove for any $y \in X_S$ and any $z \in X_{N\setminus S}$, $u^F_i((y, z'), N) \leq u^F_i((y, z), N)$ holds for any $i \in S$, where $z' = (v(N\setminus S), 0, \ldots, 0)$. Namely, we will show z' is a dominant punishment imputation of S.

Fix any $i \in S$. First of all, we will show

$$\sum_{j \in N\setminus S} \max\{y_i - z_j, 0\} \leq \sum_{j \in N\setminus S} \max\{y_i - z'_j, 0\}$$

holds for any $y \in X_S$ and any $z \in X_{N\setminus S}$. Consider a function $f : X_{N\setminus S} \to \mathbb{R}$ such that

$$f(z) = \sum_{j \in N\setminus S} f_j(z),$$

where $f_j : X_{N\setminus S} \to \mathbb{R}$ is defined as $f_j(z) = \max\{y_i - z_j, 0\}$. Since f_j is a convex function, f is a convex function. Then,

$$f(tz + (1-t)z') \leq tf(z) + (1-t)f(z') \leq \max\{f(z), f(z')\}$$

for any $z, z' \in X_{N\setminus S}$. Since $X_{N\setminus S} = \{z \in \mathbb{R}^{n-|S|} | 0 \leq z_j \leq v(S), \sum_{j \in N\setminus S} x_j = v(N \setminus S)\}$, a maximizer of $f(z)$ is one of $n - |S|$ points which satisfy all components except only one component are 0 like $(0, \ldots, 0, v(N \setminus S), 0, \ldots, 0)$. By definition, f takes the same value among these $n - |S|$ points. Then, if $y_i \geq v(N \setminus S)$, $u^F_i((y, z'), N) \leq u^F_i((y, z), N)$ holds by the definition of the F-S utility functions.

Assume $y_i < v(N \setminus S)$. The sum of the difference between the player i's share and the other's shares which are larger than the i's share under z is

$$\sum_{j \in N \setminus S} \max\{z_j - y_i, 0\} \leq \left[\sum_{j \in N \setminus S} z_j\right] - y_i = v(N \setminus S) - y_i = \sum_{j \in N \setminus S} \max\{z'_j - y_i, 0\}.$$

As a result, we proved there exists a dominant punishment imputation. $\square$

Next, we will compare the F-S core and the F-S alpha-core. Intuitively, since the remaining players punish the blocking coalition, it is more difficult for a coalition to block the grand coalition, and the F-S alpha-core seems to include the F-S core. This intuition is correct, as the next proposition reveals.

Proposition 5. For any α and any β, $C_{\alpha,\beta} \subset C^A_{\alpha,\beta}$ holds.

Proof. We assume that $x \notin C^A_{\alpha,\beta}$. By the definition of the F-S alpha-core, there exists $y \in X_S$ such that for all $z \in X_{N\setminus S}$,

$$u_i^F((y,z),N) > u_i^F(x,N)$$

for all $i \in S$. For any $z \in X_{N\setminus S}$,

$$\begin{aligned}
u_i^F(y,S) &= y_i - \frac{\alpha}{n-1}\sum_{j\in S\setminus\{i\}} max[y_j - y_i, 0] - \frac{\beta}{n-1}\sum_{j\in S\setminus\{i\}} max[y_i - y_j, 0] \\
&\geq u_i^F((y,z),N) \\
&= y_i - \frac{\alpha}{n-1}\sum_{j\in S\setminus\{i\}} max[y_j - y_i, 0] - \frac{\beta}{n-1}\sum_{j\in S\setminus\{i\}} max[y_i - y_j, 0] \\
&\quad -\frac{\alpha}{n-1}\sum_{j\in N\setminus S} max[z_j - y_i, 0] - \frac{\beta}{n-1}\sum_{j\in N\setminus S} max[y_i - z_j, 0] > u_i^F(x,N),
\end{aligned}$$

and therefore $x \notin C_{\alpha,\beta}$ holds. $\square$

This proposition implies that the condition of non-emptiness of the F-S core is stronger than that of the F-S alpha-core. The F-S core tends to exclude unequal imputations which might be included in the F-S alpha-core.

Note that the F-S core is included in the standard core with a traditional utility ($C_{0,0}$). However, the F-S alpha-core is not included in the standard core as the following example shows:

Assume $N = \{1,2,3\}, v(N) = 21, v(\{1,2\}) = 20, v(\{1,3\}) = v(\{2,3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, and $\beta > 0.6$. If $x = (7,7,7)$, the utility of each player is 7. We can describe any $y \in X_{\{1,2\}}$ as $(y_1, 20-y_1)$, where $y_1 \in [0,20]$. We only have to show that there is no $(y_1, 20 - y_1)$ which dominates x under $y_1 \leq 10$. When player 1 and player 2 join the coalition $\{1, 2\}$, player 3 gets $v(\{3\}) = 0$, and then, the utility of player 1 under $(y, 0)$ is

$$y_1 - \frac{\alpha}{2}\{(20 - y_1) - y_1\} - \frac{\beta}{2}(y_1 - 0) = \left(1 + \alpha - \frac{\beta}{2}\right) y_1 - 10\alpha.$$

By $y_1 \leq 10$ and $\beta > 0.6$,

$$u_1^F((y, 0), N) < (1 + \alpha - 0.3)10 - 10\alpha = 7$$

holds. Then, the coalition $\{1, 2\}$ can not alpha-improve upon x. By $v(\{1, 3\}) = v(\{2, 3\}) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, x is in the F-S alpha-core. However, x is not in the standard core because $x_1 + x_2 = 14 < v(\{1, 2\}) = 20$. Then, $C^A_{\alpha,\beta} \subset C_{0,0}$ does not always hold.

5.2 The alpha- and beta-core with the C-R preferences

In this section, we consider the C-R alpha-core and the C-R beta-core. First, we will show the C-R alpha-core coincides with C-R beta-core like the case of the F-S preferences.

Theorem 7. $C^A_{\gamma,\delta} = C^B_{\gamma,\delta}$ holds.

Proof. We only have to prove there is a dominant punishment imputation for any coalition. Fix any $S \subset N$. We will prove for any $y \in X_S$ and any $z \in X_{N\setminus S}$, $u_i^C((y, z^0), N) \leq u_i^C((y, z), N)$ holds for any $i \in S$, where $z^0 \in X_{N\setminus S}$ satisfies $\min_{i\in N\setminus S} z_i^0 = 0$. Namely, we will show z^0 is a dominant punishment imputation of S.

By the definition of the C-R utility functions,

$$\begin{aligned} u_i^C((y, z^0), N) &= (1 - \gamma)y_i + \gamma[0 + (1 - \delta)(v(S) + v(N \setminus S)] \leq u_i^C((y, z), N) \\ &= (1 - \gamma)y_i + \gamma\delta \min\{\min_{j\in S} y_j, \min_{j\in N\setminus S} z_j\} \\ &\quad +\gamma(1 - \delta)\{v(S) + v(N \setminus S)\}, \end{aligned}$$

and we complete the proof. □

In the case of the C-R preferences, there are no inclusion relationships among the standard core with a traditional utility, the C-R core, and the C-R alpha-core. In contrast, the F-S core is included by the standard core and the F-S alpha-core. The reason for this contrast is that the C-R utility function of the players in a coalition S may either increase or decrease when the number of players whom one cares about increases. In particular, the social welfare may either increase or decrease because the total welfare does not decrease but the minimum income does not increase. This property does not hold with the F-S preferences. The F-S utility of a player (weakly) decreases whenever the number of players whom the player cares about increases. That is why the different result occurs between the F-S and C-R preferences.

6 Concern network and new core concepts

In this section, we consider the case where each player is concerned with not only the players in their coalition but also some players who do not belong to their coalition. In Sections 3 and 4, we assume there are no such players, and in Section 5, each player is concerned with all players. The case we study in this section, therefore, can be regarded as an intermediate case between Sections 3 and 5. To describe this case, let us consider a network of players, denoted by a directed graph $G \subset N \times N$. We say the player i is concerned with the player j if $(i, j) \in G$. Then, if $G = \emptyset$, this case coincides with the case in Sections 3 and 4, and if G is a complete graph, this case coincides with the case in Section 5. Notice that we allow the case where a player is concerned with another player but not vice versa. If these two players are concerned with each other, we interpret their relationship is very intimate like a family.

Definition 12. $i, j \in S$ are intimate if and only if $(i, j) \in G$ and $(j, i) \in G$.

Notations are as follows. For each $i \in N$, let $S_i = \{j \in N | (i, j) \in G\}$, that is, S_i means the set of the players whom player i is concerned with. For any $S \subset N$, let $S^c = \{i \in N \setminus S|$ For some j $\in S, i$ and j are intimate$\}$, that is, S^c is the set of the players who are intimate with some player in S, and let $S^{nc} = N \setminus (S \cup S^c)$, that is, S^{nc} is the set of the players who are not in S and are not intimate to any player in S. Finally, we define $e_S = (\frac{v(S)}{|S|}, \dots, \frac{v(S)}{|S|}) \in X_S$ for each $S \subset N$.

If there is such a network of players, it is natural to imagine that a player keeps caring about some other players after he/she blocked an imputation of the grand coalition when they have an intimate relationship. The alpha-core or the beta-core allows them to punish the players in the blocking coalition, but they do not want to punish their intimate players. Rather, they may wish to help the intimate players. Then, we propose a new core concept, that is, G-alpha-core.

In this section, we focus only on the F-S utility functions. We can define the G-alpha-core under the C-R utility functions, but there are no inclusion relationships among the C-R core, the C-R alpha-core, and the G-alpha-core under the C-R utility functions by the same reason in the last paragraph at Section 5.2.

Definition 13. $S \subset N$ can G-alpha-improve upon $x \in X_N$ if and only if there exists $y \in X_S$ such that for all $z \in X_{S^{nc}}$ $u_i^F((y, z, e_{S^c}), S \cup S_i) > u_i^F(x, N)$ for all $i \in S$.

Definition 14. The G-alpha-core, denoted by $C_{\alpha,\beta}^{AG}$, is the set of all elements in X_N which can not be G-alpha-improved through any $S \subset N$.

An important difference between the alpha-core and the G-alpha-core is that in the G-alpha-core, if some coalition S tries to block the grand coalition, the players in S^c cooperate and allocate themselves the equal imputation. They cooperate and allocate themselves the equal imputation because they will not try to punish the blocking coalition S and are basically egalitarian. This idea comes from Equal Devision Core that is proposed by Selten (1972) to explain experimental outcomes.

Moreover, we can also define the G-beta-core:

Definition 15. $S \subset N$ can G-beta-improve upon $x \in X_N$ if and only if for all $z \in X_{S^{nc}}$, there exists $y \in X_S$ such that $u_i((y, z, e_{S^c}), S \cup S_i) > u_i(x, N)$ for all $i \in S$.

Definition 16. The G-beta-core, denoted by C^{BG}, is the set of all elements in X_N which can not be G-beta-improved through any $S \subset N$.

The G-alpha-core does not coincide with the G-beta-core because there might be no dominant punishment imputation unless G is the complete graph. Consider the following example. Assume $N = \{1, 2, 3, 4\}$, $\alpha = \beta = 1/2$, $G = \{(3, 1), (4, 2)\}$, and

$$v(S) = \begin{cases} 24 \text{ if } S = N \\ 12 \text{ if } |S| = 3 \\ 12 \text{ if } S = \{1, 2\} \\ 8 \text{ if } S = \{3, 4\} \\ 0 \text{ otherwise.} \end{cases}$$

Let $x = (8, 8, 4, 4) \in X_N$. Coalitions except $\{3, 4\}$ will not block x. The utility of player 3 and that of player 4 is the same value under x, and its value is

$$4 - \frac{1}{3} \cdot \frac{1}{2}\{(8 - 4) + (8 - 4)\} = \frac{8}{3}.$$

Fix any $z = (z_1, z_2) \in X_{\{1,2\}}$. Without loss of generality, assume $z_1 \geq z_2$. If $z_1 \neq 12$,

$$\begin{aligned} u_k^F((z, 4, 4), \{3, 4\} \cup S_k) &= 4 - \frac{1}{3} \cdot \frac{1}{2}|z_l - 4| \\ &> 4 - \frac{1}{3} \cdot \frac{1}{2}(12 - 4) = \frac{8}{3} \end{aligned}$$

for any $k = 3, 4$, where $l \in S_k$. If $z_1 = 12$,

$$\begin{aligned} u_3^F((z, 4.1, 3.9), \{3,4\} \cup \{1\}) &= 4.1 - \frac{1}{3} \cdot \frac{1}{2}(12 - 4.1 + 4.1 - 3.9) \\ &= 2.75 > \frac{8}{3} \end{aligned}$$

and

$$\begin{aligned} u_4^F((z, 4.1, 3.9), \{3,4\} \cup \{2\}) &= 3.9 - \frac{1}{3} \cdot \frac{1}{2}(3.9 - 0 + 4.1 - 3.9) \\ &= \frac{193}{60} > \frac{8}{3}. \end{aligned}$$

Hence, x is not in the G-beta-core.

However, x is in the G-alpha-core. Fix any $y = (y_3, y_4) \in X_{\{3,4\}}$. Without loss of generality, assume $y_3 \geq y_4$. Then,

$$u_4^F((0, 12, y), \{3,4\} \cup \{2\}) = y_4 - \frac{1}{3} \cdot \frac{1}{2}(12 - y_4 + y_3 - y_4) \leq \frac{8}{3}$$

because $y_3 \geq 4$ and $y_4 \leq 4$. Therefore, the G-alpha-core does not coincide with the G-beta-core in this game, and this implies that there is no dominant punishment imputation.

In the case of the ordinary α- and β-core studied in the previous section, the players in the remaining coalition are symmetric, but the players are not symmetric because of the network. Then, the worst imputation of S^{nc} for a blocking coalition S depends on the imputation of S. That is why there might be no dominant punishment imputation.

We will compare the F-S core and the G-beta-core. In the case of the G-beta-core, some remaining players punish the blocking coalition, and then, we expect the G-beta-core includes the F-S core. These intuitions are correct, as the next proposition shows. Moreover, the next proposition shows that the G-beta-core is included by the G-alpha-core.

Proposition 6. $C_{\alpha,\beta} \subset C^{BG}_{\alpha,\beta} \subset C^{AG}_{\alpha,\beta}$.

Proof. First of all, we will show $C_{\alpha,\beta} \subset C^{BG}_{\alpha,\beta}$. We assume that $x \notin C^{BG}_{\alpha,\beta}$. By the definition of "$G$-beta-improve", for all $z \in X_{N \setminus S}$, there exist $S \subset N$ and $y \in X_S$ such that

$$u_i^F((y, z, e_{S^c}), S \cup S_i) > u_i^F(x, N)$$

for any $i \in S$. For any $z \in X_{N \setminus S}$,

$$\begin{aligned} u_i^F(y, S) &= y_i - \frac{\alpha}{n-1} \sum_{j \in S \setminus \{i\}} max[y_j - y_i, 0] - \frac{\beta}{n-1} \sum_{j \in S \setminus \{i\}} max[y_i - y_j, 0] \\ &\geq u_i^F((y, z, e_{S^c}), S \cup S_i) > u_i^F(x, N), \end{aligned}$$

for any $i \in S$. Then, y dominates x and $x \notin C_{\alpha,\beta}$ holds.

Secondly, we will show $C^{BG}_{\alpha,\beta} \subset C^{AG}_{\alpha,\beta}$. We assume that $x \notin C^{AG}_{\alpha,\beta}$. By the definition of "$G$-alpha-improve", there exist $S \subset N$ and $y \in X_S$ such that for all $z \in X_{N\setminus S}$,

$$u^F_i((y, z, e_{S^c}), S \cup S_i) > u^F_i(x, N)$$

for any $i \in S$. Then, for all $z \in X_{N\setminus S}$, $S \subset N$ and $y \in X_S$ satisfies the above inequality. Therefore, $x \notin C^{BG}_{\alpha,\beta}$. □

The result that the G-beta-core is included by the G-alpha-core also holds under the model of coalitional strategic games, which has already been shown in previous studies. The idea of the G-alpha- or G-beta-core is an intermediate case between the F-S core and the F-S alpha- or F-S beta-core, respectively. In the F-S core, the players do not care about the other players who are not in their coalition, and this proposition shows that the F-S core is included by the G-alpha- and G-beta-core respectively, where the players would care about the other players who is not in their coalition.

In contrast, there is no inclusion relationship between the F-S alpha-core and the G-alpha-core. See the following example. Notice that the examples also hold when we compare the F-S beta-core and the G-beta-core.

Assume $N = \{1, 2, 3\}, v(N) = 10, v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3, v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \alpha = 0.5$ and $\beta = 0$. If $x = (8/7, 31/7, 31/7) \in X_N$, the utility of player 1 is

$$\frac{8}{7} - \frac{\alpha}{2}\left\{\frac{31}{7} - \frac{8}{7} + \frac{31}{7} - \frac{8}{7}\right\} = -0.5.$$

It is readily seen that all coalitions except $\{1\}$ are not a blocking coalition under both the G-alpha-core and the F-S alpha-core because when the coalitions try to block x, player 2 or player 3 can not get their utility of x, $31/7$. Then, consider $\{1\}$. First, we will show x is in the F-S alpha-core. If $\{1\}$ tries to block x, then player 1's utility is

$$0 - \frac{\alpha}{2}(3 - 0) = -0.75 < -0.5$$

whatever an imputation of $\{2, 3\}$ is. Therefore, x is in the F-S alpha-core because $\{1\}$ is not a blocking coalition. Next, we will prove x is not in the G-alpha-core, where we assume $G = \{(1, 2), (2, 1), (1, 3)\}$. Then, player 1 and player 2 are intimate. The utility of x for player 1 is the same as the case of the F-S alpha-core. If $\{1\}$ tries to block x, then player 2 and player 3 do not make the coalition because player 2 is an intimate player for player 1 but player 3 is not, and then player 1's utility is 0. Therefore, x is not in the G-alpha-core. As a result, $C^{A}_{\alpha,\beta} \subset C^{AG}_{\alpha,\beta}$ does not always hold.

We will show $C^{AG}_{\alpha,\beta} \subset C^{A}_{\alpha,\beta}$ does not always hold. See the following example: Assume $N = \{1,2,3\}, v(N) = 10, v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 3, v(\{1\}) = 4, v(\{2\}) = v(\{3\}) = 0$, and $\alpha = \beta = 0.3$. If $x = (3, 3.5, 3.5)$, the utility of player 1 is

$$3 - \frac{\alpha}{2}\{(3.5-3) + (3.5-3)\} = 2.85.$$

It is readily seen that all coalitions except $\{1\}$ are not a blocking coalition under both the G-alpha-core and the F-S alpha-core. This is because the utilities of x for player 2 and player 3 are

$$3.5 - \frac{\beta}{2}\{3.5-3\} = 3.425.$$

Consider $\{1\}$. First, we will show x is not in the F-S alpha-core. If $\{1\}$ tries to block x, the utility of player 1 is

$$4 - \frac{\beta}{2}(4-3+4-0) = 3.25 > 2.85$$

whatever an imputation of $\{2,3\}$ is. Therefore, x is not in the F-S alpha-core because $\{1\}$ is a blocking coalition. Next, we will prove x is in the G-alpha-core, where we assume $G = \{(1,2),(2,1),(1,3)\}$. Then, player 1 and player 2 are intimate. The utility of x for player 1 is the same as the case of the F-S alpha-core. If $\{1\}$ tries to block x, player 2 and player 3 do not make the coalition because player 2 is an intimate player for player 1 but player 3 is not, and then player 1's utility is

$$4 - \frac{\beta}{2}(4-0+4-0) = 2.8 < 2.85.$$

Therefore, x is in the G-alpha-core. As a result, $C^{AG}_{\alpha,\beta} \subset C^{A}_{\alpha,\beta}$ does not always hold.

By the definition of the graph, this graph prevents the remaining players making a coalition. It makes the blocking coalition to get larger shares or get smaller shares. As a result, there is no inclusion relationship between the F-S alpha-core and the G-alpha-core.

7 Conclusion

In this paper, we analyzed core concepts when players have the Fehr-Schmidt preferences or the Charness-Rabin preferences, respectively. We characterized the new cores by a set of inequalities called "coalitional rationality" which is an extension of that on ordinary TU games when players care about those who are in their coalition.

We analyzed not only the case where players are concerned with those who are in their coalition but also the case where players are concerned with other players in two different ways. One is that players are concerned with all the players in the game. The other is that players are concerned with some specific players indicated by a directed graph G and the players in their coalition. In the case where players are concerned with all players, the alpha-core coincides with the beta-core but this may not hold in the other case. The failure of coincidence is due to the structure of a graph G. It is interesting to ask what type of the graph G make the G-alpha-core coincide with the G-beta-core. Or one can ask what type of G makes the cores bigger or smaller.

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A Appendix

A.1 Proof of Proposition 1

If $S = N$ and $\beta = \frac{n-1}{n}$ (hence $A_0 = 0$),

$$\begin{aligned} |S|u_{i_1}^F(y,S) &= |S|\left\{y_{i_1} - \frac{\beta}{n-1}\sum_{k=1}^{|S|}(y_{i_1} - y_{i_k})\right\} \\ &= |S|\left\{A_0 y_{i_1} + \frac{\beta}{n-1}v(S)\right\} \\ &= v(N). \end{aligned}$$

Assume $S \neq N$ or $\beta < \frac{n-1}{n}$ (hence $A_0 \neq 0$). For all $i_j \in S$,

$$u_{i_j}^F(y,S) = y_{i_j} - \frac{\alpha}{n-1}\sum_{k=1}^{j-1}(y_{i_k} - y_{i_j}) - \frac{\beta}{n-1}\sum_{k=j+1}^{|S|}(y_{i_j} - y_{i_k})$$

holds. Then,

$$\begin{aligned} u_{i_{j-1}}^F(y,S) - u_{i_j}^F(y,S) &= y_{i_{j-1}} - \frac{\alpha}{n-1}\sum_{k=1}^{j-1}(y_{i_k} - y_{i_{j-1}}) - \frac{\beta}{n-1}\sum_{k=j}^{|S|}(y_{i_{j-1}} - y_{i_k}) \\ &\quad - \left\{y_{i_j} - \frac{\alpha}{n-1}\sum_{k=1}^{j-1}(y_{i_k} - y_{i_j}) - \frac{\beta}{n-1}\sum_{k=j}^{|S|}(y_{i_j} - y_{i_k})\right\} \\ &= (y_{i_{j-1}} - y_{i_j}) + \frac{j-1}{n-1}\alpha(y_{i_{j-1}} - y_{i_j}) - \frac{|S|-j+1}{n-1}\beta(y_{i_{j-1}} - y_{i_j}) \\ &= A_{j-1}(y_{i_{j-1}} - y_{i_j}) \end{aligned}$$

holds. Thus,

$$y_{i_1} - y_{i_{|S|}} = \sum_{j=2}^{|S|} (y_{i_{j-1}} - y_{i_j})$$
$$= \frac{u^F_{i_1}(y,S)}{A_1} + \sum_{j=2}^{|S|-1} \left(\frac{1}{A_j} - \frac{1}{A_{j-1}} \right) u^F_{i_j}(y,S) - \frac{u^F_{i_{|S|}}(y,S)}{A_{|S|-1}}. \quad \text{Eq. (A.1)}$$

Moreover,

$$u^F_{i_1}(y,S) = A_0 y_{i_1} + \frac{\beta}{n-1} v(S)$$

and

$$\begin{aligned} u^F_{i_{|S|}}(y,S) &= y_{i_{|S|}} - \frac{\alpha}{n-1} \sum_{k=1}^{|S|} (y_{i_k} - y_{i_{|S|}}) \\ &= A_{|S|} y_{i_{|S|}} - \frac{\alpha}{n-1} v(S) \end{aligned}$$

hold because of $\sum_{i \in S} y_i = v(S)$. Therefore,

$$y_{i_1} - y_{i_{|S|}} = \frac{u^F_{i_1}(y,S) - \frac{\beta}{n-1} v(S)}{A_0} - \frac{u^F_{i_{|S|}}(y,S) + \frac{\alpha}{n-1} v(S)}{A_{|S|}} \quad \text{Eq. (A.2)}$$

holds. By Eq. (A.1) and Eq. (A.2),

$$\sum_{j=1}^{|S|} \left(\frac{1}{A_{j-1}} - \frac{1}{A_j} \right) u^F_{i_j}(y,S) = \left(\frac{\beta}{A_0} + \frac{\alpha}{A_{|S|}} \right) \frac{v(S)}{n-1}$$

holds. By the definition of A_j, we rearrange this and get the following equality:

$$\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_j A_{j-1}} u^F_{i_j}(y,S) = v(S).$$

A.2 Proof of Theorem 2

First of all, we will show the next lemma, which introduces useful formulas.

Lemma 4. For any α and any β,

$$(j+k)A_j - jA_{j+k} = kA_0$$

holds, where $A_j = 1 + \frac{j}{n-1}\alpha - \frac{|S|-j}{n-1}\beta$.

Proof. By the definition of A_j,

$$(j+k)A_j - jA_{j+k} = (j+k)\left(1+\frac{j}{n-1}\alpha - \frac{|S|-j}{n-1}\beta\right) - j\left(1+\frac{j+k}{n-1}\alpha - \frac{|S|-j-k}{n-1}\beta\right)$$
$$= k - \frac{k|S|}{n-1}\beta = kA_0.$$

□

Secondly, we will show that x is not in the F-S core if and only if there exists some $S = \{i_1, \dots, i_{|S|}\} \subsetneq N$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$ such that $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1}A_j} u_{i_j}^F(x, N) < v(S)$, and we will complete the proof.
($\Rightarrow$) Assume x is not in the F-S core. Then, there exist $S \subset N$ and $y \in X_S$ such that y dominates x through S. Let $\{i_1, \dots, i_{|S|}\} = S$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$. Thus, $u_{i_j}^F(y, S) > u_{i_j}^F(x, N)$ holds for all $j = 1, \dots, |S|$. Besides, $S \neq N$ holds by Proposition 2, and therefore $A_0 \neq 0$ holds. Since $\frac{A_0 A_{|S|}}{A_{j-1}A_j} > 0$ for all $j = 1, \dots, |S|$ by the definition of A_j, $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1}A_j}\{u_{i_j}^F(y, S) - u_{i_j}^F(x, N)\} > 0$ holds. By Lemma 2, we can assume $y_{i_1} \geq \cdots \geq y_{i_{|S|}}$ without loss of generality. Then, $v(S) > \sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1}A_j} u_{i_j}^F(x, N)$ holds by Proposition 1.
($\Leftarrow$) If there exists some $S = \{i_1, \dots, i_{|S|}\} \subsetneq N$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$ such that $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1}A_j} u_{i_j}^F(x, N) < v(S)$, there exists some $\lambda > 0$ such that $\sum_{j=1}^{|S|} \frac{A_0 A_{|S|}}{A_{j-1}A_j}(u_{i_j}^F(x, N) + \lambda) = v(S)$. Let $\tilde{u}_{i_j} = u_{i_j}^F(x, N) + \lambda$. By Lemma 1, $\tilde{u}_{i_1} \geq \tilde{u}_{i_2} \geq \cdots \geq \tilde{u}_{i_{|S|}}$ holds. Moreover, if $\tilde{u}_{i_{|S|}} < 0$, the player $i_{|S|}$ can block x through $\{i_{|S|}\}$, and x is not in the F-S core. Then, assume $\tilde{u}_{i_{|S|}} \geq 0$.

Define $y = (y_{i_1}, y_{i_2}, \dots, y_{i_{|S|}})$ as follows:

$$\begin{aligned} y_{i_{|S|}} &= \frac{\tilde{u}_{i_{|S|}} + \frac{\alpha}{n-1}v(S)}{A_{|S|}} \\ y_{i_j} &= y_{i_{j+1}} + \frac{\tilde{u}_{i_j} - \tilde{u}_{i_{j+1}}}{A_j} \end{aligned}$$

for all $j = 1, \dots, |S| - 1$.

We will confirm that y is in X_S. $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_{|S|}}$ holds because of $\tilde{u}_{i_1} \geq \tilde{u}_{i_2} \geq \cdots \geq \tilde{u}_{i_{|S|}}$. Since $y_{i_{|S|}} \geq 0$ because of $\tilde{u}_{i_{|S|}} \geq 0$, $y_{i_j} \geq 0$ holds for all $j = 1, \dots, |S|$. Therefore, we only have to show $\sum_{j=1}^{|S|} y_{i_j} = v(S)$. We can get this equality as follows:

$$\begin{aligned}
\sum_{j=1}^{|S|} y_{i_j} &= \frac{\tilde{u}_{i_{|S|}} + \frac{\alpha}{n-1}v(S)}{A_{|S|}} + \sum_{j=1}^{|S|-1}\left(\frac{\tilde{u}_{i_{|S|}} + \frac{\alpha}{n-1}v(S)}{A_{|S|}} + \sum_{k=|S|-j}^{|S|-1}\frac{\tilde{u}_{i_k} - \tilde{u}_{i_{k+1}}}{A_k}\right) \\
&= |S|\frac{\tilde{u}_{i_{|S|}} + \frac{\alpha}{n-1}v(S)}{A_{|S|}} + \sum_{j=1}^{|S|-1} j\left(\frac{\tilde{u}_{i_j} - \tilde{u}_{i_{j+1}}}{A_j}\right) \\
&= \frac{|S|\alpha}{A_{|S|}(n-1)}v(S) + \frac{1}{A_{|S|}}\sum_{j=1}^{|S|}\frac{A_0 A_{|S|}}{A_j A_{j-1}}\tilde{u}_{i_j} \\
&= v(S).
\end{aligned}$$

The third equality comes from $\frac{j}{A_j} - \frac{j-1}{A_{j-1}} = \frac{A_0}{A_j A_{j-1}}$ which holds by Lemma 4, and the last equality comes from the definitions of $A_{|S|}$ and $\tilde{u}_{i_j}$.

Therefore, if y satisfies $u^F_{i_j}(y,S) = \tilde{u}_{i_j}(= u^F_{i_j}(x,N)+\lambda)$ for all $j = 1,\ldots,|S|$, the proof completes. By the definition of y,

$$\begin{aligned}
& y_{i_{|S|}} = \frac{\tilde{u}_{i_{|S|}} + \frac{\alpha}{n-1}v(S)}{A_{|S|}} \\
\Leftrightarrow \quad & u^F_{i_{|S|}}(y,S) = \tilde{u}_{i_{|S|}}
\end{aligned}$$

holds, and for all $j \leq |S|-1$,

$$\begin{aligned}
& y_{i_j} = y_{i_{j+1}} + \frac{\tilde{u}_{i_j} - \tilde{u}_{i_{j+1}}}{A_j} \\
\Leftrightarrow \quad & u^F_{i_j}(y,S) - u^F_{i_{j+1}}(y,S) = \tilde{u}_{i_j} - \tilde{u}_{i_{j+1}}
\end{aligned}$$

holds. Therefore, we can prove $u^F_{i_j}(y,S) = \tilde{u}_{i_j}$ inductively.

As a result, we proved Theorem 2.

A.3 Proof of Theorem 3

By Theorem 1, if $\beta > \frac{n-1}{n}$, the F-S core does not depend on α. Then, we must show the statement of the theorem when $\beta \leq \frac{n-1}{n}$. Assume $\beta \leq \frac{n-1}{n}$. Fix any $S = \{i_1,\ldots,i_{|S|}\} \subsetneq N$ and any $x \in X_N$ with $x_{i_1} \geq \cdots \geq x_{i_{|S|}}$. By Theorem 2, we have to prove only $\frac{\partial T}{\partial \alpha} \leq 0$, where $T = \sum_{j=1}^{|S|}\frac{A_0 A_{|S|}}{A_{j-1}A_j}u^F_{i_j}(x,N)$.

Let $B_j = \frac{A_0 A_{|S|}}{A_{j-1}A_j}$. By $\sum_{j=1}^{|S|}\frac{A_0 A_{|S|}}{A_j A_{j-1}} = |S|$, $\sum_{j=1}^{|S|} B_j = |S|$ holds. Then,

$\sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} = 0$ holds, and therefore

$$\begin{aligned}
\sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} u_{i_j}^F(x,N) =& \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} \left(x_{i_j} - \frac{\alpha}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] - \frac{\beta}{n-1} \sum_{k \in N} max[x_{i_j} - x_k, 0] \right) \\
=& \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} \left(x_{i_j} - \frac{\alpha+\beta}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] \right. \\
& \left. + \frac{\beta}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] - \frac{\beta}{n-1} \sum_{k \in N} max[x_{i_j} - x_k, 0] \right) \\
=& \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} \left(x_{i_j} + \frac{\beta}{n-1} \sum_{k \in N} (x_k - x_{i_j}) - \frac{\alpha+\beta}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] \right) \\
=& \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} \left(\left(1 - \frac{n}{n-1}\beta \right) x_{i_j} + \frac{\beta}{n-1} v(N) - \frac{\alpha+\beta}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] \right) \\
=& \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} \left(\left(1 - \frac{n}{n-1}\beta \right) x_{i_j} - \frac{\alpha+\beta}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{\partial T}{\partial \alpha} \leq 0 \\
\Leftrightarrow & \sum_{j=1}^{|S|} \left\{ \frac{\partial B_j}{\partial \alpha} u_{i_j}^F(x,N) - \frac{B_j}{(n-1)} \sum_{k \in N} max[x_k - x_{i_j}, 0] \right\} \leq 0 \\
\Leftrightarrow & \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} u_{i_j}^F(x,N) \leq \sum_{j=1}^{|S|} \frac{B_j}{n-1} \sum_{k \in N} max[x_k - x_{i_j}, 0] \\
\Leftrightarrow & \left(1 - \frac{n}{n-1}\beta \right) \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} x_{i_j} \leq \sum_{j=1}^{|S|} \left\{ \frac{\alpha+\beta}{n-1} \frac{\partial B_j}{\partial \alpha} + \frac{B_j}{n-1} \right\} \sum_{k \in N} max[x_k - x_{i_j}, 0].
\end{aligned}$$

Eq. (A.3)

Furthermore, we can claim $C_j \geq 0$, where $C_j = \frac{\alpha+\beta}{n-1} \frac{\partial B_j}{\partial \alpha} + \frac{B_j}{n-1}$. This is because

$$\begin{aligned}
\frac{\partial B_j}{\partial \alpha} &= A_0 \frac{|S| A_j A_{j-1} - j A_{|S|} A_{j-1} - (j-1) A_{|S|} A_j}{(A_{j-1} A_j)^2 (n-1)} \\
&= B_j \frac{|S| A_j A_{j-1} - j A_{|S|} A_{j-1} - (j-1) A_{|S|} A_j}{A_{|S|} A_{j-1} A_j (n-1)}.
\end{aligned}$$

Then, since $|S|A_j - jA_{|S|} = (|S| - j)A_0$ holds by Lemma 4,

$$\begin{aligned} C_j &= \frac{B_j}{n-1} \frac{\frac{\alpha+\beta}{n-1}(|S|A_jA_{j-1} - jA_{|S|}A_{j-1} - (j-1)A_{|S|}A_j) + A_{j-1}A_jA_{|S|}}{A_{j-1}A_jA_{|S|}} \\ &= \frac{B_j}{n-1} \frac{\frac{\alpha+\beta}{n-1}((|S| - j)A_0A_{j-1} - (j-1)A_{|S|}A_j) + A_{j-1}A_jA_{|S|}}{A_{j-1}A_jA_{|S|}} \\ &= \frac{B_j}{A_{j-1}A_jA_{|S|}(n-1)} \left(\frac{\alpha+\beta}{n-1}(|S| - j)A_0A_{j-1} + A_0A_{|S|}A_j \right) \geq 0 \end{aligned}$$

holds by $A_{j-1} + \frac{\alpha+\beta}{n-1} = A_j$ for all $j = 1, \ldots, |S|$.

Therefore, to show the following inequality is sufficient to show Eq. (A.3):

$$\left(1 - \frac{n}{n-1}\beta\right) \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} x_{i_j} \leq \sum_{j=1}^{|S|} \left\{ \frac{\alpha+\beta}{n-1} \frac{\partial B_j}{\partial \alpha} + \frac{B_j}{n-1} \right\} \sum_{k=1}^{j-1} (x_{i_k} - x_{i_j}).$$

Let $D_j = (j-1)C_j + (1 - \frac{n}{n-1}\beta)\frac{\partial B_j}{\partial \alpha}$. Then, the above inequality is equivalent to

$$\begin{aligned} & \sum_{j=1}^{|S|} \left(D_j x_{i_j} - C_j \sum_{k=1}^{j-1} x_{i_k} \right) \leq 0 \\ \Leftrightarrow \; & \sum_{j=1}^{|S|} D_j x_{i_j} - \sum_{j=1}^{|S|} \sum_{k=1}^{j-1} C_j x_{i_k} \leq 0 \\ \Leftrightarrow \; & \sum_{j=1}^{|S|} D_j x_{i_j} - \sum_{k=1}^{|S|} \sum_{j=k+1}^{|S|} C_j x_{i_k} \leq 0 \\ \Leftrightarrow \; & \sum_{j=1}^{|S|} \left(D_j - \sum_{k=j+1}^{|S|} C_k \right) x_{i_j} \leq 0. \end{aligned}$$

Besides, since $\frac{\alpha+\beta}{n-1} = A_j - A_{j-1}$ holds for all $j = 1, \ldots, |S|$,

$$\begin{aligned} D_j =& \frac{\partial}{\partial \alpha} \{ (\frac{\alpha+\beta}{n-1}(j-1) + 1 - \frac{n\beta}{n-1})B_j \} \\ =& \frac{\partial}{\partial \alpha} \{ (A_{j-1} - \frac{n - |S|}{n-1}\beta)B_j \} \\ =& \frac{\partial}{\partial \alpha} \{ \frac{A_0 A_{|S|}}{A_j} - \frac{n - |S|}{n-1}\beta B_j \} \end{aligned}$$

and

$$C_j = \frac{\partial}{\partial \alpha}(A_j - A_{j-1})B_j = \frac{\partial}{\partial \alpha}(\frac{1}{A_{j-1}} - \frac{1}{A_j})A_0 A_{|S|}.$$

Thus,

$$\begin{aligned} D_j - \sum_{k=j+1}^{|S|} C_k &= \frac{\partial}{\partial \alpha}\left(A_0 - \frac{n-|S|}{n-1}\beta B_j\right) \\ &= -\frac{n-|S|}{n-1}\beta \frac{\partial B_j}{\partial \alpha}. \end{aligned}$$

Hence,

$$\begin{aligned} & \sum_{j=1}^{|S|}\left(D_j - \sum_{k=j+1}^{|S|} C_k\right) x_{i_j} \leq 0 \\ \Leftrightarrow \quad & \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} x_{i_j} \geq 0. \end{aligned}$$

If $|S| = 1$, $B_1 = 1$ and $\frac{\partial B_1}{\partial \alpha} = 0$. Then, $\sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} x_{i_j} = 0$.

Assume $|S| \geq 2$. There exists some $s \in \mathbb{N}$ that satisfies $\frac{\partial B_s}{\partial \alpha} \geq 0 > \frac{\partial B_{s+1}}{\partial \alpha}$ because $\frac{\partial B_1}{\partial \alpha} = \frac{|S|-1}{n-1}\frac{A_0}{A_1^2} > 0$ and $\frac{\partial B_{|S|}}{\partial \alpha} = -\frac{|S|-1}{n-1}\frac{A_0}{A_{|S|-1}^2} < 0$ hold (note that $A_0 > 0$ because $S \neq N$). Moreover, s is unique because $\frac{\partial B_j}{\partial \alpha}$ has the same sign as $|S|A_jA_{j-1} - jA_{|S|}A_{j-1} - (j-1)A_{|S|}A_j$ which is a quadratic function of j.

As a result,

$$\sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} x_{i_j} \geq x_{i_s} \sum_{j=1}^{|S|} \frac{\partial B_j}{\partial \alpha} = 0,$$

and we proved the theorem.

A.4 Proof of Theorem 4

We will show that x is not in the C-R core if and only if there exists some $S = \{i_1, i_2, \ldots, i_{|S|}\} \subset N$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$ such that

$$\frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S}\sum_{j=1}^{|S|-m_S} u^C_{i_j}(x, N) + \frac{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|}{(1-\gamma)\Gamma_S} u^C_{i_{|S|-m_S+1}}(x, N) < v(S) \qquad \text{Eq. (A.4)}$$

holds.

($\Rightarrow$) Assume x is not in the C-R core. Then, there exist $S \subset N$ and $y \in X_S$ such that y dominates x through S. Let $\{i_1, \ldots, i_{|S|}\} = S$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$.

Thus, $u^C_{i_j}(y,S) > u^C_{i_j}(x,N)$ holds for all $j = 1,\dots,|S|$. Since $\frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S} > 0$ and $\frac{(1-\gamma+\gamma\delta)m_S-\gamma\delta|S|}{(1-\gamma)\Gamma_S} > 0$ hold by the definition of m_S,

$$\frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S}\sum_{j=1}^{|S|-m_S}\{u^C_{i_j}(y,S) - u^C_{i_j}(x,N)\}$$
$$+ \quad \frac{(1-\gamma+\gamma\delta)m_S-\gamma\delta|S|}{(1-\gamma)\Gamma_S}\{u^C_{i_{|S|-m_S+1}}(y,S) - u^C_{i_{|S|-m_S+1}}(x,N)\} > 0$$

holds. By Lemma 3, we can assume $y_{i_1} \geq \cdots \geq y_{i_{|S|-m_S+1}} = \cdots = y_{i_{|S|}}$ without loss of generality. Then, Eq. (A.4) holds by Proposition 4.

($\Leftarrow$) If there exists some $S = \{i_1,\dots,i_{|S|}\} \subset N$ with $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_{|S|}}$ such that Eq. (A.4) holds, there exists some $\lambda > 0$ such that

$$v(S) = \frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S}\sum_{j=1}^{|S|-m_S}\{u^C_{i_j}(x,N)+\lambda\} + \frac{(1-\gamma+\gamma\delta)m_S-\gamma\delta|S|}{(1-\gamma)\Gamma_S}\{u^C_{i_{|S|-m_S+1}}(x,N)+\lambda\} \qquad \text{Eq. (A.5)}$$

because $\gamma < 1$ and $(1-\gamma+\gamma\delta)m_S - \gamma\delta|S| > 0$ hold+.

Let $\tilde{u}_{i_j} = u^C_{i_j}(x,N) + \lambda$ for all $j = 1,\dots,|S|-m_S+1$. By the definition of the C-R utility functions, $\tilde{u}_{i_1} \geq \tilde{u}_{i_2} \geq \cdots \geq \tilde{u}_{i_{|S|-m_S+1}}$ holds.

Define $y = (y_{i_1}, y_{i_2}, \dots, y_{i_{|S|}})$ as follows:

$$y_{i_j} = \frac{\tilde{u}_{i_{|S|-m_S+1}} - \gamma(1-\delta)v(S)}{1-\gamma+\gamma\delta} \quad \text{if } j \geq |S|-m_S+1$$
$$y_{i_j} = y_{i_{|S|-m_S+1}} + \frac{\tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}}}{1-\gamma} \quad \text{otherwise.}$$

We will confirm that y is in X_S. First, we will show $\sum_{j=1}^{|S|} y_{i_j} = v(S)$.

$$\sum_{j=1}^{|S|}(y_{i_j} - y_{i_{|S|-m_S+1}}) = \frac{\sum_{j=1}^{|S|-m_S}(\tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}})}{1-\gamma} \qquad \text{Eq. (A.6)}$$

holds by the definition of y. Then,

$$\begin{aligned}\sum_{j=1}^{|S|} y_{i_j} &= \frac{\sum_{j=1}^{|S|-m_S}(\tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}})}{1-\gamma} + |S| y_{i_{|S|-m_S+1}} \\ &= \frac{(1-\gamma+\gamma\delta)\sum_{j=1}^{|S|-m_S}(\tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m+1}}) + |S|(1-\gamma)\{\tilde{u}_{i_{|S|-m_S+1}} - \gamma(1-\delta)v(S)\}}{(1-\gamma)(1-\gamma+\gamma\delta)} \\ &= \Gamma_S \frac{(1-\gamma+\gamma\delta)\sum_{j=1}^{|S|-m_S}\tilde{u}_{i_j} + \{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|\}\tilde{u}_{i_{|S|-m_S+1}}}{(1-\gamma)(1-\gamma+\gamma\delta)\Gamma_S} \\ &\quad - \frac{\gamma(1-\gamma)(1-\delta)|S|}{(1-\gamma)(1-\gamma+\gamma\delta)} v(S)\end{aligned}$$

holds. Therefore, by Eq. (A.5),

$$\begin{aligned}\sum_{j=1}^{|S|} y_{i_j} &= \frac{(1-\gamma)\Gamma_S - \gamma(1-\gamma)(1-\delta)|S|}{(1-\gamma)(1-\gamma+\gamma\delta)} v(S) \\ &= v(S).\end{aligned}$$

Next, we will show $\min_{j \in S} y_j \geq 0$. $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_{|S|-m_S+1}} = \cdots = y_{i_{|S|}}$ holds because of $\tilde{u}_{i_1} \geq \tilde{u}_{i_2} \geq \cdots \geq \tilde{u}_{i_{|S|-m_S+1}}$. Then, we will prove $y_{i_{|S|-m_S+1}} \geq 0$. By the definition of y and $\sum_{j=1}^{|S|} y_{i_j} = v(S)$,

$$\begin{aligned}&(1-\gamma+\gamma\delta) y_{i_{|S|-m_S+1}} = \tilde{u}_{i_{|S|-m_S+1}} - \gamma(1-\delta)\sum_{j=1}^{|S|} y_{i_j} \\ \Leftrightarrow\quad &\Gamma_S y_{i_{|S|-m_S+1}} = \tilde{u}_{i_{|S|-m_S+1}} - \gamma(1-\delta)\sum_{j=1}^{|S|}(y_{i_j} - y_{i_{|S|-m_S+1}})\end{aligned}$$

holds. Then, by Eq. (A.6),

$$\begin{aligned}
(1-\gamma)\Gamma_S y_{i_{|S|-m_S+1}} &= (1-\gamma)\tilde{u}_{i_{|S|-m_S+1}} - \gamma(1-\delta)\sum_{j=1}^{|S|-m_S}(\tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}}) \\
&\geq (1-\gamma)\left\{(1-\gamma)x_{i_{|S|-m_S+1}} + \gamma\left[\delta\min_{j\in N} x_j + (1-\delta)\sum_{j\in N} x_j\right]\right. \\
&\quad \left. - \gamma(1-\delta)\sum_{j=1}^{|S|-m_S}(x_{i_j} - x_{i_{|S|-m_S+1}})\right\} \\
&\geq (1-\gamma)\left\{(1-\gamma)x_{i_{|S|-m_S+1}} + \gamma\left[\delta\min_{j\in N} x_j + (1-\delta)\sum_{j\in N} x_j\right]\right. \\
&\quad \left. - \gamma(1-\delta)\sum_{j\in N} x_j + \gamma(1-\delta)(|S|-m_S)x_{i_{|S|-m_S+1}}\right\} \\
&= (1-\gamma)\{(1-\gamma)x_{i_{|S|-m_S+1}} \\
&\quad +\gamma[\delta\min_{j\in N} x_j + (1-\delta)(|S|-m_S)x_{i_{|S|-m_S+1}}]\} \geq 0
\end{aligned}$$

holds, and we got $y_{i_{|S|-m_S+1}} \geq 0$ because of $\Gamma_S > 0$.

Therefore, if y satisfies $u^C_{i_j}(y,S) = \tilde{u}_{i_j} (= u^C_{i_j}(x,N)+\lambda)$ for all $j = 1,\ldots,|S|-m_S+1$, then y dominates x because $u^C_{i_j}(y,S) > u^C_{i_j}(x,N)$ for all $j = 1,\ldots,|S|-m_S+1$ and $u^C_{i_j}(y,S) = u^C_{i_{|S|-m_S+1}}(y,S) > u^C_{i_{|S|-m_S+1}}(x,N) \geq u^C_{i_j}(x,N)$ for all $j = |S|-m_S+2,\ldots,|S|$. Then, we will show $u^C_{i_j}(y,S) = \tilde{u}_{i_j}$ for all $j = 1,\ldots,|S|-m_S+1$. $u^C_{i_{|S|-m_S+1}}(y,S) = \tilde{u}_{i_{|S|-m_S+1}}$ holds by the definition of y. Similarly,

$$\begin{aligned}
& y_{i_j} = y_{i_{|S|-m_S+1}} + \frac{\tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}}}{1-\gamma} \\
\Leftrightarrow\ & \left\{(1-\gamma)y_{i_j} + \gamma\left[\delta\min_{j=1,\ldots,|S|} y_{i_j} + (1-\delta)\sum_{j=1}^{|S|} y_{i_j}\right]\right\} \\
& - \left\{(1-\gamma)y_{i_{|S|-m_S+1}} + \gamma\left[\delta\min_{j=1,\ldots,|S|} y_{i_j} + (1-\delta)\sum_{j=1}^{|S|} y_{i_j}\right]\right\} = \tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}} \\
\Leftrightarrow\ & u^C_{i_j}(y,S) - u^C_{i_{|S|-m_S+1}}(y,S) = \tilde{u}_{i_j} - \tilde{u}_{i_{|S|-m_S+1}}
\end{aligned}$$

holds by the definition of y. Therefore, we can prove $u^C_{i_j}(y,S) = \tilde{u}_{i_j}$ for any $j \leq |S|-m_S$ by the definition of the case of $j = |S|-m_S+1$.

As a result, we proved Theorem 4.

A.5 Proof of Theorem 5

Fix any $S = \{i_1, \dots, i_{|S|}\} \subset N$ and any $x \in X_N$ with $x_{i_1} \geq \cdots \geq x_{i_{|S|}}$. Let $x_{min} = \min_{i \in N} x_i$. Define a function of δ denoted by U as

$$U(\delta) = \frac{1-\gamma+\gamma\delta}{(1-\gamma)\Gamma_S} \sum_{j=1}^{|S|-m_S} u^C_{i_j}(x,N) + \frac{(1-\gamma+\gamma\delta)m_S - \gamma\delta|S|}{(1-\gamma)\Gamma_S} u^C_{i_{|S|-m_S+1}}(x,N).$$

Let $\delta_m = \frac{m(1-\gamma)}{\gamma(|S|-m)}$ for each $m = 1, \dots, |S|-1$. We can easily show that U is differentiable except at δ_m for all $m = 1, \dots, |S|-1$ by the definition of m_S. We will show U is continuous at δ_m for all $m = 1, \dots, |S|-1$. Since $\lim_{\delta \to \delta_m + 0} U(\delta) = U(\delta_m)$, we will prove $\lim_{\delta \to \delta_m - 0} U(\delta) = U(\delta_m)$. Since

$$\begin{aligned} U(\delta) &= \frac{\frac{1-\gamma+\gamma\delta}{1-\gamma} \sum_{j=1}^{|S|-m_S+1} (u^C_{i_j}(x,N) - u^C_{i_{|S|-m_S+1}}(x,N)) + |S| u^C_{i_{|S|-m_S+1}}(x,N)}{\Gamma_S} \\ &= \frac{(1-\gamma+\gamma\delta) \sum_{j=1}^{|S|-m_S+1} (x_{i_j} - x_{i_{|S|-m_S+1}}) + |S| u^C_{i_{|S|-m_S+1}}(x,N)}{\Gamma_S} \end{aligned}$$

holds,

$$\begin{aligned} \lim_{\delta \to \delta_m - 0} U(\delta) - U(\delta_m) &= \frac{\frac{(1-\gamma)|S|}{|S|-m} \sum_{j=1}^{|S|-m} (x_{i_{|S|-m}} - x_{i_{|S|-m+1}})}{\Gamma_S} \\ &\quad + \frac{|S| \{u^C_{i_{|S|-m+1}}(x,N) - u^C_{i_{|S|-m}}(x,N)\}}{\Gamma_S} \\ &= \frac{(1-\gamma)|S|(x_{i_{|S|-m}} - x_{i_{|S|-m+1}})}{\Gamma_S} \\ &\quad + \frac{(1-\gamma)|S|\{x_{i_{|S|-m+1}}(x,N) - x_{i_{|S|-m}}(x,N)\}}{\Gamma_S} = 0. \end{aligned}$$

Then, we proved continuity of U. Therefore, by Theorem 4, we only have to prove $\frac{\partial U(\delta)}{\partial \delta} \leq 0$ at $\delta \neq \delta_m$ for all $m = 1, \dots, |S|$. Since

$$\begin{aligned} \frac{\partial U}{\partial \delta} &= \frac{\gamma\Gamma_S + \gamma(|S|-1)(1-\gamma+\gamma\delta)}{\Gamma_S^2} \sum_{j=1}^{|S|-m_S+1} (x_{i_j} - x_{i_{|S|-m_S+1}}) \\ &\quad + \frac{\frac{\partial u^C_{i_{|S|-m_S+1}}}{\partial \delta} \Gamma_S + \gamma(|S|-1)\{(1-\gamma)x_{i_{|S|-m_S+1}} + \gamma(\delta x_{min} + (1-\delta)v(N))\}}{\Gamma_S^2} |S| \\ &= \frac{\gamma|S|}{\Gamma_S^2} \sum_{j=1}^{|S|-m_S+1} (x_{i_j} - x_{i_{|S|-m_S+1}}) \\ &\quad + \frac{-\gamma(1-\gamma)(x_{i_{|S|-m_S+1}} - x_{min}) + \gamma|S|\{(1-\gamma)x_{i_{|S|-m_S+1}} + \gamma x_{min}\} - \gamma v(N)}{\Gamma_S^2} |S| \end{aligned}$$

holds,

$$\begin{aligned}
\frac{\partial U}{\partial \delta} &\le \frac{\gamma|S|}{\Gamma_S^2}\left\{\left(v(N)-\sum_{j=|S|-m_S+2}^{|S|} x_{i_j}\right)-(|S|-m_S+1)x_{i_{|S|-m_S+1}}\right\} \\
&\quad + \frac{-\gamma(1-\gamma)(x_{i_{|S|-m_S+1}}-x_{min})+\gamma|S|\{(1-\gamma)x_{i_{|S|-m_S+1}}+\gamma x_{min}\}-\gamma v(N)}{\Gamma_S^2}|S| \\
&= \frac{\sum_{j=|S|-m_S+2}^{|S|}(x_{i_{|S|-m_S+1}}-x_{i_j})-(1-\gamma+\gamma|S|)(x_{i_{|S|-m_S+1}}-x_{min})}{\Gamma_S^2}\gamma|S| \\
&\le \frac{[m_S-1-(1-\gamma+\gamma|S|)](x_{i_{|S|-m_S+1}}-x_{min})}{\Gamma_S^2}\gamma|S|
\end{aligned}$$

holds. The integer $m_S - 1$ satisfies

$$m_S - 1 \le \frac{\gamma\delta|S|}{1-\gamma+\gamma\delta} \le \frac{\gamma|S|}{1-\gamma+\gamma} = \gamma|S|$$

because $\frac{\gamma\delta|S|}{1-\gamma+\gamma\delta}$ is a non-decreasing function of δ and $\delta \in (0, 1)$. Therefore,

$$\begin{aligned}
\frac{\partial U}{\partial \delta} &\le \frac{[\gamma|S|-(1-\gamma+\gamma|S|)](x_{i_{|S|-m_S+1}}-x_{min})\gamma|S|}{\Gamma_S^2} \\
&= -\frac{(1-\gamma)(x_{i_{|S|-m_S+1}}-x_{min})\gamma|S|}{\Gamma_S^2} \le 0
\end{aligned}$$

holds. As a result, we proved the theorem.