A summary of "Construction of general symplectic field theory"

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The aim of this paper is to provide a construction of symplectic field theory (SFT). SFT is a theory of contact manifolds and symplectic manifolds with cylindrical ends proposed by Eliashberg, Givental and Hofer in [3]. It is a generalization of contact homology and Gromov-Witten invariant, and it is constructed by counting the number of appropriate pseudo-holomorphic curves in the symplectization of a contact manifold or a symplectic manifold with cylindrical ends. In general, we need perturbation to obtain transversality of moduli spaces of pseudo-holomorphic curves, and it was a difficult problem to carry out perturbation with compatibility conditions required for the construction of the algebras. There were various attempts to overcome this difficulty. For example, Hofer, Wysocki and Zehnder developed the theory of polyfold ([6]-[10]) for a systematic construction. However, they have not yet published a complete proof of the construction of SFT. Recently, contact homology (this is a part of SFT) was constructed by Pardon [11] and Bao and Honda [1] independently. However, the general SFT has not yet been fully constructed.

The main result of this paper is construction of SFT in full generality.

Theorem 1. *For each closed contact manifold (Y, ξ) and each finite subset $\overline{K}^0 \subset H_*(Y, \mathbb{Q})$, we can define SFT cohomology $H^*_{\mathrm{SFT}}(Y, \xi, \overline{K}^0)$, rational SFT cohomology $H^*_{\mathrm{RSFT}}(Y, \xi, \overline{K}^0)$ and contact homology $H^*_{\mathrm{CH}}(Y, \xi, \overline{K}^0)$ as invariants of $(Y, \xi, \overline{K}^0)$.*

In fact, we construct generating functions defined in [3] for contact manifolds and symplectic manifolds with cylindrical ends and prove all of their properties explained in [3].

For the construction, we use Kuranishi theory, a theory developed by Fukaya and Ono. This is one of the general techniques to overcome the transversality problem and it was first used in [4] for the construction of Gromov-Witten invariant and Hamiltonian Floer Homology of symplectic manifolds. We mainly follow the argument of [4].

The compactification of the space of pseudoholomorphic curves consists of holomorphic buildings. For the construction of the Kuranishi neighborhood of a holomorphic building (especially the case of genus > 0), we need to treat the deformation of the target space as well as the deformation of the domain curve. Hence we introduce a new space which parametrizes the deformations of both of the domain curve and the target space.

We also improve the theory of Kuranishi structure and introduce the new notion of pre-Kuranishi structure and its weakly good coordinate system. In [4], they used the notion of good coordinate system for the construction of compatible perturbed multisection. It was a nice notion for their construction, but it has a disadvantage for our construction. The product of good coordinate systems is not a good coordinate system. This makes the argument highly complicated for our case. Hence we introduce the notion of weakly good coordinate system. Weakly good coordinate system is defined for a pre-Kuranishi structure, it is closed with respect to product, and we can use their product directly for the product of pre-Kuranishi spaces. We also alter the definition of multisection, and explain about the compatibility condition of multisections in details.

We also explain a new way to prove the smoothness of pre-Kuranishi structure by using the estimates of the differentials of implicit functions. In the theory of Kuranishi structure, the smoothness of the Kuranishi structre was one of its difficult part. One proof was given by Fukaya, Oh, Ohta and Ono in [5], but our way is easier than theirs. It is enough to estimate the implicit functions by direct calculations using appropriate coordinates.

We also deal with Bott-Morse case. Some easy cases of Bott-Morse case was studied by Bourgeois in [2]. We use the chain complex of triangulation of the space of periodic orbits instead of Morse chain complex used in [2].

We treat symplectic field theory of Bott-Morse case by using a triangulation of the space of periodic orbits. Using the chain complex of this simplicial complex, we treat the most general case where bad orbits appear as a subcomplex of the space of periodic orbits. To construct the algebras by counting intersection numbers with simplices, we need to use correction terms which correspond to cascades in [2]. Since the algebra of SFT is more complicated than that of contact homology, the correction terms are also complicated. Hence we also solve algebraic equations to define appropriate correction terms.

Using this Bott-Morse case, we can calculate the SFT cohomology of a contact manifold with S^1-action generated by the Reeb vector field. Then we can prove the following.

Theorem 2. *Assume that (Y, ξ) admits a contact form λ whose Reeb flow defines a locally free S^1-action on Y. We also assume that all cycles in $\overline{K}^0$ are S^1-invariant. Let $\overline{P}$ be the space of non-parametrized periodic orbits. Then $H^*_{\mathrm{SFT}}(Y, \xi, \overline{K}^0)$ is the algebra generated by $H_*(\overline{P}; \mathbb{R})$, $H^*_c(\overline{P}; \mathbb{R})$ and the variables t_x $(x \in \overline{K}^0)$, $\hbar$ with the product defined by the following commutative relations: all variables are super-commutative except*

$$[p_c, q_\alpha] = \langle c, \alpha \rangle \hbar$$

for all $c \in H_*(\overline{P}; \mathbb{R})$ and $\alpha \in H^*_c(\overline{P}; \mathbb{R})$, where we denote the elements corresponding to c or α by p_c or q_α.

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