A SUMMARY OF “RECONSTRUCTION OF INVARIANTS OF CONFIGURATION SPACES OF HYPERBOLIC CURVES FROM ASSOCIATED LIE ALGEBRAS”

KOICHIRO SAWADA

This is a summary of the paper [6], entitled “Reconstruction of invariants of configuration spaces of hyperbolic curves from associated Lie algebras”. In [6], we discuss various algorithms for reconstructing some objects of the configuration space of a hyperbolic curve from its associated Lie algebra.

Let K be a field of characteristic zero, X a hyperbolic curve of type (g,r) over K, n a positive integer, l a prime number, and Σ a set of prime numbers which coincides with either $\{l\}$ or the set of all prime numbers. Write X_n for the n-th configuration space of X over K, i.e., the complement of the union of various weak diagonals in the product of n copies of X.

If K is algebraically closed, then write Π_n^Σ for the maximal pro-Σ quotient of the étale fundamental group $\pi_1(X_n)$. In [2], certain explicit group-theoretic algorithms for reconstructing some objects from Π_n^Σ are given. For instance, n and the set of generalized fiber subgroups of co-length m of Π_n^Σ (that is, the kernel of the natural outer surjection $\Pi_n^\Sigma \twoheadrightarrow \Pi_m^\Sigma$ induced by a “generalized projection morphism” $X_n \to X_m$) can be reconstructed from Π_n^Σ, and, moreover, if $n \geq 2$, then (g,r) can be reconstructed from Π_n^Σ (cf. [2] Theorem 2.5).

To reconstruct the set of (generalized) fiber subgroups of co-length one, they proved the following: For a normal closed subgroup $H \subset \Pi_n^\Sigma$ of Π_n^Σ, if Π_n^Σ/H is a pro-Σ surface group (i.e., isomorphic to “Π_1^Σ” for some hyperbolic curve over an algebraically closed field of characteristic zero) which is not isomorphic to a free pro-Σ group of rank two, then H contains a fiber subgroup of co-length one (that is, the kernel of the natural outer surjection $\Pi_n^\Sigma \twoheadrightarrow \Pi_1^\Sigma$ induced by a projection morphism $X_n \to X_1$).

By using this, we can easily verify that, if $(g,r) \notin \{(0,3),(1,1)\}$, then a normal closed subgroup $N \subset \Pi_n^\Sigma$ of Π_n^Σ is a fiber subgroup of co-length one if and only if $\Pi_n^\Sigma/N \cong \Pi_1^\Sigma$.

In the paper [6], at first we consider the Lie algebra analogues of there results. Here, we shall recall the definition of the Lie algebra associated to a configuration space of a hyperbolic curve over an algebraically closed field of characteristic zero. We shall write

$$\Pi_n^l(1) := \Pi_n^l (:= \Pi_n^{\{l\}}),$$

$$\Pi_n^l(2) := \ker(\Pi_n^l \to (\pi_1(\overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n})^l)^{\mathrm{ab}}),$$

$$\Pi_n^l(m) := \overline{\langle [\Pi_n^l(m_1), \Pi_n^l(m_2)] \mid m_1 + m_2 = m,\ m_1 \geq 1,\ m_2 \geq 1 \rangle}\ (m \geq 3),$$

where $(\pi_1(\overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n})^l)^{\mathrm{ab}}$ is the abelianization of $\pi_1(\overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n})^l$ and $\Pi_n^l \to (\pi_1(\overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n})^l)^{\mathrm{ab}}$ is the homomorphism induced by the open immersion $X_n \hookrightarrow \overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n}$ (cf. [1] Definitions 1.1, 1.4). Moreover, we shall write

$$\mathrm{Gr}^m(\Pi_n^l) := \Pi_n^l(m)/\Pi_n^l(m+1) \ (m \geq 1),$$
$$\mathrm{Gr}(\Pi_n^l)(d) := \bigoplus_{m \geq d} \mathrm{Gr}^m(\Pi_n^l) \ (d \geq 1),$$
$$\mathrm{Gr}(\Pi_n^l) := \mathrm{Gr}(\Pi_n^l)(1).$$

We define the Lie bracket $[A, B] \in \mathrm{Gr}^{m_1+m_2}(\Pi_n^l)$ of $A \in \mathrm{Gr}^{m_1}(\Pi_n^l)$ and $b \in \mathrm{Gr}^{m_2}(\Pi_n^l)$ by $[A, B] = aba^{-1}b^{-1} \mod \Pi_n^l(m_1 + m_2 + 1)$, where $a \in \Pi_n^l(m_1)$ and $b \in \Pi_n^l(m_2)$ are representatives of A, B, respectively. Then (the Lie bracket is well-defined and) $\mathrm{Gr}(\Pi_n^l)$ is a $\mathbb{Z}_{>0}$-graded Lie algebra over $\mathbb{Z}_l$. Moreover, for each positive integer d, $\mathrm{Gr}(\Pi_n^l)(d)$ is a Lie ideal of $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$. We shall refer to the kernel of the natural surjection $\mathrm{Gr}(\Pi_n^l) \twoheadrightarrow \mathrm{Gr}(\Pi_m^l)$ induced by a "generalized projection morphism" $X_n \to X_m$ as a *generalized fiber ideal of co-length m*.

In [6], we prove the following:

Theorem A (cf. [6] Theorems 3.10, 4.3, 4.7)**.** *There exist algorithms for reconstructing n and the set of generalized fiber ideals of given co-length from (the Lie algebra obtained by forgetting the grading of) the graded Lie algebra $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$. Moreover, if $n \geq 2$, then there exist algorithms for reconstructing (g, r) and $\mathrm{Gr}(\Pi_n^l)(m)$ for $m \geq 1$ from (the Lie algebra obtained by forgetting the grading of) the graded Lie algebra $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$.*

Theorem B (cf. [6] Theorem 3.5(ii))**.** *Let $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l)$ be a (not necessarily homogeneous) Lie ideal of $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$. Then $\mathfrak{i}$ is a generalized fiber ideal of co-length one if and only if $\mathrm{Gr}(\Pi_n^l)/\mathfrak{i}$ is isomorphic to $\mathrm{Gr}(\Pi_1^l)$ as abstract Lie algebras over $\mathbb{Z}_l$.*

To prove Theorem B, by using an explicit presentation of $\mathrm{Gr}(\Pi_n^l)$ (cf. e.g., [4] (3.2)), we classify surjective homomorphisms (as abstract Lie algebras over $\mathbb{Z}_l$) from $\mathrm{Gr}(\Pi_n^l)$ to a surface algebra, that is, a Lie algebra over $\mathbb{Z}_l$ isomorphic to "$\mathrm{Gr}(\Pi_1^l)$" for some hyperbolic curve. This classification is obtained in [6] Theorem 3.1. The algorithm for reconstructing the set of generalized fiber ideals in Theorem A is obtained by Theorem B and the reconstruction of (the isomorphism class of) $\mathrm{Gr}(\Pi_1^l)$, which is also obtained by the above classification. Moreover, to reconstruct (g, r) and $\mathrm{Gr}(\Pi_n^l)(m)$ $(m \geq 1)$, we use the following idea: Let $\mathfrak{i}, \mathfrak{j}$ be distinct generalized fiber ideals of co-length $n-1$ of $\mathrm{Gr}(\Pi_n^l)$. If $g \neq 0$, then $a \in \mathfrak{i}$ is contained in $\mathrm{Gr}(\Pi_n^l)(2)$ if and only if $[a, b] \in [\mathrm{Gr}(\Pi_n^l), [\mathrm{Gr}(\Pi_n^l), \mathrm{Gr}(\Pi_n^l)]]$ for any $b \in \mathfrak{j}$.

In the paper [6], we also give some applications of the consideration of Lie algebras. One of the applications is giving a group-theoretic algorithm for reconstructing the set of generalized fiber subgroups of co-length one in a unified way (i.e., without depending on invariants, especially on (g, r)). This algorithm is obtained by proving the following group-theoretic analogue of Theorem B:

Theorem C (cf. [6] Theorem 5.12)**.** *Let $N \subset \Pi_n^\Sigma$ be a normal closed subgroup of Π_n^Σ. Then N is a generalized fiber subgroup of co-length one if and only if Π_n^Σ/N is isomorphic to Π_1^Σ as profinite groups.*

If $g = 0$ or $r \leq 1$, then the Lie algebra $\mathrm{Gr}(\Pi_n^l)$ is isomorphic to the Lie algebra (as an abstract Lie algebra over $\mathbb{Z}_l$) obtained by the lower central series of Π_n^l. Thus, we can prove Theorem C in the case $g = 0$ or $r \leq 1$ by using Theorem B. On the other hand, as mentioned above, Theorem C in the case $(g, r) \notin \{(0, 3), (1, 1)\}$ is essentially proved in [2].

Let us observe that Theorem C asserts that a normal closed subgroup N of Π_n^Σ such that the quotient Π_n^Σ/N is isomorphic to the surface group Π_1^Σ must be "geometric" in some sense. By using some results in [3], [2] and the classification of surjective homomorphisms from $\mathrm{Gr}(\Pi_n^l)$ to a surface algebra, we consider normal closed subgroups of Π_n^Σ such that the quotient is isomorphic to a surface group more closely. As a consequence, we obtain the following Grothendieck conjecture-type result for a configuration space and a hyperbolic polycurve:

Theorem D (cf. [6] Theorem 7.14)**.** *Suppose that K is generalized sub-l-adic (i.e., isomorphic to a subfield of a finitely generated extension of the quotient field of the ring of Witt vectors with coefficients in $\overline{\mathbb{F}_l}$). Let $(Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to Z_{(1)} \to \operatorname{Spec} K = Z_{(0)})$ be a parametrized hyperbolic polycurve of dimension n over K (see [5] Definition 2.3) and $\varphi : \pi_1(X_n) \xrightarrow{\sim} \pi_1(Z)$ an isomorphism from $\pi_1(X_n)$ to $\pi_1(Z)$ over the absolute Galois group G_K of K. If $(g, r) = (0, 3)$, then suppose that $X \cong \mathbb{P}_K^1 \setminus \{0, 1, \infty\}$. Moreover, suppose that at least one of the following holds:*

- *$g \geq 2$.*
- *For each integer m such that $1 \leq m \leq n$, the hyperbolic curve $Z_{(m)}/Z_{(m-1)}$ is not of type $(0, 3), (1, 1)$.*

Then there exist generalized projection morphisms $p_m : X_m \to X_{m-1}$ ($1 \leq m \leq n$) such that φ arises from a unique isomorphism $(X_n, X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} \operatorname{Spec} K = X_0) \xrightarrow{\sim} (Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to \operatorname{Spec} K = Z_{(0)})$ of parametrized hyperbolic polycurves over K. In particular, the natural map

$$\mathrm{Isom}_K(X_n, Z) \to \mathrm{Isom}_{G_K}(\pi_1(X_n), \pi_1(Z))/\operatorname{Inn}(\pi_1(Z \times_K \overline{K}))$$

is bijective.

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