A summary of "Random walks on random trees and hyperbolic groups: trace processes on boundaries at infinity and the speed of biased random walks"

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This is a summary of the authors thesis [18], entitled "Random walks on random trees and hyperbolic groups: trace processes on boundaries at infinity and the speed of biased random walks".

Consider a transient random walk (Z_n) on an infinite graph. As the time n tends to ∞, Z_n escapes to infinity due to its transience. It is the aim of this thesis to study several problems which arise from the transience and investigate its deeper aspects.

The first object we will study is *trace processes at infinity*. We begin with introducing a classical example called the *Douglas integral* which gives an intuition for the construction of trace processes on the boundary. Consider the reflected Brownian motion on the unit disc $\mathbb{D} := \{(x,y) \in \mathbb{R}^2 \ ; \ x^2+y^2<1\}$ starting at the origin. The corresponding Dirichlet form (E,F) is given by

$$E(u,v) := \int_{\mathbb{D}} \int_{\mathbb{D}} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dm(x)dm(y),$$
$$F := \left\{ u \in L^2(\mathbb{D},m) \ ; \ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\mathbb{D},m) \right\},$$

where m is the Lebesgue measure on $\mathbb{D}$. Let H' be a map which sends a function φ on $\partial\mathbb{D}$ to a function $H'\varphi$ on $\mathbb{D}$ defined by

$$H'\varphi(re^{i\theta}) := \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\varphi(\theta')d\kappa(\theta'),$$

where κ is the first hitting distribution of the reflecting Brownian motion to $\partial\mathbb{D}$, which coincides with the normalized uniform measure on $\partial\mathbb{D}$ due to its rotation invariance. This integral is called the *Poisson integral*, and it is known that $H'\varphi$ gives a solution to the Dirichlet problem on $\mathbb{D}$. Now we define a quadratic form $(E_{\partial\mathbb{D}}, F_{\partial\mathbb{D}})$ on $\partial\mathbb{D}$ by

$$E_{\partial\mathbb{D}}(\varphi,\psi) := E(H'\varphi, H'\psi),$$
$$F_{\partial\mathbb{D}} := \{\varphi \in L^2(\partial\mathbb{D},\kappa) \ ; \ H\varphi \in F\}.$$

It is known that $(E_{\partial\mathbb{D}}, F_{\partial\mathbb{D}})$ is a regular Dirichlet form on $L^2(\partial\mathbb{D}, \kappa)$ which corresponds to the trace process of the reflecting Brownian motion on $\partial\mathbb{D}$, and $E_{\partial\mathbb{D}}$ has the following explicit expression:

$$E_{\partial\mathbb{D}}(\varphi, \psi) = \frac{\pi}{4} \int_0^{2\pi} \int_0^{2\pi} \frac{(\varphi(\theta) - \varphi(\theta'))(\psi(\theta) - \psi(\theta'))}{\sin^2(\frac{\theta-\theta'}{2})} d\kappa(\theta) d\kappa(\theta').$$

We can consider the discrete version of the above construction on general graphs. Let $G = (V, E)$ be an infinite connected graph with vertex set V and edge set E and $c : E \to \mathbb{R}$ be a weight function on E. We now define a random walk (Z_n) on G whose transition probabilities $\{p(x, y)\}_{x,y\in V}$ are given by

$$p(x, y) := \frac{c([x, y])}{\sum_{y'\in V;[x,y']\in E} c([x, y'])},$$

where $[x, y]$ denotes the edge connecting x and y. Let $(\mathcal{E}, \mathcal{F})$ be a quadratic form on G defined by

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x,y\in V} c([x, y])(f(x) - f(y))(g(x) - g(y)),$$
$$\mathcal{F} := \{f : V \to \mathbb{R}\ ;\ \mathcal{E}(f, f) < \infty\}.$$

When (Z_n) is transient, (Z_n) converges to a random point Z_∞ of the *Martin boundary* M of the weighted graph (G, c). We denote by ν the distribution of Z_∞, which is called the *harmonic measure* of (Z_n). It is a well-known fact in the theory of the Martin boundary that there exists a map H which sends a function u on M to a harmonic function Hu on G which has the boundary value u on M. We now introduce a bilinear form $(\mathcal{E}_M, \mathcal{F}_M)$ on M defined by

$$\mathcal{E}_M(u, v) := \mathcal{E}(Hu, Hv),$$
$$\mathcal{F}_M := \{u \in L^2(M, \nu)\ ;\ Hu \in \mathcal{F}\}.$$

The function Hu gives a solution to the Dirichlet problem at infinity, and $(\mathcal{E}_M, \mathcal{F}_M)$ is called the *trace* of $(\mathcal{E}, \mathcal{F})$ on M. In order to construct a stochastic process associated to $(\mathcal{E}_M, \mathcal{F}_M)$, we need to prove the regularity of $(\mathcal{E}_M, \mathcal{F}_M)$, which roughly means that the domain $\mathcal{F}_M$ contains sufficiently many continuous functions on M. The proof requires a good understanding of the topology of M. When the graph G has some special structure, we sometimes have a concrete description of M by using the structure of G. In this thesis, we will focus on the following graphs;

(1) infinite trees,

(2) non-elementary hyperbolic groups.

We first explain the background of **(1)**. It is shown in [5] that the Martin boundary of an infinite transient weighted tree (T, c) is homeomorphic to the set of geodesic rays of T. By utilizing this concrete geometric description of the Martin boundary, Kigami [11] proved the regularity of $(\mathcal{E}_M, \mathcal{F}_M)$. Therefore, there exists a Hunt process on M corresponding to $(\mathcal{E}_M, \mathcal{F}_M)$. Moreover, he obtained the two-sided estimates of the heat kernel associated to $(\mathcal{E}_M, \mathcal{F}_M)$ under the assumption that the harmonic measure ν satisfies the volume doubling property with respect to a certain metric on M. It is one of the aims of this thesis to extend this result to *Galton-Watson trees*, which are randomly generated trees by the branching mechanism. We will choose λ-*biased random walks* on Galton Watson trees as a transient random walk on them. The proof of the regularity of $(\mathcal{E}_M, \mathcal{F}_M)$ in [11] also applies to Galton- Watson trees, but we cannot use the heat kernel estimates shown in [11] because in general we do not have the volume doubling property of the harmonic measure for Galton-Watson trees.

We now define Galton-Watson trees and λ-biased random walks. Let $\{p_k\}_{k\geq 0}$ be a probability measure on $\mathbb{Z}_+$, namely, $p_k \geq 0$ for any $k \in \mathbb{Z}_+$ and $\sum_{k=0}^{\infty} p_k = 1$. The Galton-Watson tree with offspring distribution $\{p_k\}_{k\geq 0}$ is a random tree constructed in the following manner: we start with one distinguished individual o called the *root*, and the root gives birth to children whose number is determined by the offspring distribution, namely, it has k children with probability p_k. Each of those children gives birth to its children again independently with the same distribution. We let this procedure continue and the family tree $\mathbf{T}$ constructed in this manner is called the Galton-Watson tree with offspring distribution $\{p_k\}$. We denote the distribution of a Galton-Watson tree by $\mathbb{P}$. Throughout this thesis, we will assume $p_0 = 0$ and $p_1 < 1$ so that $\mathbf{T}$ is an infinite tree with probability 1 and $m := \sum_{k=0}^{\infty} kp_k > 1$.

We introduce the definition of λ-biased random walks. Choose an infinite rooted tree (T, o) without leaves. We define the λ-biased random walk on T under the probability measure P_λ^T in the following manner: for $\lambda > 0$, let $\{Z_n^\lambda\}_{n\geq 0}$ be a random walk on T such that if $v \neq o$, v has k children $v_1, ..., v_k$ and the parent $\pi(v)$, then

$$\begin{aligned} P_\lambda^T(Z_{n+1}^\lambda = \pi(v) \mid Z_n^\lambda = v) &= \frac{\lambda}{\lambda + k}, \\ P_\lambda^T(Z_{n+1}^\lambda = v_i \mid Z_n^\lambda = v) &= \frac{1}{\lambda + k}, \quad \text{for } 1 \leq i \leq k, \end{aligned}$$

and if $v = o$, the random walk moves to its children equally likely. It is proved in [12] that the λ-biased random walk on a Galton-Watson tree $\mathbf{T}$ is transient for almost every $\mathbf{T}$ if and only if $0 < \lambda < m$.

We next explain the background of **(2)**. A geodesic metric space (X, d) is said to be *hyperbolic in the sense of Gromov* if there exist a point $o \in X$ and a constant $\delta > 0$ such that for any $x, y, z \in X$ we have

$$(x, z)_o \geq \min\{(x, y)_o, (y, z)_o\} - \delta,$$

where

$$(x, y)_o := \frac{d(o, x) + d(o, y) - d(x, y)}{2}.$$

The quantity $(x, y)_o$ is called the *Gromov product* of x and y with respect to the base point o. A finitely generated group Γ is said to be hyperbolic if the Cayley graph of Γ endowed with a word metric is hyperbolic in the sense of Gromov. A hyperbolic groups is called *non-elementary* if it is non-amenable. One important feature of a non-elementary hyperbolic group is the existence of the *Gromov boundary* $\partial\Gamma$, which is a kind of the boundary at infinity canonically defined by using the hyperbolicity of Γ. We next quickly explain the definition of random walks on Γ. Let μ be a probability measure on Γ whose support generates Γ. We now define a random walk (Z_n) on Γ in such a way that

$$P(Z_{n+1} = y \mid Z_n = x) = \mu(x^{-1}y).$$

Thus, the distribution of Z_n is identical to μ^{*n}, which is the n-th fold convolution power of μ. All random walks on Γ defined in this way are known to be transient due to the non-amenability of Γ. It is a natural question to study relations between the Gromov boundary $\partial\Gamma$ and the Martin boundary of (Γ, μ). Actually, it is proved in [2] that $\partial\Gamma$ is homeomorphic to the Martin boundary of (Γ, μ) when μ has a finite support. Later, this result is extended in [9], and it is shown that the same result holds when μ satisfies the following moment estimate: for any $a > 0$, it holds that

$$\sum_{x\in\Gamma} e^{a|x|}\mu(x) < \infty,$$

where $|x|$ denotes the word metric between the identity of Γ and x. The results explained above gives the relation between $\partial\Gamma$ and the Martin boundary from the topological point of view. On the other hand, it is proved in [10] that under the assumption that μ has a finite first moment, $(\partial\Gamma, \nu)$ is isomorphic to the Martin boundary endowed with ν as measurable spaces. However, to our best knowledge, there have not been any results which study the trace form on the boundary for non-elementary hyperbolic groups in general.

The second problem arising from the transience which we will address in this thesis is the analysis of the *speed* of transient random walks. Here we are interested in random walks with very strong transience in a sense that the distance between the starting point and the position of a random walk at time n grows linearly in n. The speed of λ-biased random walks on Galton-Watson trees is an interesting example of this topic. It is shown in [14] that when $p_0 = 0$ and $0 < \lambda < m$, the sequence $n^{-1}d(Z_n^\lambda)$ converges to a deterministic strictly positive constant, denoted by v_λ, P_λ-almost surely and in $L^1(P_\lambda)$, where $d(Z_n^\lambda)$ is the distance between the root o and the vertex Z_n^λ, and P_λ is the *annealed measure* defined by

$$P_\lambda(\cdot) := \int \mathbb{P}(\mathrm{d}\mathbf{T})P_\lambda^{\mathbf{T}}(\cdot).$$

The constant v_λ is called the *speed* of (Z_n^λ), and the following conjecture about its behavior is posed in [14].

Conjecture 0.1. *The function $\lambda \mapsto v_\lambda$ is monotonically decreasing on $\lambda \in (0, m)$.*

This conjecture is called the *monotonicity conjecture* and has been unsolved for more than twenty years though there are several partial results. In [3], the authors proved the monotonicity of the speed for $\lambda \in (0, 1/1160)$. Later, in the unpublished note [1], this results was extended to $\lambda \in (0, 1/2)$.

We conclude this section with explaining *regeneration times*, introduced in [14] for Galton-Watson trees, which will play an important role in this thesis. Regeneration times enable us to decompose paths of random walks into *i.i.d.* components. We introduce below the definition and basic properties of regeneration times.

Definition 0.2. *A time n is called a regeneration time of the λ-biased random walk (Z_n^λ) if $Z_n^\lambda \neq Z_k^\lambda$ for any $0 \leq k < n$ and $Z_{n-1}^\lambda \neq Z_l^\lambda$ for any $l \geq n$.*

Proposition 0.3. *[14]*

- *For any $0 < \lambda < m$, there exist infinitely many regeneration times $0 =: \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < ....$ P_λ almost surely.*
- *For any $0 < \lambda < m$, under the annealed measure P_λ, the sequences $\{\tau_{i+1} - \tau_i\}_{i\geq 1}$ and $\{d(Z^\lambda_{\tau_{i+1}}) - d(Z^\lambda_{\tau_i})\}_{i\geq 1}$ are i.i.d. random variables.*

1 Main results of this thesis

This thesis consists of the following three papers:

(1) Chapter 2: Y. Tokushige. Jump processes on the boundaries of random trees, arXiv:1708.08075.

(2) Chapter 3: P. Mathieu and Y. Tokushige. An approach to traces on the boundary of random walks on a hyperbolic group via reflected Dirichlet spaces, preprint.

(3) Chapter 4: Y. Tokushige. Regularity results of the speed of biased random walks on Galton-Watson trees, arXiv:1811.04849.

The rest of this chapter is devoted to the summary of the main results of this thesis.

1.1 Results in Chapter 2

In Chapter 2, we study the asymptotics of the heat kernels associated to the trace forms for λ-biased random walks on Galton-Watson trees. As explained in Section 1.1, the λ-biased random walk on a Galton-Watson tree $\mathbf{T}$ is $\mathbb{P}$-almost surely transient when $0 < \lambda < m$. Thus, for $0 < \lambda < m$, we have the harmonic measure $\mathrm{HARM}_{\mathbf{T}}^\lambda$, the heat kernel $p_t^\lambda(\cdot,\cdot)$ and the Markov process (X_t^λ) on the Martin boundary M associated to the trace form $(\mathcal{E}_M, \mathcal{F}_M)$. In [13, 14], the authors proved that $\beta_\lambda := \dim \mathrm{HARM}_{\mathbf{T}}^\lambda$ is a positive deterministic constant satisfying $0 < \beta_\lambda < \log m$. In Chapter 2, we will prove the following estimates of β_λ and $p_t^\lambda(\cdot,\cdot)$. Note that $d(\cdot,\cdot)$ is the natural metric on M.

Theorem 1.1. *(Theorem 2.1.1, Theorem 2.1.2 and Corollary 2.3.7 in Chapter 2.)*

- *For $0 < \lambda < m$, we have $\beta_\lambda > \log \lambda \vee 0$.*

- *For $0 < \lambda < m$, the following holds $\mathbb{P}$-a.s.*

$$-\lim_{t\to 0}\frac{\log p_t^\lambda(\omega,\omega)}{\log t}=\frac{\beta_\lambda}{\beta_\lambda-\log\lambda},\qquad \mathrm{HARM}_{\mathbf{T}}^\lambda \text{ a.e.-}\omega,$$

and for any $\gamma > 0$

$$\lim_{t\to 0}\frac{\log E_\omega[d(\omega,X_t^\lambda)^\gamma]}{\log t}=\left(\frac{\gamma}{\beta_\lambda-\log\lambda}\right)\wedge 1,\qquad \mathrm{HARM}_{\mathbf{T}}^\lambda \text{ a.e.-}\omega.$$

In the proof, we utilize an explicit expression of $p_t^\lambda(\cdot,\cdot)$ obtained in [11] which involves the harmonic measure and the effective resistance of the electric network on an infinite tree corresponding to the transient random walk on it. Therefore, in order to prove the above theorem, it is important to obtain controls of the harmonic measure $\mathrm{HARM}_{\mathbf{T}}$ and the effective resistance. As for the control of the harmonic measure, we employ results in [13, 14] which study the speed of the decay of $\mathrm{HARM}_{\mathbf{T}}^\lambda(A_n)$ when (A_n) is a decreasing sequence of balls of the Martin boundary M. As for the control of the effective resistance, we will determine the asymptotic of the values of the effective resistance along geodesic rays of a Galton-Watson tree in Proposition 2.3.5 in Chapter 4. In the proof of Proposition 2.3.5, we will use the ergodic theory on the space of trees developed in [13, 14].

1.2 Results in Chapter 3

In Chapter 3, we will study the trace forms on the Gromov boundaries of non-elementary hyperbolic groups Γ. Let $(\mathcal{E}_M,\mathcal{F}_M)$ be the trace of the random walk driven by a probability measure μ and ν be the corresponding harmonic measure. We first prove the following expression of $(\mathcal{E}_M,\mathcal{F}_M)$ which uses the *Naïm kernel* $\Theta^\mu(\cdot,\cdot)$ introduced in [16] in the Euclidean setting and extended in [17] to the discrete setting.

Proposition 1.2. *(Proposition 3.4.5 in Chapter 3.) When a probability measure μ admits a finite first moment, we have*

$$\mathcal{E}_M(u,u)=\int_{Z\times Z}(u(\xi)-u(\eta))^2\Theta^\mu(\xi,\eta)d\nu(\xi)d\nu(\eta)\,,$$
$$\mathcal{F}_M=\{u\in L^2(M,\nu)\ ;\ \mathcal{E}_M(u,u)<\infty\}.$$

We next prove the regularity of $(\mathcal{E}_M,\mathcal{F}_M)$ under the assumption that a probability measure μ satisfies the finite second moment condition and the *Ahlfors-regular conformal dimension* of the Gromov boundary $\partial\Gamma$ is strictly less than 2.

Theorem 1.3. *(Theorem 3.5.1 in Chapter 3.) Assume the Ahlfors-regular conformal dimension of $\partial\Gamma$ is strictly less than 2. When a probability measure μ admits a finite second moment, $(\mathcal{E}_M,\mathcal{F}_M)$ is a regular Dirichlet form on $L^2(M,\nu)$.*

To prove the above theorem, we will compare $(\mathcal{E}_M, \mathcal{F}_M)$ with the Besov spaces on $\partial\Gamma$ introduced in [4]. In [4], the authors constructed the Besov space on $\partial\Gamma$ for each metric in the *Ahlfors-regular conformal gauge* of $\partial\Gamma$, and proved that those Besov spaces are all isomorphic as Banach spaces. Denote by $C(\partial\Gamma)$ the set of continuous functions on $\partial\Gamma$. In Chapter 3, we will show that the intersection of the domain of a Besov space in [4] and $C(\partial\Gamma)$ does not depend on the choice of metrics in the Ahlfors-regular conformal gauge. We will denote by $\mathcal{C}$ the common set. Moreover, we will prove that the intersection $C(\partial\Gamma) \cap \mathcal{F}_M$ also coincides with $\mathcal{C}$ whenever μ admits a finite second moment. By using these observations, we also prove that the Besov spaces in [4] give rise to regular Dirichlet forms.

The next results in Chapter 3 is about a potential theoretic property of the harmonic measure ν. A positive Radon measure m is said to be *smooth* with respect to $(\mathcal{E}, \mathcal{F})$ if m charges no sets of zero capacity and satisfies an additional technical condition. (See Subsection 3.5.2 for the precise definition.) We present below one of the main results in Chapter 3.

Theorem 1.4. *(Theorem 3.5.16 in Chapter 3.) Assume that the Ahlfors-regular conformal dimension of $\partial\Gamma$ is strictly less than 2 and a probability measure μ admits a finite first moment. Then, any harmonic measure ν of a random walk driven by a probability measure μ is smooth with respect to $(\mathcal{E}_M, \mathcal{F}_M)$.*

We prove the above theorem in a somewhat indirect way. For a given regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, we consider the set of all smooth measures with respect to $(\mathcal{E}, \mathcal{F})$. We first show that all the sets of smooth measures with respect to the Besov spaces in [4] coincide. Let $\mathcal{S}(\partial\Gamma)$ be this common set of smooth measures. We next prove that whenever μ admits a finite second moment, the set of all smooth measures with respect to $(\mathcal{E}_M, \mathcal{F}_M)$ coincides with $\mathcal{S}(\partial\Gamma)$. By using the above claims, we can easily prove Theorem 1.4 when μ admits a finite second moment. We finally extend this result to probability measures with a finite first moment by employing the heat kernel estimates for non-local Dirichlet forms obtained in [7] and the deviation inequality in [15] which controls the deviation of paths of random walks on Γ from geodesics.

At the end of Chapter 3, we will give a concrete probabilistic interpretation of the Markov process (X_t) on $\partial\Gamma$ associated to $(\mathcal{E}_M, \mathcal{F}_M)$. We first construct the reflecting random walk (W_t) on Γ, which is an extension of the transient random walk on Γ, by using the theory of *reflected Dirichelt spaces* introduced in [6]. See Theorem 3.7.1 for the precise statement. We then prove that (X_t) coincides with a certain time-change process of (W_t) in Theorem 3.7.3.

1.3 Results in Chapter 4

In Chapter 4, we study the speed of λ-biased random walks on Galton-Watson trees. Specifically, we will prove the differentiablity of the function $\lambda \mapsto v_\lambda$ on $\lambda \in (0,1)$, and an expression of the derivative using a two dimensional Gaussian random variable (X, Y).

Let us explain why the two dimensional Gaussian random variable (X, Y) arises in this context. The Gaussian random variables X and Y are both related to the distance function $d(Z_n^\lambda)$. We first explain the definition of X in the first coordinate. Recall that $\{d(Z_{\tau_{i+1}}^\lambda) - d(Z_{\tau_1}^\lambda)\}_{i\geq 1}$ are *i.i.d.* random variables, where τ_i is the i-th regeneration time. Thus, when τ_i satisfies a finite $(2+\varepsilon)$-th moment for some $\varepsilon > 0$, $(d(Z_{\tau_n}^\lambda) - \tau_n v_\lambda)/\sqrt{n}$ converges to a Gaussian random variable as n tends to ∞ under the annealed measure P_λ due to the CLT. It is not difficult to show the CLT for $(d(Z_n^\lambda) - nv_\lambda)/\sqrt{n}$ by using the above fact, and X is given by its limit.

We next explain the definition of Y in the second coordinate. Define P_n by

$$P_n = \frac{d(Z_n) - \sum_{k=1}^{n-1} E^{\mathbf{T}}_{\lambda, Z_k}[d(Z_1) - d(Z_0)]}{2\lambda}.$$

It is clear from the definition that P_n is a martingale under the quenched measure $P^{\mathbf{T}}_\lambda$. By using regeneration times again, we can show the CLT for $P_n/\sqrt{n}$, and Y is given by its limit. Since the proofs of the CLT for $(d(Z_n^\lambda) - nv_\lambda)/\sqrt{n}$ and $P_n/\sqrt{n}$ both rely on the regeneration structure of Galton-Watson trees, we actually have the joint CLT for the random vector $((d(Z_n^\lambda) - nv_\lambda)/\sqrt{n}, P_n/\sqrt{n})$. We now state the main result in Chapter 4.

Theorem 1.5. *(Theorem 4.3.5 in Chapter 4.) The function $\lambda \mapsto v_\lambda$ is differentiable on $(0,1)$. Moreover, the derivative of the speed v'_λ can be expressed as the covariance of a 2-dimensional Gaussian random variable, namely, there exists a centered 2-dimensional Gaussian random vector (X, Y) with the covariance matrix Σ_λ such that $v'_\lambda = E_\lambda[XY]$.*

The differentiability of the speed v_λ on $\lambda \in (0,1)$ is already proved in [1], but the expression of the derivative in [1] is completely different from ours.

We conclude this section by explaining why we need to assume $\lambda \in (0,1)$. In the proof of Theorem 1.2.5, we need a finite exponential moment of regeneration times τ_i. To obtain the finite exponential moment of τ_i, we utilize the estimate of slow-down probabilities of $d(Z_n^\lambda)/n$ obtained in [8]. However, it is also shown in [8] that the slow-down probabilities do not decay exponentially fast in general when $\lambda \geq 1$.

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