Stability of Characteristic Curves of Nonlinear Resistive Circuits

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Abstract. In this paper, we discuss the stability of the characteristic curves for nonlinear resistive circuits including parasitic elements[1]. Although the DC solution is determined by analyzing the nonlinear resistive circuit, its equilibrium point will be stable or unstable because every resistive element has a small parasitic component in practice. We consider here two parasitic elements: a *capacitor* between every resistor node and ground, and an *inductor* in series with each resistor. Of course, the stability can be decided by solving the variational equation at each equilibrium point obtained by the DC analysis, however, this is very time-consuming. We show here that the stability is *mainly* changed at the boundary of the presence of negative differential resistance (NDR) and the bifurcation points such as turning and pitchfork points on the DC characteristic curves, so that the instability regions of the solution curve are easily found by both the locations of bifurcation points and NDR regions of the nonlinear resistors.

1. Introduction

The DC analysis of nonlinear resistive networks is the most important problem for the design of electronic circuits, where the solutions correspond to the operating points for a DC bias. There have been many papers published about algorithms for calculating the multiple solutions of nonlinear resistive circuits [1-2]. In practical circuits, however, the equilibrium points obtained by the DC analysis can be described as stable or unstable, because every resistive element has a parasitic component [3]. In this paper, instability of the equilibrium points is investigated by the Liapunov's direct method, where we assume that the ratio L_p/C_p of the parasitic elements may take any positive value even if the parasitic elements L_p, C_p themselves are arbitrarily small. Now, we define the stability conditions of resistive circuits as follows.

Definition 1 [4]: An equilibrium point $x^\star$ is said to be *stable* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $\| x(t) - x^\star \| < \epsilon$ for all $t \geq t_0$, whenever $\| x(t_0) - x^\star \| < \delta$. Otherwise, the equilibrium point is said to be *unstable.*

Definition 2: A DC circuit's equilibrium point is said to be *totally stable* if the point of any dynamic circuit created by augmenting the DC circuit with an arbitrary set of parasitic capacitors between every node and ground, and inductors in series with each resistor is stable.

Definition 3: A resistive circuit's equilibrium point is said to be *k-th order saddle-node* unstable if the characteristic equation obtained from the variational dynamic circuit has k positive real roots. An equilibrium point is said to be *occasionally* unstable if

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the above dynamic circuit is unstable only for some set of the parasitic capacitors and inductors and is stable for some other parasitic elements. We also define *NDR unstable* region of the solution curves where it has one or more negative differential resistances (NDRs)[2] .

There are many papers discussing the stability of nonlinear networks. In references [5-6], a condition for global asymptotic stability is discussed for nonlinear dynamic networks in a qualitative manner. In reference [4], a simple technique is proposed to identify unstable DC operating points, where the instability is investigated by the sign of the determinant of the Jacobian matrix. In this paper, we show in *Theorem 2* that the regions on the solution curve correspond to the parts where $dv_{in}/ds < 0$ for the input v_{in} and the arc-length s from the starting point, and the equilibrium point will be a saddle-node unstable point if the starting point of the arc-length method [2] is stable. In reference [7-8], the *open-circuit (short-circuit)* instability is defined for the driving-point characteristic curves for nonlinear one-port resistive circuits.

In this paper, we discuss how to identify the unstable regions of solution curves such as driving-point and transfer characteristic curves, where we assume the ratios of the parasitic elements can take any positive value. The first result is NDR unstable regions (*Theorem 1*) where one or more differential resistances has negative value. The second result is saddle-node unstable regions (*Theorem 2*), where $dv_{in}/ds < 0$ on the solution curve and the stability changes at the turning point ($dv_i/dv_{in} = 0$, for input v_{in}). The third result is also saddle-node unstable regions (*Theorem 3*), such that the stability changes at the branch points [9-10] where two solution curves intersect at a point. Thus, unstable regions of the solution curve can be easily identified without investigating the variational equation. Thus, our technique is very useful for identifying the instabilty regions of solution curves.

2. Stability of nonlinear reciprocal resistive circuits

Consider a nonlinear reciprocal resistive circuit having both voltage- and current-controlled nonlinear resistors. To obtain the DC circuit equation, partition the circuit into two groups as follows:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad i = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \quad A = (A_1 A_2) \tag{1}$$

where A is the incidence matrix, and the subscript "1" indicates the elements of voltage-controlled resistors (including linear resistors), and "2" those of current-controlled resistors, respectively.
Then, we have the following circuit equation using the *tableau approach* [6].

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -A_1^T \\ 0 & 0 & 0 & 1 & -A_2^T \\ A_1 & A_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ v_1 \\ v_2 \\ v_n \end{pmatrix} - \begin{pmatrix} g_v(v_1) \\ r_i(i_2) \\ E_1 \\ E_2 \\ AJ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

[2]It has been proven in Theorem 1 that, for reciprocal resistive circuits with parasitic elements, if the circuit has one or more NDR, the equil ibrium point may become unstable depending on the choice of the parasitic elemen t, which is classified as an *occasionally* unstable or saddle-node point.

where v_n denotes the vector of node voltage. From the first and third rows, we have

$$i_1 = g_v(A_1^T v_n + E_1) \tag{3}$$

Thus, we have the following relations from the second and fourth rows in (2):

$$A_2 i_2 = -A_1 g_v(A_1^T v_n + E_1) + AJ, \quad A_2^T v_n + E_2 = r_i(i_2) \tag{4}$$

The first equation corresponds to the *nodal equation*, and the second one is obtained by the *Kirchhoff's voltage law* for the loops containing the current-controlled resistors.

Now, let us derive the dynamic equation by considering the parasitic elements. We assume a parasitic capacitor between every resistor and ground, and a parasitic inductor in series with each resistor. Then, we have

$$C_p \frac{dv_n}{dt} = -A_2 i_2 - A_1 g_v(v_1) + AJ \tag{5.1}$$

$$L_{p1} \frac{dg_v(v_1)}{dt} = -v_1 + A_1^T v_n + E_1 \tag{5.2}$$

$$L_{p2} \frac{di_2}{dt} = A_2^T v_n + E_2 - r_i(i_2) \tag{5.3}$$

Observe that the right-hand side is exactly equal to the DC equations (3) and (4). Thus, stability of the equilibrium point can be determined by the variational equation. Let the equilibrium point be (v_{n0}, v_{10}, i_{20}), and put

$$v_n = v_{n0} + \Delta v_n, \quad v_1 = v_{10} + \Delta v_1 \quad i_2 = i_{20} + \Delta i_2 \tag{6}$$

We further transform the variables as follows:

$$x_1 = \sqrt{C_p}\Delta v_n, \quad x_2 = \sqrt{L_{p1}} G_v \Delta v_1 \quad x_2 = \sqrt{L_{p2}} \Delta i_2 \tag{7}$$

Then, we have from (5)

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = P \begin{pmatrix} 0 & -A_1 G_v & -A_2 \\ G_v A_1^T & -G_v & 0 \\ A_2^T & 0 & -R_i \end{pmatrix} P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{8}$$

where $P = \mathrm{diag}(C_p^{-\frac{1}{2}}, L_{p1}^{-\frac{1}{2}} G_v^{-1}, L_{p2}^{-\frac{1}{2}})$, and $G_v = \left.\frac{\partial g_v(v)}{\partial v}\right|_{v=v_{10}}, \quad R_i = \left.\frac{\partial r_i(i)}{\partial i}\right|_{i=i_{20}}$

Now, let us define the Liapunov function as follows:

$$V = \left(x_1^T x_2^T x_3^T\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{9}$$

Then, we have

$$\frac{dV}{dt} = -2\left(x_1^T x_2^T x_3^T\right) P \begin{pmatrix} 0 & 0 & 0 \\ 0 & G_v & 0 \\ 0 & 0 & R_i \end{pmatrix} P \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{10}$$

Now, we have the following instability condition for the nonlinear reciprocal resistive circuits.

Theorem 1 *For reciprocal circuits, if all of G_v and R_i have positive differential resistances at the equilibrium point, those parts of the characteristic curve will be totally stable. On the other hand, the equilibrium point will be NDR unstable if one or more of the differential resistances has negative value at the point.*

Proof: See reference [12].

3. Unstable regions of solution curves

Now, consider more general circuits containing bipolar transistors, FETs and so on. The solution curve can be calculated by solving a resistive circuit composed of n equations in $(n+1)$ variables [3]

$$f(x) = 0, \quad f : R^{n+1} \longrightarrow R^n \tag{11}$$

where x_{n+1} is an additional variable, and sometimes chosen as the DC bias or forced input. Assume $f(x)$ is C^2 continuous in $x \in R^{n+1}$. Let us describe the variable by $x = x(s)$ as a function of *arc-length* s from the starting point x_0 [2]. Then, the solution of (11) satisfies the following set of algebraic-differential equations:

$$\Gamma(x) \equiv \begin{pmatrix} f(x) \\ \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \cdots + \left(\frac{dx_{n+1}}{ds}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{12}$$

Since the solution curve $\Gamma(x)$ is a continuous function of s even at the *turning point*, we have from (12)

$$D\Gamma(x) \begin{pmatrix} \frac{dx_1}{ds} \\ \cdots \\ \frac{dx_n}{ds} \\ \frac{dx_{n+1}}{ds} \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ 1 \end{pmatrix}, \quad \text{where } D\Gamma(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \\ \frac{dx_1}{ds} & \cdots & \frac{dx_n}{ds} & \frac{dx_{n+1}}{ds} \end{pmatrix}$$

Observe that the first $n \times (n+1)$ submatrix corresponds to the Jacobian matrix of $f(x)$, and the last row shows the derivatives of the curve. Our curve tracing algorithm [2] can efficiently trace the solution curve satisfying (12). In this case, it is proved that whenever the rank of Jacobian matrix of $f(x)$ is n, the coefficient matrix $D\Gamma(x)$ is nonsingular. So, we can trace even for the *turning points*. Thus, we have the following relation by the Cramer's formula to $(n+1)$th variable

$$\frac{dx_{n+1}}{ds} = \frac{\det|D_n f(x)|}{\det|D\Gamma(x)|} \tag{13}$$

where $D_n f(x)$ is the Jacobian matrix for the variable $\{x_1, x_2, \ldots, x_n\}$

This relation plays a very important role in investigating the stability of the solution

[3]Note that the relations (3) and (4) can be applied to *nonreciprocal circuits*, where the nonlinear resistors are described by

$$i_1 = g_v(v, i), \quad v_2 = r_i(v, i)$$

curves. Now, assume that we have the following dynamic equation by considering parasitic elements:

$$Q\frac{dx}{dt} = f(x, x_{n+1}), \quad \text{where } Q = \begin{pmatrix} C_p & 0 \\ 0 & L_p \end{pmatrix} \tag{14}$$

Then, the variational equation at an equilibrium point x is given by

$$Q\frac{d\Delta x}{dt} = D_n f(x)\Delta x \tag{15}$$

Thus, the stability condition of the resistive circuits is decided by the eigenvalues of the Jacobian matrix $D_n f(x)$. We have the following stability property around the *turning point.*

Theorem 2 *When the solution curve has passed through the turning point, one of three bifurcations of the stability may occur at the point as follows: a totally stable to a saddle-node, an unstable focal to a saddle-node, and a saddle-node to a different order of saddle-node.*

Proof. The direction of the solution curve is changed at a turning point, so that we have $dx_{n+1}/ds = 0$ at the point. This means that the sign of $\det|D_n f(x)|$ is changed after passing through the turning point because of the nonsingularity of $D\Gamma(x)$ in (13) [10]. Here, we transform (15) as follows:

$$\frac{d\Delta x}{dt} = Q^{-1}D_n f(x)\Delta x$$

The eigenvalues of the variational equation satisfy the following relation [4]:

$$\det|Q^{-1}D_n f(x)| = \det|Q^{-1}|\prod_{i=1}^{n}\lambda_i \tag{16}$$

We assume that $\det|Q^{-1}| \neq 0$ holds, so that the stability depends on the eigenvalues of $D_n f(x)$, where $\lambda_i (i = 1, 2, \ldots, n)$ are the eigenvalues composed of real and/or complex conjugates. Thus, the change of sign (13) means that an odd number of the real eigenvalues has changed signs after passing through the turning point. Therefore, the type of stability is changed at the point. Note that for the complex conjugate eigenvalues, the sign of (16) does not change even if the real parts have changed sign, so that one of three bifurcations given in the *Theorem 2* will be possible.

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Next, we consider the stability of the solution curve around the *branch bifurcation point* [9], where two solution curves cross at a point. It is known that the rank of the Jacobian matrix of (11) for $\{x_1, x_2, \ldots, x_{n+1}\}$

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial x_{n+1}} \end{pmatrix}$$

is reduced to less than n. Hence, the matrix $D\Gamma(x)$ becomes singular at the bifurcation point. We have the following theorem around the point.

Theorem 3 *Let $\Gamma(x)$ be a smooth solution curve passing through the branch bifurcation point. Then, the stability of solution is changed at the point.*
Proof: For simplicity, put

$$d_1(x_{n+1}) \equiv \det|D\Gamma(x)| \tag{17}$$

Now, applying Taylor expansion to $d_1(x_{n+1})$ at two points $x^*_{n+1} - \Delta x_{n+1}$ and $x^*_{n+1} + \Delta x_{n+1}$ before and after the bifurcation point x^*, we have

$$d_1(x^*_{n+1} - \Delta x_{n+1}) = d_1(x^*_{n+1}) - d_1{}'(x^*_{n+1})\Delta x_{n+1} + \cdots \tag{18.1}$$

$$d_1(x^*_{n+1} + \Delta x_{n+1}) = d_1(x^*_{n+1}) + d_1{}'(x^*_{n+1})\Delta x_{n+1} + \cdots \tag{18.2}$$

where $'$ indicates the derivative with respect to x_{n+1}. At the branching point x^*, the following relations hold [9]

$$\mathrm{rank}(Df(x^*)) = n-1, \quad d_1(x^*_{n+1}) = 0, \quad d_1{}'(x^*_{n+1}) \neq 0 \tag{19}$$

Multiplying the two equations in (18), we obtain

$$d_1(x^*_{n+1} - \Delta x_{n+1})d_1(x^*_{n+1} + \Delta x_{n+1}) \cong -[d_1{}'(x^*_{n+1})]^2 \Delta x^2_{n+1}$$

Thus, the sign of the denominator of (13) is changed whenever it passes through the point. We have the same result as $\det|D\Gamma(x)|$ for

$$d_2(x_{n+1}) \equiv \det|D_n f(x)| \tag{20}$$

because the rank of $Df(x)$ is $n-1$ at the bifurcation point.
Thus, both signs of the denominator and numerator of (13) are changed at the point, so that the direction of solution curve dx_{n+1}/ds will not be changed at the branch bifurcation point. But the stability of the solution curve is changed. The instability of the equilibrium point after the bifurcation point will be a saddle-node type.

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As a special case, there are many symmetric circuits such as a Flip-Flop circuit. In this case, they sometimes have an interesting property such that one of the solution curves is *symmetric* with respect to another one. This type of bifurcation is termed a *pitchfork* bifurcation [9].
Corollary 1 *At a pitchfork point, one of the solution curves changes stability at the point, while the other keeps the same stability passing through the point.*

4. An illustrative example

Hopfield neural networks are sometimes applied to solve combinatorial problems such as the traveling salesman problem, the layout of VLSI circuits and so on. Now, consider the circuits containing 6 synapses whose equation is given by

$$\frac{du_i}{dt} = \sum_{j=1}^{6} w_{ij}x_j - \frac{a}{2}\log\frac{x_i}{1-x_i} + I_i, \quad i = 1, 2, \ldots, 6$$

where

$$W = \begin{pmatrix} 0 & 1 & -2 & -2 & -2 & -2 \\ 1 & 0 & -2 & -2 & -2 & -2 \\ -2 & -2 & 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & -2 & 0 & 1 \\ -2 & -2 & -2 & -2 & 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 3.5 & 3.5 & 5.0 & 5.0 & 3.5 & 3.5 \end{pmatrix}^T$$

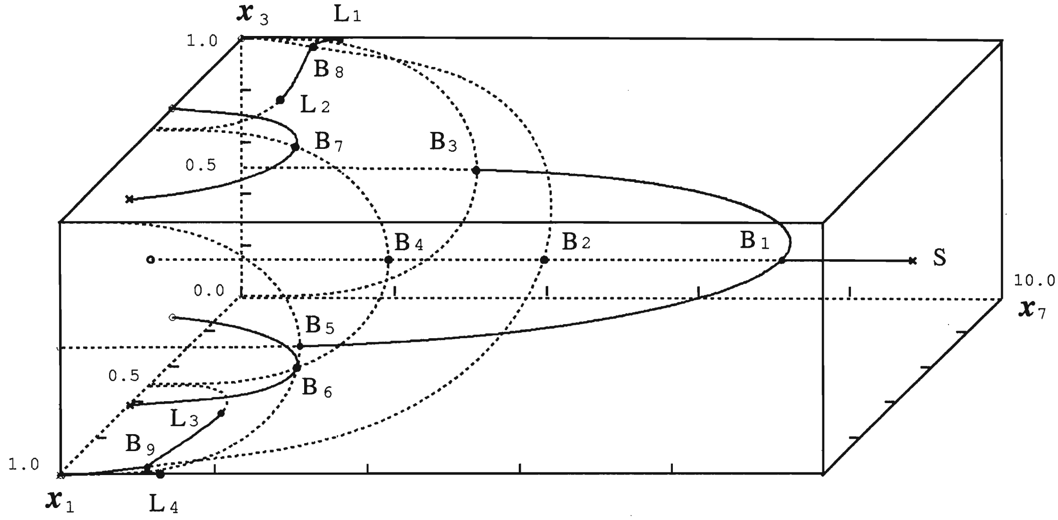

Figure 1 Stability of the solution curve for a Hopfield network

Setting $du_i/dt = 0$, the stationary solutions are obtained. Choosing a as an additional variable, we have a set of 6 algebraic equations with 7 variables. The solution curves are obtained starting from $a = 0.1$ [10]. The curves in the (x_1, x_3, x_7)-plane are shown in the Figure, where we choose $a = 0.29x_7 + 0.1$. We found 9 pitchfork points and 4 turning points.

Note that since the coefficient matrix W is symmetric, all of the eigenvalues are real, and the equilibrium points belong to the saddle-node points. We show the unstable curves by dotted lines. Therefore, all of the stabilities are determined by the application of *Theorem 2* and *Corollary 1.* Note that, in this case, the equilibrium points in region (B_1, B_2) are 1st order saddle-node points because those of (S, B_1) are stable. On the other hand, those in the region (B_2, B_3) are 2nd order saddle-node points, because the order is changed by one at the pitchfork B_2. We can see that most of the branches are unstable.

5. Conclusions and remarks

In this paper, the stability of DC solution curves is examined by introducing parasitic elements, in the form of a small capacitor between every resistor and ground, and an inductor in series with every resistor. The ratios of the capacitors and inductors play a very important role in the stability. We have assumed that the ratios C_p/L_p may take any value from zero to infinity.

We have proved three theorems and one corollary which are very useful for checking the instability regions of the solution curves. We have two main results. The first result is that instability may be occurred in the negative differential resistance(NDR) regions depending on the choice of parasitic elements. The second result is that the stability is also changed at the bifurcation points such as a turning, branch and pitchfork. Thus, we can easily find the instability regions of the solution curves without investigating the variational equation.

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