SELF-EXCITED OSCILLATIONS AND THEIR BIFURCATIONS IN SYSTEMS DESCRIBED BY NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract

Experimental studies of the steady states described by some nonlinear differential-difference equations have been carried out. Various types of self-excited oscillations and their bifurcations in computer simulated systems are surveyed.

1. INTRODUCTION

This report deals with oscillatory steady states which occur in systems described by the following two differential-difference equations:

$$\frac{d\theta(t)}{dt} + \sin\theta(t - L) = \delta, \tag{1}$$

$$\frac{d^2\theta(t)}{dt^2} + \beta\frac{d\theta(t)}{dt} + (2\zeta - \beta)\cos\theta(t - L)\frac{d\theta(t - L)}{dt} + \sin\theta(t - L) = \delta. \tag{2}$$

These equations are mathematical models of phase-locked-loops (PLL) with time delay. Synchronized states of the PLL are represented by the equilibrium points of the equations. Firstly, the parameter regions are important from the engineering point of view, in which all initial conditions lead to quiescent steady states. The regions, i.e., pull-in ranges of the PLL, were already reported in [1]. Secondly, oscillatory steady states are of scientific interest. By using a hybrid computer, this report surveys self-excited oscillations and their bifurcations obtained by varying the system parameters.

2. EXPERIMENTAL RESULTS

In this report, steady states will mean asymptotically stable equilibrium points, periodic solutions, and so forth of (1) and (2). In the following the experimental results leading to the steady states of (1) and (2) will be described. Before giving the details, the following remarks should be noted:

(i) The hybrid, i.e., analog-digital, computer is used to solve equations (1) and (2). An outline of the computer block diagram is given in [1].

(ii) When we compute the solutions, the initial condition is usually given by

$\theta(t)$ = constant for $-L \leqq t \leqq 0$.

Fixing the boundaries of the regions for the different types of steady states, we change the type of steady state by slow variation of the parameter δ.

2.1 CLASSIFICATION OF STEADY STATES

The trajectories of the steady states of (1) and (2) can be represented on the cylindrical phase plane $(\theta, \dot{\theta})$ as shown in Figure 1. The types of steady states in the computer simulated systems can be roughly classified as follows:

(a) Equilibrium point: There are two equilibrium points for (1) and (2).

$$\theta = \sin^{-1}\delta, \tag{3}$$

$$\theta = \pi - \sin^{-1}\delta. \tag{4}$$

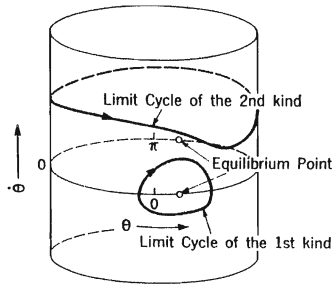


Figure 1. Cylindrical phase plane (θ, $\dot{\theta}$).

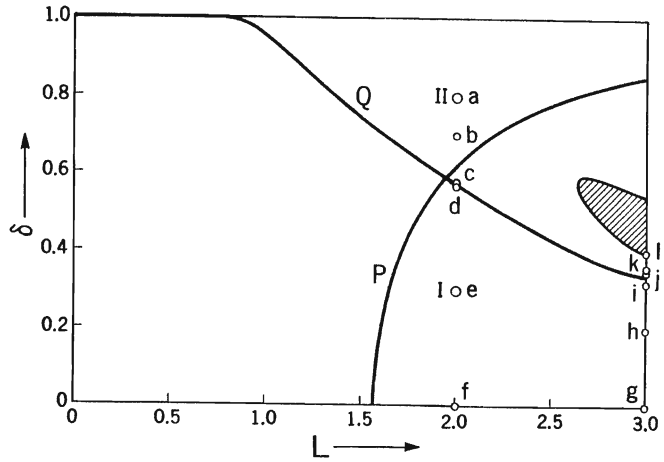


Figure 2. Regions of different steady states for the system represented by (1).

The one given by (4) is always unstable, while the stability of the point given by (3) depends on the values of the system parameters.

(b) Limit cycle of the first kind: The closed orbit represented by a periodic solution

$$\theta(t) = \phi(t),\ \phi(t + T) = \phi(t), \qquad (5)$$

is called a limit cycle of the first kind provided every solution of (1) or (2) tends to this orbit as $t \to \infty$.

(c) Limit cycle of the second kind: When $\theta(t)$ along a limit cycle is represented by

$$\theta(t) = \phi(t) \pm 2\pi t/T,\ \phi(t + T) = \phi(t), \qquad (6)$$

the corresponding orbit is called a limit cycle of the second kind.

Examples of equilibrium points, limit cycles of the first and second kinds are illustrated in Figure 1.

(c') N-kai* limit cycle of the second kind: Special cases of limit cycles of the second kind are closed orbits with phase space periods which are multiples of 2π. In these cases the corresponding steady states take the form

$$\theta(t) = \phi_N(t) \pm 2N\pi t/T_N,\ \phi_N(t + T_N) = \phi_N(t), \qquad (7)$$

for some least integer N.

(d) Chaotic steady states: There sometimes appear chaotic steady states in spite of the perfectly deterministic nature of the equations. Similar to ordinary differential equations (ODE), a chaotic steady state is considered to be represented by a bundle of orbits where the bundle is asymptotically orbitally stable but contains infinitely many unstable periodic orbits.

* We introduce the Japanese word "kai" here which means "times" or "rounds". It has the features of being short and gives exactly the correct meaning desired.

2.2 REGIONS FOR DIFFERENT TYPES OF STEADY STATES AND BIFURCATIONS BETWEEN THEM

Figure 2 shows the regions on the (L, δ)-plane in which different steady states are observed in the computer simulated system described by (1). Figures 3a and 3b are the regions for (2).

In these figures, stable equilibrium points exist to the left of the boundary curves P. These equilibrium points become unstable to the right of P casting off a stable limit cycle of the first kind which surrounds it. This bifurcation shows a close resemblance to the Hopf bifurcation in the ODE systems. Above the boundary curves Q, there exists a limit cycle of the second kind. It disappears below Q.

For large values of L, various steady states have been observed. Some of them are so complicated that it is almost impossible to locate their boundaries. Under such situations, only representative boundaries are depicted in Figures 2 and 3. In these figures, hatched lines are used with the same meaning and will be seen in connection with the ex-

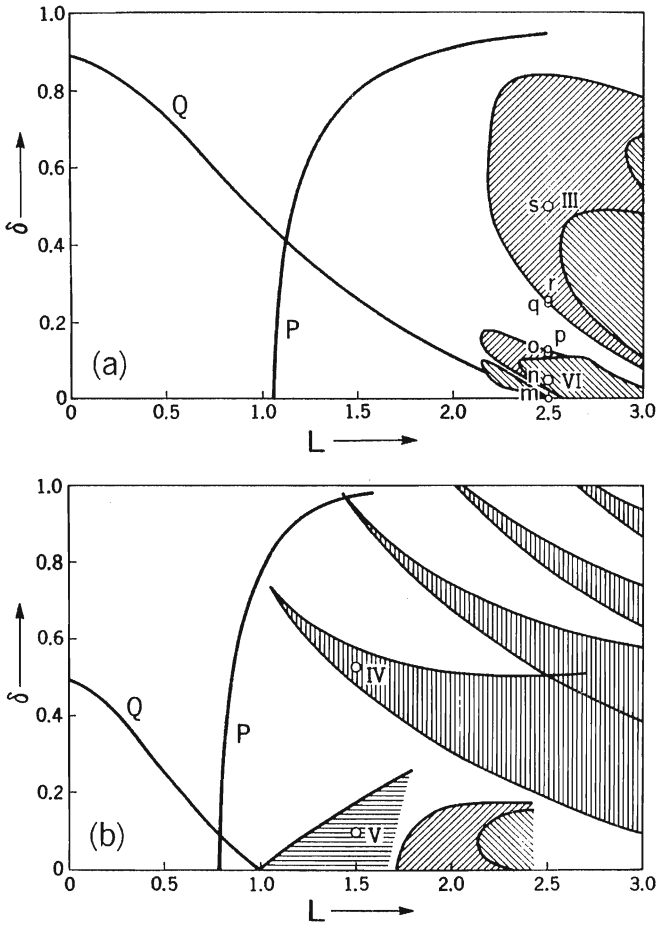


Figure 3. Regions of different steady states for the system represented by (2).
(a) $\zeta = 1/\sqrt{2}$ and $\beta = 0.5$.
(b) $\zeta = 1/\sqrt{2}$ and $\beta = 0.1$.

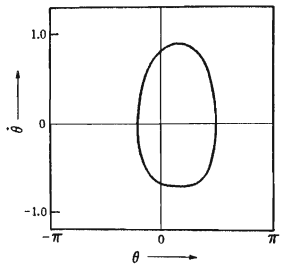


Figure 4-I. Steady state at the point I in Fugure 2 ($\delta = 0.3$ and $L = 2.0$).

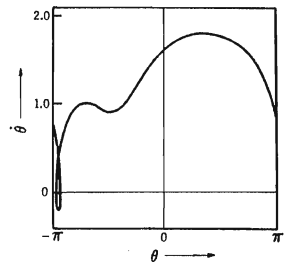


Figure 4-II. Steady state at the point II in Figure 2 ($\delta = 0.8$ and $L = 2.0$).

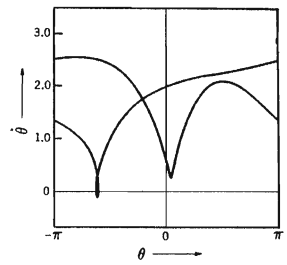


Figure 4-III. Steady state at the point III in Figure 3a ($\delta = 0.5$ and $L = 2.5$).

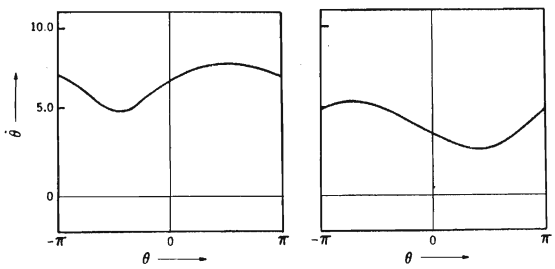


Figure 4-IV. Steady states at the point IV in Figure 3b ($\delta = 0.53$ and $L = 1.5$).

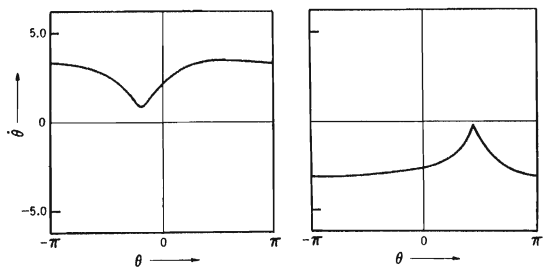
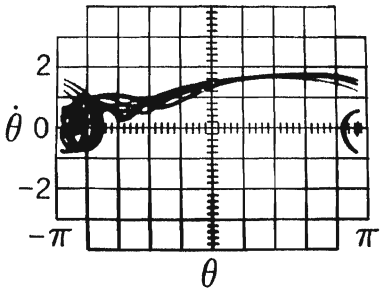


Figure 4-V. Steady states at the point V in Figure 3b ($\delta = 0.1$ and $L = 1.5$).



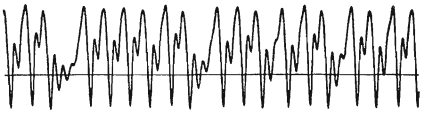


Figure 4-VI. Steady state at the point VI in Figure 3a (δ = 0.05 and L = 2.5).

amples of steady states in the following section.

2.3 REPRESENTATIVE EXAMPLES OF SELF-EXCITED OSCILLATIONS

In order to illustrate the regions of the (L, δ)-plane, we show the steady states at six representative sets of parameters (L, δ) indicated by I, II, ...VI in Figures 2 and 3.

Figure 4-I shows a limit cycle of the first kind. The mathematical proof of the existence of the periodic solution for this case was proved by Furumochi [2]. Figure 4-II shows a limit cycle of the second kind. An example of a 2-kai limit cycle of the second kind is shown in Figure 4-III. In the cases of Figures 4-IV and V, there are two possible steady states and which one occurs depends on the initial conditions. Figure 4-VI shows a chaotic steady state with a waveform $\dot{\theta}(t)$.

2.4 BIFURCATIONS OF THE STEADY STATES

Figure 5a shows an example of a bifurcation in which the parameter δ is decreased with L = 2.0 in (1). The locations of those parameter values are indicated by a, b,.. f in Figure 2. We can see that the limit cycle of the second kind disappears at some value of δ between 0.58 and 0.57 and a limit cycle of the first kind then appears. This situation has some resemblance to the case of ODE systems in which limit cycles approach a saddle to saddle separatrix and then cease to exist. Increasing δ from zero, the limit cycle of the first kind converges to the stable equilibrium point on the boundary P and the point continues to exist until δ reaches 1.0.

Figure 5b shows another example of a bifurcation in which δ is increased with L = 3.0. The locations of the parameters are also indicated in Figure 2. This example demonstrates the great complexity of the phenomenon.

Figure 6 is another example in which δ is increased with L = 2.5 in (2). The locations of the parameters are indicated by m, n,... s in Figure 3a. These figures show the trajectory doubling bifurcations explicitly and it seems that chaotic steady states occurs through a cascade of the bifurcations. These situations also closely resemble the case of ODE systems.

3. CONCLUSION

By using a hybrid computer, experimental studies of the steady states of nonlinear differential-difference equations (1) and (2) have been carried out. From these results, various types of self-excited oscillations and their bifurcations under variations of the system parameters have been clarified. The results reported here deserve further mathematical attention.

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[2] T. Furumochi: Tohoku Mathematical Journal, The Second Series, Vol. 30, No. 1, pp. 13-35, March 1978.

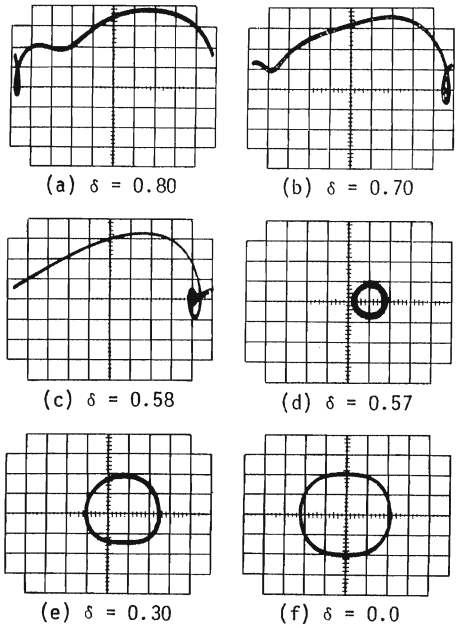

Figure 5a. Bifurcation for decreasing δ with L = 2.0 in (1).

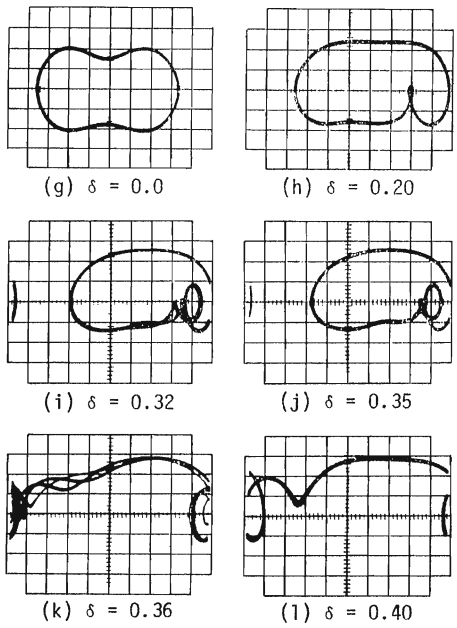

Figure 5b. Bifurcation for increasing δ with L = 3.0 in (1).

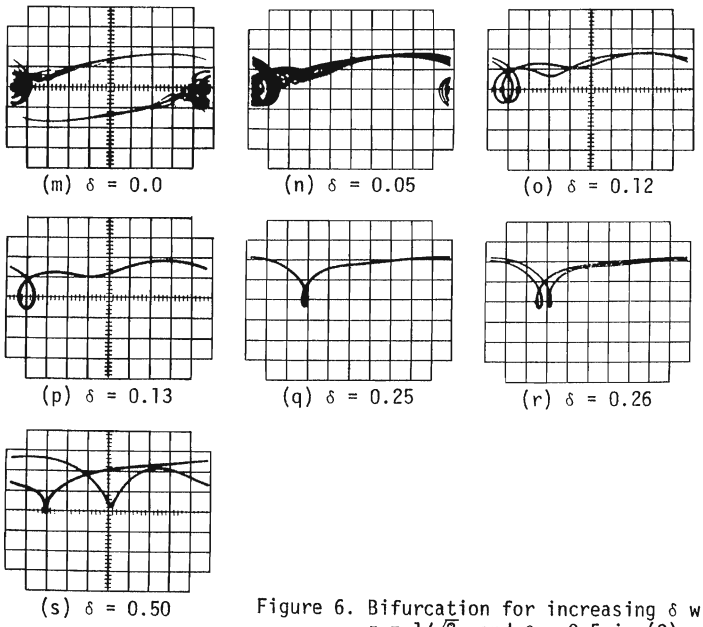

Figure 6. Bifurcation for increasing δ with L = 2.5, $\zeta = 1/\sqrt{2}$, and β = 0.5 in (2).