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Several results in classical and modern harmonic analysis in mixed Lebesgue spaces

By

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Abstract

Mixed Lebesgue spaces have attracted the interest of harmonic analysts since the early sixties. These spaces naturally appear when considering functions with different quantitive behavior on different sets of variables on which they depend. For example, this is the case when studying functions with physical relevance like the solutions of partial differential equations with time and space dependence. Mixed Lebesgue spaces can also be seen as vector-valued Lebesgue spaces. Using this point of view we revisit some classical results in the literature and survey newer ones about Leibniz's rule for fractional derivatives, bilinear null forms, sampling, Calderóns reproducing formula, and wavelets in the context of mixed norms.

§ 1. Intorduction

This article is an expansion of the talk presented by the first named author at the workshop on *Harmonic Analysis and Nonlinear Partial Differential Equations* held at the *Research Institute for Mathematical Sciences (RIMS)*, Kyoto University, Kyoto, Japan, from July 4 to 6, 2017. As such, it is a survey of both classical and more recent results involving tools of harmonic analysis and their applications in the context of *mixed Lebesgue spaces*. All the results presented here have appeared in one way or another elsewhere, and we reproduce them without proof but providing a brief historical account of their development and pointing to their motivation and the main references in the literature. In particular we will be borrowing substantially from [41]. Our hope

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is to provide a comprehensive summary of results in the subject including several which have been, perhaps, overlooked in more recent works, but which have a lot of potential for further analysis involving mixed norm estimates.

The continuing interest in mixed Lebesgue spaces has two main motivations. The most classical one is related to the fact that these spaces represent one of the most simple examples of vector valued L^p spaces. This has been the focus of the original works of Benedek-Panzone [7] and Benedek-Calderón-Panzone [6], where the basic properties of the spaces and the natural extension of results for classical singular integrals were first considered. Later works with mixed norms providing evidence of the intimate connection between vector valued estimates, singular integrals and Littlewood-Paley theory include those by Rubio de Francia-Ruiz-Torrea [37], and Fernández [19], this last one also relating the study of mixed Lebesgue spaces to multiparameter analysis. The more modern interest in mixed norms has arisen from problems in partial differential equations. In particular, in time dependent equations it is often necessary to consider function of a time variable t and a space variable x with different quantitative and regularity properties in each variable. Most notably among motivating examples for our results, we mention the mixed estimates works of Kenig-Ponce-Vega [29] using *Leibniz's rules* in their study of well-posedness of the generalized KdV equation, and those involving estimates for *null forms* by Foschi-Klainerman [20] and then by Planchon [36] in their study of homogeneous wave equations. The latter works motivated the work by Stefanov-Torres [39] on null forms (from which we will also borrow for this note) while the former provided the inspiration for the results in [41] on Leibniz's rule on mixed Lebesgue spaces. Several fundamental tools in harmonic analysis need to be extended to the mixed norm setting to deal with the above type of estimates. Once these extension are achieved one can then apply them to other common problems of today's harmonic analysis, such as those involving decomposition and characterization of other function spaces, sampling and wavelet representations. Such studies started in [42] and [41], but a lot of work still remains to be done.

This paper is organized as follows. In the next section we collect the basic definitions and notation that we shall employ. In Section 3 we present results about vector valued estimates for Calderón-Zygmund operators and consequences of them, while in Section 4 we recall some results about weighted norm inequalities in mixed Lebesgue spaces which were obtained by Kurtz in [31] (see also the work of Moen [32]). Section 5 collects several result about bilinear estimates for null forms and fractional derivatives. Finally Section 6 contains some results in the literature related to sampling and wavelets in mixed norm spaces.

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§ 2. Definitions and notation

For convenience, we will consider the mixed Lebesgue spaces $L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^n)$, or simply $L^p_t L^q_x(\mathbb{R}^{n+1})$, for $0 < p, q < \infty$, which for us will be defined by the quasi-norms

$$\|f\|_{L^p_t L^q_x(\mathbb{R}^{n+1})} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |f(t,x)|^q \, dx \right)^{p/q} dt \right)^{1/p}.$$

Although reversing the order of the norms clearly produce different spaces (unless of course $p = q$), all what we present here could be done for other version of the spaces, grouping any number of variables and considering successive L^p norms on them. Our choice, however, is motivated by the appearance of these spaces in PDEs, where the first variable t is viewed as time and has a distinctive role. Also,when the role of variables and space dimensions are clear we may just write $L^p L^q$.

The mixed Lebesgue spaces are Banach spaces when $1 \le p, q < \infty$ and quasi-Banach spaces otherwise. They enjoy very natural duality, density, and interpolation properties analogous to those for Lebesgue spaces and certainly other underlying products of measure spaces could be used too. Although mixed Lebesgue spaces may have appeared in the literature early on, many of their basic properties were proved in great detail in [7], to where we refer the reader.

More generally, we will need to consider *vector valued Lebesgue spaces* $L^p(\mathbb{R}^m, \mathbf{B})$ of measurable functions F in $\mathbb{R}^m$ taking values in a Banach space $\mathbf{B}$ and defined by the norm

$$\|f\|_{L^p(\mathbb{R}^m, \mathbf{B})} = \left(\int_{\mathbb{R}^m} \|f(y)\|^p_{\mathbf{B}} \, dy \right)^{1/p}.$$

In this sense one has the identification $L^p_t L^q_x(\mathbb{R}^{n+1}) = L^p(\mathbb{R}, L^q(\mathbb{R}^n))$. Even in greater generality one may consider *vector valued mixed Lebesgue spaces* $L^p_t L^q_x(\mathbb{R}^{n+1}, \mathbf{B}) = L^p(\mathbb{R}, L^q(\mathbb{R}^n, \mathbf{B}))$. The most important example for our purposes will be $L^p_t L^q_x(\mathbb{R}^{n+1}, \ell^2)$ consisting to sequences of functions $F(t,x) = \{f_j(t,x)\}_j$ in $\mathbb{R}^{n+1}$ such that

$$\|F\|_{L^p_t L^q_x(\mathbb{R}^{n+1}, \ell^2)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} \left(\sum_j |f_j(t,x)|^2 \right)^{q/2} dx \right)^{p/q} dt \right)^{1/p} < \infty.$$

As customarily done, we denote by $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$ the space of bounded linear operators from a Banach space $\mathbf{B}_1$ to another Banach space $\mathbf{B}_2$.

When convenient, we write $z \in \mathbb{R}^{n+1}$ as either $z = (x_0, x)$ or $z = (t, x)$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Likewise, we will denote the full Fourier transform on all $(n+1)$ variables by

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^{n+1}} f(z) e^{-iz\zeta}\, dz,$$

with frequency variable $\zeta = (\tau, \xi)$ or $\zeta = (\xi_0, \xi)$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Our notation for dilations will denote normalization in $L^1(\mathbb{R}^{n+1})$, i.e. for any $r > 0$ and $z \in \mathbb{R}^{n+1}$, $f_r(z) = r^{-(n+1)} f(r^{-1}z)$. Hence $\widehat{f_r}(\zeta) = \widehat{f}(r\zeta)$.

Both the Fourier transform and several operators presented below are assumed to be a priori given for sufficiently nice functions so that their expressions make sense and are extended then by continuity to larger spaces of functions or distributions when allowed to do so by the estimates obtained.

We will consider the *(full) homogeneous fractional derivatives* $|\nabla|^s$ defined for $s \in \mathbb{R}$ by

$$\widehat{|\nabla|^s f}(\zeta) = |\zeta|^s \widehat{f}(\zeta).$$

Note that we will allow s to be negative, hence we will actually consider *fractional integration* as well. We also have the *(partial) homogeneous fractional derivatives* $|\nabla_x|^s$ and $|\nabla_t|^s$ defined by

$$\widehat{|\nabla_x|^s f}(\tau, \xi) = |\xi|^s \widehat{f}(\tau, \xi) \quad \text{and} \quad \widehat{|\nabla_t|^s f}(\tau, \xi) = |\tau|^s \widehat{f}(\tau, \xi).$$

As usual the *inhomogeneous fractional derivative* operators J^s are defined by

$$\widehat{J^s f}(\zeta) = (1 + |\zeta|^2)^{s/2} \widehat{f}(\zeta).$$

The *null forms* Q_{ij} for $i, j = 0, \dots, n$ and Q_0, are the bilinear differential operators given by

$$Q_{ij}(f, g)(x_0, x) = \partial_i f(x_0, x) \partial_j g(x_0, x) - \partial_j f(x_0, x) \partial_i g(x_0, x)$$

and

$$\begin{aligned} Q_0(f, g)(x_0, x) &= \partial_0 f(x_0, x) \partial_0 g(x_0, x) - \nabla_x f(x_0, x) \cdot \nabla_x g(x_0, x) \\ &= \partial_0 f(x_0, x) \partial_0 g(x_0, x) - \sum_{i=1}^{n} \partial_i f(x_0, x) \partial_i g(x_0, x). \end{aligned}$$

Note that we allow in the definition of Q_{ij} derivatives in the time variable $t = x_0$.

§ 3. Vector valued Calderón-Zygmund operators

The vector valued version of the Calderón-Zygmund theory is by now very well understood. At least for convolution operators it goes back to [6] and more general versions can be found in [37]. The books by Stein [38] and by Duoandikoetxea [16] also present it in great detail. The main result is the following which we state as in [16, Theorem 5.17]

Theorem 3.1.
Let $\mathbf{K}$ *be an* $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$*-valued function defined at least on* $\mathbb{R}^m \times \mathbb{R}^m \setminus \{(x, y) : x = y\}$ *and such that it satisfies the regularity estimates*

$$\int_{|x-y|\geq 2|y-z|} \|\mathbf{K}(x,y) - \mathbf{K}(x,z)\|_{\mathbf{B}_1\to\mathbf{B}_2} dx \leq C, \tag{3.1}$$

$$\int_{|x-y|\geq 2|w-x|} \|\mathbf{K}(x,y) - \mathbf{K}(w,y)\|_{\mathbf{B}_1\to\mathbf{B}_2} dy \leq C. \tag{3.2}$$

Let $\mathbf{T}$ *be bounded from* $L^q(\mathbb{R}^m, \mathbf{B}_1)$ *to* $L^q(\mathbb{R}^m, \mathbf{B}_2)$ *for some* $1 < q < \infty$ *and assume that it is associated with* $\mathbf{K}$*, in the sense that*

$$\mathbf{Tf}(x) = \int_{\mathbb{R}^m} \mathbf{K}(x,y)(\mathbf{f}(y))dy$$

for all compactly supported $\mathbf{f} \in L^\infty(\mathbb{R}^m, \mathbf{B}_1)$ *and* $x \notin \operatorname{supp} \mathbf{f}$*. Then* $\mathbf{T}$ *is bounded from* $L^p(\mathbb{R}^m, \mathbf{B}_1)$ *to* $L^p(\mathbb{R}^m, \mathbf{B}_2)$ *for all* $1 < p < \infty$*. Moreover* $\mathbf{T}$ *satisfies the weak-type (1,1) estimate*

$$|\{x : \|\mathbf{Tf}(x)\|_{\mathbf{B}_2} > \lambda\}| \leq \frac{C}{\lambda}\|\mathbf{f}\|_{L^1(\mathbb{R}^m,\mathbf{B}_1)}.$$

The above theorem states that, even in the vector valued case, the crux of the (scalar-valued) Calderón-Zygmund theory still holds. Namely, the boundedness of the operator on one L^q space combined with the regularity of its kernel gives the boundedness on all L^p spaces in the range indicated.

A simple consequence of this result is that we can go further and obtain from the boundedness on vector-valued L^r spaces the one on the vector-valued L^pL^q spaces. In the case of convolution operators the result is in [6]. A scalar-valued version for non-convolution operators was in the thesis work by Moen [32], while the form we present here is from [41, Corollary 2.3].

Corollary 3.2. *Let* $\mathbf{T}$ *be a bounded operator from* $L^r(\mathbb{R}^{n+1}, \mathbf{B}_1)$ *to* $L^r(\mathbb{R}^{n+1}, \mathbf{B}_2)$ *for all* r*,* $1 < r < \infty$*, which is associated to a kernel* $\mathbf{K}$ *in the sense of Theorem* (3.1)*. Suppose also that* $\mathbf{K}$ *satisfies the size estimate*

$$\|\mathbf{K}(x,y)\|_{\mathbf{B}_1\to\mathbf{B}_2} \leq \frac{C}{|x-y|^{n+1}}, \tag{3.3}$$

and for some $0 < \delta < 1$*, the regularity estimates*

$$\|\mathbf{K}(x,y) - \mathbf{K}(x,z)\|_{\mathbf{B}_1\to\mathbf{B}_2} \leq C\frac{|y-z|^\delta}{|x-y|^{n+1+\delta}} \qquad \text{for } |x-y| \geq 2|y-z|, \tag{3.4}$$

$$\|\mathbf{K}(x,y)-\mathbf{K}(w,y)\|_{\mathbf{B}_1\to\mathbf{B}_2} \le C\frac{|x-w|^\delta}{|x-y|^{n+1+\delta}} \qquad \text{for } |x-y|\ge 2|x-w|. \tag{3.5}$$

Then $\mathbf{T}$ *extends to a bounded linear operator from* $L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathbf{B}_1)$ *to* $L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^n,\mathbf{B}_2)$ *for all* $1<p,q<\infty$.

The proof consists in identifying $L^p_tL^q_x(\mathbb{R}^{n+1},\mathbf{B}) = L^p(\mathbb{R},L^q(\mathbb{R}^n,\mathbf{B}))$ and using (3.3)-(3.5) to establish (3.1)-(3.2) via an application of Schur's test, to then invoke Theorem 3.1. See the Appendix in [41] for details.

§ 3.1. Examples and applications

By Corollary 3.2 all classical Calderón-Zygmund operators are bounded on mixed Lebesgue spaces. Of relevance to the study of null forms, as we will explain later, are the *Riesz transform operators* in $\mathbb{R}^{n+1}$, R_j, $j=0,\dots,n$, which are the principal valued singular integrals defined, following the notation in [39], by the kernels

$$k_0(t,x)=c_n\frac{t}{(|t|^2+|x|^2)^{(n+2)/2}}$$

and

$$k_j(t,x)=c_n\frac{x_j}{(|t|^2+|x|^2)^{(n+2)/2}}.$$

Here the constant c_n is properly selected so that

$$\widehat{R_0f}(\tau,\xi)=-i\tau|(\tau,\xi)|^{-1}\widehat{f}(\tau,\xi)$$

and

$$\widehat{R_jf}(\tau,\xi)=-i\xi_j|(\tau,\xi)|^{-1}\widehat{f}(\tau,\xi).$$

It follows that for all $j=0,\dots,n$ and all $1<p,q<\infty$,

$$R_j : L^p_tL^q_x(\mathbb{R}^{n+1})\to L^p_tL^q_x(\mathbb{R}^{n+1}), \tag{3.6}$$

since the kernels k_j clearly satisfy (3.3)-(3.5) for $B_1=B_2=\mathbb{C}$. Moreover, as observed in [41, Corollary 2.5], if T_m is a *multiplier operator* given by

$$\widehat{T_mf}(\tau,\xi)=m(\tau,\xi)\widehat{f}(\tau,\xi)$$

and

$$|\partial^\alpha m(\tau,\xi)|\lesssim_\alpha \frac{1}{|(\tau,\xi)|^{|\alpha|}}. \tag{3.7}$$

for all multi-indexes α, then

$$T_m : L^p_tL^q_x(\mathbb{R}^{n+1})\to L^p_tL^q_x(\mathbb{R}^{n+1})$$

for all $1 < p, q < \infty$.

We note that recently Antonić and Ivec [2] improved on this result by requiring the Mihlin condition (3.7) only for all $|\alpha| \le [\frac{n+1}{2}] + 1$, as in the classical case of Lebesgue spaces; or more generally by imposing the Hörmander condition

$$\sup_{r>0} r^{-(n+1)+2|\alpha|} \int_{r<|\zeta|<2r} |\partial^\alpha m(\zeta)|^2 \, d\zeta \le C < \infty,$$

also for all $|\alpha| \le [\frac{n+1}{2}] + 1$.

One of the most useful applications of the vector valued Calderón-Zygmund theory is the Littlewood-Paley characterization of Lebesgue spaces by realizing the square function as an ℓ^2 valued singular integral. Corollary 3.2 allows to do the same in the mixed Lebesgue case. The following result is from [41, Corollary 2.4].

Corollary 3.3. *Let $1 < p, q < \infty$. Suppose that Ψ is a function on $\mathbb{R}^{n+1}$ which satisfies, for all $|\alpha| \le 1$,*

$$|\partial^\alpha \Psi(x)| \lesssim \frac{1}{(1 + |(t,x)|)^{n+2+|\alpha|}}. \tag{3.8}$$

and has mean value zero. Then for all $f \in L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^n)$,

$$\left\| \left(\sum_{j\in\mathbb{Z}} |f * \Psi_{2^{-j}}|^2 \right)^{1/2} \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)}. \tag{3.9}$$

Conversely, if $\Psi \in \mathcal{S}(\mathbb{R}^{n+1})$ is such that supp $\widehat{\Psi} \subset \{\frac{\pi}{4} < |\xi| < \pi\}$ *and $\widehat{\Psi} > c > 0$ on $\{\frac{\pi}{4} + \epsilon < |\xi| < \pi - \epsilon\}$, then*

$$\|f\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f * \Psi_{2^{-j}}|^2 \right)^{1/2} \right\|_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^n)}. \tag{3.10}$$

The proof of (3.9) given in [41] uses the fact from the classical case that the operator $\mathbf{T}$ with kernel

$$\mathbf{K}(t,x) : \mathbb{C} \to l^2$$

$$\mathbf{K}(t,x)(v) = \{\Psi_{2^{-j}}(t,x)v\}$$

is a bounded vector valued Calderón-Zygmund operator from the space $L^p(\mathbb{R}^{n+1}, \mathbb{C})$ to the space $L^p(\mathbb{R}^{n+1}, l^2)$, $1 < p < \infty$; see e.g. the book by Grafakos [24, Theorem 5.1.2] for very detailed arguments. It follows by Corollary 3.2 that $\mathbf{T}$ is also bounded from $L^p_t L^q_x(\mathbb{R}^{n+1}, \mathbb{C})$ to $L^p_t L^q_x(\mathbb{R}^{n+1}, l^2)$. The converse inequality (3.10) actually follows from

(3.9) by standard duality arguments and the fact that the extra condition on Ψ allows one to construct a reproducing formula of the form

$$f = \sum_{j\in\mathbb{Z}} \varphi_{2^{-j}} * \Psi_{2^{-j}} * f,$$

where φ is another function with similar properties as Ψ.

§ 4. Maximal operatorss and weighted estimates

Let M be the Hardy-Littlewood maximal operator in $\mathbb{R}^{n+1}$ defined on cubes. The results in [19] and [31] extended the boundedness of M and its Fefferman-Stein [17] vector-valued version to $L^p_t L^q_x(\mathbb{R}^{n+1})$. In particular, see [19, Theorem 4.2], for $1 < p, q, r < \infty$.

$$\left\| \left(\sum_j |M(f_j)|^r \right)^{1/r} \right\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \lesssim \left\| \left(\sum_j |f_j|^r \right)^{1/r} \right\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)}. \tag{4.1}$$

Actually both [19] and [31] consider the strong maximal operator on $\mathbb{R}\times\mathbb{R}^n$,

$$M_S g(t,x) = \sup_{R\ni(t,x)} \frac{1}{|R|}\int_R |g(s,y)|dyds$$

where $R = I\times Q$ and I is an interval in $\mathbb{R}$ and Q is a cube in $\mathbb{R}^n$.

The following weights are considered in [31]. A nonnegative function w is in $A_p(A_q)(\mathbb{R}^{n+1})$, $1 < p, q < \infty$, if

$$\left(\int_I \left(\int_Q w(t,x)\,dx \right)^{p/q} dt \right) \left(\int_I \left(\int_Q w(t,x)^{1-q'}\,dx \right)^{p'/q'} dt \right)^{p-1} \le C|I\times Q|^p \tag{4.2}$$

for all I and Q. Note that for $p = q$ this is the A_p condition on $\mathbb{R}^{n+1}$ but on rectangles of the form $I\times Q$. The smallest constant in the right-hand side of (4.2) is denoted by $\|w\|_{A_p(A_q)}$.

The following extrapolation result of Kurtz [31, Theorem 2] is an extension of the classical result of Rubio de Francia and an extremely powerful tool to obtain boundedness results in mixed Lebesgue spaces.

Theorem 4.1. *Let T be a sublinear operator bounded on $L^s(\mathbb{R}^{n+1}, w)$ for some $1 < s < \infty$ and all $w \in A_s(A_s)$, with a norm that depends only on $\|w\|_{A_s(A_s)}$. Then, for any $1 < p, q < \infty$, T is bounded on $L^pL^q(\mathbb{R}^{n+1}, w)$ for all $w \in A_p(A_q)$ of the form $w(t,x) = u(t)v(x)$ (and with a norm that depends only on $\|w\|_{A_p(A_q)}$).*

Note that $A_s(A_s)(\mathbb{R}^{n+1}) \subset A_s(\mathbb{R}^{n+1})$, the regular A_s class, so the above result allows to extended any operator bounded on one weighted Lebesgue space $L^s(\mathbb{R}^{n+1}, w)$ for all $w \in A_s(\mathbb{R}^{n+1})$ to, in particular, all unweighted mixed Lebesgue spaces $L_t^p L_x^q(\mathbb{R}^{n+1})$. The extrapolation result of Kurtz can be used then to provide an alternative proof of the boundedness of Calderón-Zygmund operators on mixed Lebesgue spaces. With the same reasoning, we can make the following observation for the commutator with pointwise multiplication with function in the John-Nirenberg space BMO.

Corollary 4.2. *Let T be a bounded Calderon-Zygmund operator in $L^s(\mathbb{R}^{n+1})$ and $b \in BMO(\mathbb{R}^{n+1})$. Then the commutator*

$$[T, b]f = T(bf) - bT(f)$$

is bounded on $L_t^p L_x^q(\mathbb{R}^{n+1})$ for all $1 < p, q < \infty$.

The fact that Theorem 4.1 actually involves rectangular weights make it also applicable in product-type situations. In particular it is used in [31, Theorem 4] to obtain the boundedness on mixed Lebesgue spaces of the product-type Calderón-Zygmund operators of Fefferman-Stein [18]. See those references for further details.

We conclude this section by mentioning also that an off-diagonal extrapolation theorem on mixed Lebesgue spaces was obtained by Moen [32, Theorem 5.2], allowing one to consider also operators like fractional integration in $L_t^p L_x^q(\mathbb{R}^{n+1})$ (and corresponding weighted versions as well). The following corollary is also a consequence of [32, Corollary 5.3.3], but this unweighted version was previously obtained in [7, p.321]. See also the work of Adams and Bagby [1] for a different approach, a more general range of exponents, and further references to earlier versions of the result.

Corollary 4.3. *Assume $1 < p_1, p_2, q_1, q_2 < \infty$ and $0 < s < n + 1$ so that $1/p_1 - 1/p_2 = s/(n + 1)$ and $1/q_1 - 1/q_2 = s/(n + 1)$. Then*

$$|\nabla|^{-s} : L^{p_1} L^{q_1}(\mathbb{R}^{n+1}) \to L^{p_2} L^{q_2}(\mathbb{R}^{n+1}).$$

§ 5. Bilinear estimates involving derivatives

§ 5.1. Null forms

As mentioned in the introduction, Foschi and Klainerman [20] used several estimates for null forms in their study of homogeneous wave equations. Moreover they conjecture several estimates for $\left\| |\nabla|^\beta Q(f, g) \right\|_{L^q L^r}$ in terms of certain wave-Sobolev space norms of f and g. Then in [36], Planchon used Littlewood-Paley argument to establish, among other things, the estimate

$$\left\| |\nabla_x|^{-s} Q_{ij}(f, g) \right\|_{L^1(\mathbf{R}^{n+1})} \lesssim \left\| |\nabla_x|^{1-s/2} f \right\|_{L^2(\mathbf{R}^{n+1})} \left\| |\nabla_x|^{1-s/2} g \right\|_{L^2(\mathbf{R}^{n+1})}, \tag{5.1}$$

for $0 < s < 1$, and a Besov-type estimate

$$\big\||\nabla_x|^{-s}Q_{ij}(f,g)\big\|^2_{L^q_tL^r_x} \lesssim \sum_{k\in\mathbb{Z}}\left(2^{k(1-s/2)}\|\Delta_k f\|_{L^{q_1}_tL^{r_1}_x}\right)^2\sum_{k\in\mathbb{Z}}\left(2^{k(1-s/2)}\|\Delta_k g\|_{L^{q_2}_tL^{r_2}_x}\right)^2, \tag{5.2}$$

for appropriate q, r, s with $1/q = 1/q_1 + 1/q_2$, $1/r = 1/r_1 + 1/r_2$ and $0 \le s \le 1$. Here $\Delta_k f = \psi_{2^{-k}} f$ for some ψ as in Corollary 3.3. The work in [39] further generalizes and extends this estimate using the boundedness of the Riesz transforms in mixed Lebesgue spaces. This approach also allows the consideration of time derivatives.

The relation between the Riesz transforms and the null forms is very simple. Note that

$$(\widehat{|\nabla|R_jf})(\xi) = -i\xi_j\widehat{f}(\xi),$$

so

$$|\nabla|R_j = \partial_j.$$

Hence, writing $u = |\nabla|f$ and $v = |\nabla|g$, transforms

$$Q_{ij}(f,g) = \partial_i f\partial_j g - \partial_j f\partial_i g$$

into

$$R_iuR_jv - R_juR_iv.$$

It follows that to prove an estimate of the form

$$\|Q_{ij}(f,g)\|_{X_1} \lesssim \||\nabla|f\|_{X_2}\||\nabla|g\|_{X_3}$$

for some spaces X_1, X_2 and X_3, is then equivalent to prove

$$\|R_iuR_jv - R_juR_iv\|_{X_1} \lesssim \|u\|_{X_2}\|v\|_{X_3}.$$

Using this approach Stefanov-Torres obtained the following result in [39, Theorem 2].

Theorem 5.1. *Let $(n+1)/(n+2) < r < \infty$ and $s_0(r) = \min(n+2-(n+1)/r, 1)$. Then*

$$\big\||\nabla|^{-s}Q_{ij}(f,g)\big\|_{\mathcal{H}^r(\mathbf{R}^{n+1})} \lesssim \Big\||\nabla|^{1-s/2}f\Big\|_{L^p(\mathbf{R}^{n+1})}\Big\||\nabla|^{1-s/2}g\Big\|_{L^q(\mathbf{R}^{n+1})} \tag{5.3}$$

for all $p, q > 1$ such that $1/p + 1/q = 1/r$ and $0 \le s < s_0(r)$. If $r > 1$, then (5.3) also holds for $s = s_0 = 1$. The estimate (5.3) is sharp in the sense that for any $(n+1)/(n+2) < r \le 1$, there are Schwartz functions f and g with $|\nabla|^{-s_0(r)}Q_{ij}(f,g) \notin \mathcal{H}^r(\mathbf{R}^{n+1})$.

Moreover, if $1 < q, r < \infty$, $1 < q_1, q_2, r_1, r_2 < \infty$, $1/q = 1/q_1 + 1/q_2$, *and* $1/r = 1/r_1 + 1/r_2$. *Then,*

$$\big\||\nabla|^{-s} Q_{ij}(f,g)\big\|_{L_t^q L_x^r(\mathbf{R}^{n+1})} \lesssim \Big\||\nabla|^{1-s/2} f\Big\|_{L_t^{q_1} L_x^{r_1}(\mathbf{R}^{n+1})} \Big\||\nabla|^{1-s/2} g\Big\|_{L_t^{q_2} L_x^{r_2}(\mathbf{R}^{n+1})}, \tag{5.4}$$

for all $0 \le s \le 1$.

Here $\mathcal{H}^r(\mathbf{R}^{n+1}) = \dot{F}_r^{0,2}(\mathbf{R}^{n+1})$, i.e., Hardy spaces for $0 < r \le 1$ and Lebesgue spaces for $1 < r < \infty$. As noted in [39], (5.3) improves (5.1) because we can now take the full $|\nabla|$ in space and time variables, null forms also involving the time variable, and some values of $r < 1$. Meanwhile one can see that (5.4) is, in a sense, an improvement of (5.2). See [39] for more details.

When f and g are solutions of the wave equation $\Box u = (-\partial_t^2 + \Delta_x)u = 0$, then a version of (5.3) can be obtained for a higher power of $|\nabla|^{-1}$. Recall that (q, r) is a *wave admissible (Strichartz) pair* if it belongs to the set

$$A := \{(q,r) \,:\, 2 \le q,\, r \le \infty,\, \frac{1}{q} + \frac{n-1}{2r} \le \frac{n-1}{4}\,\} \setminus \{(2,\infty) \text{ when } n = 3\}.$$

The number $s(q,r) = n/2 - 1/q - n/r$ is called the smoothness parameter associated with the wave admissible pair (q, r). The following is from [39, Theorem 3].

Theorem 5.2. *Let* (q, r), *be wave admissible pair with* $2 < q, r < \infty$, s *be its smoothness parameter and* $0 \le \sigma < \frac{4}{(n-1)q}$. *Suppose that* f *and* g *are solutions of* $\Box f = \Box g = 0$. *Then*

$$\begin{aligned} &\big\||\nabla|^{-1-\sigma} Q_{ij}(f,g)\big\|_{L_t^{q/2} L_x^{r/2}(\mathbf{R}^{n+1})} \lesssim \\ &\qquad \|\nabla_{xt} f(0,\cdot)\|_{\dot{L}^2_{s-1/2-\sigma/2}(\mathbb{R}^n)} \|\nabla_{xt} g(0,\cdot)\|_{\dot{L}^2_{s-1/2-\sigma/2}(\mathbb{R}^n)}. \end{aligned} \tag{5.5}$$

Here for any $s \in \mathbb{R}$, $\dot{L}^2_s(\mathbb{R}^n)$ is the homogeneous Sobolev space of all distributions f such that $|\nabla|^s f \in L^2(\mathbb{R}^n)$.

Similar estimates were also obtained in [36] but only for fractional derivatives and null forms involving the space variables. The proof of the result relies on an estimate of Klainerman and Tataru [30] for solutions of the wave equation of the form

$$\big\||\nabla_x|^{-\sigma}(fg)\big\|_{L_t^{q/2} L_x^{r/2}(\mathbf{R}^{n+1})} \lesssim \|\nabla_{xt} f(0,\cdot)\|_{\dot{L}^2_{s-1-\sigma/2}(\mathbb{R}^n)} \|\nabla_{xt} g(0,\cdot)\|_{\dot{L}^2_{s-1-\sigma/2}(\mathbb{R}^n)}.$$

We refer again to [39] for details as well as further results involving the null form Q_0.

§ 5.2. Leibniz's rule for fractional derivatives

The first Leibniz's rule results for fractional derivatives go back to the work of Kato-Ponce [28] who showed that for $1 < p < \infty$, $s > 0$

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s(f)\|_{L^p} \|g\|_{L^\infty} + \|J^s(g)\|_{L^p} \|f\|_{L^\infty}.$$

It was also obtained by Christ-Weinstein [14] that for $1 < p, q, r < \infty$, $1/p + 1/q = 1/r$, $s > 0$,

$$\||\nabla|^s(fg)\|_{L^r} \lesssim \||\nabla|^s(f)\|_{L^p}\|g\|_{L^q} + \|f\|_{L^p}\||\nabla|^s(g)\|_{L^q}.$$

These types of estimates have been generalized by many authors. For example in the general bilinear estimate context, pseudodifferential versions were obtained by Bényi-Torres [9] and Bényi-Nahmod-Torres [8]; mixed derivatives estimates were considered by Muscalu-Pipher-Tao-Thiele [33]; and weighed versions by Bernicot-Maldonado-Moen-Naibo [11], to name a few. The largest and optimal range of exponents was more recently investigated by Muscalu and Schlag [34] and Grafakos-Oh [25].

In the mixed Lebesgue context, Kenig-Ponce-Vega [29] established the estimate

$$\||\nabla_x|^s(fg) - f|\nabla_x|^s(g) - g|\nabla_x|^s(f)\|_{L^p_x L^q_r} \lesssim \||\nabla_x|^{s_1}(f)\|_{L^{p_1}_x L^{q_1}_t} + \||\nabla_x|^{s_2}(g)\|_{L^{p_2}_x L^{q_2}_t},$$

for $0 < s, s_1, s_2 < 1$, $s = s_1 + s_2$, $1 < p, p_1, p_2, q, q_1, q_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, and $1/q = 1/q_1 + 1/q_2$.

The first result involving the full $|\nabla|$ was obtained by the authors of this note, [41, Thereom 3.2].

Theorem 5.3. *Let $s > 0$, $1 < p, q, p_i, q_i < \infty$ for $1 = 1, \dots 4$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}$. Then,*

$$\| \, |\nabla|^s(fg)\|_{L^p L^q} \lesssim \|f\|_{L^{p_1}L^{q_1}} \| \, |\nabla|^s g\|_{L^{p_2}L^{q_2}} + \| \, |\nabla|^s f\|_{L^{p_3}L^{q_3}} \|g\|_{L^{p_4}L^{q_4}}$$

The proof consists in using the method of Christ and Weinstein, writing

$$P_k f = \sum_{j \le k-3} \Delta_j f,$$

and the *paraproduct expansion*

$$f \cdot g = \sum_k \Delta_k g \cdot P_k f + \sum_k \Delta_k f \cdot P_k g + \sum_{|i-j| \le 2} \Delta_i f \cdot \Delta_j g.$$

The reader familiar with the subject will recall that after this representation, what is essentially needed in the L^p case is the boundedness of the square function and the Fefferman-Stein vector-valued maximal theorem. As described in previous sections, these tools have now been made available in the mixed Lebesgue setting and similar arguments can be applied for $L^p L^q$.

Theroem 5.3 has been recently extended to include mixed derivatives of the form $|\nabla_t|^\beta |\nabla_x|^\alpha (fg)$ by Benea and Muscalu [3]. Moreover, using different methods, Di Plinio and Ou considered a range of exponents in $\mathbb{R}^{1+1}$ that allows for $p > \max(1/2, 1/(1+\alpha))$ but q still greater or equal than 1; see [15, Corollary 1]. After the first version of this manuscript was submitted we became aware of new versions of preprints of Benea and Muscalu [4, 5] obtaining the largest set of exponents stated in the next theorem, which allows also for some values of $q < 1$; see in particular [5, Theorem 2].

Theorem 5.4. *Let $n = 1$. For any $\alpha, \beta > 0$ and $1 < p_1, p_2, q_1, q_2 \le \infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\max(\frac{1}{2}, \frac{1}{1+\beta}) < q < \infty$, and $\max(\frac{1}{2}, \frac{1}{1+\alpha}, \frac{1}{1+\beta}) < p < \infty$,*

$$\begin{aligned}\| \, |\nabla_t|^\beta |\nabla_x|^\alpha (fg)\|_{L^p L^q} \lesssim \|f\|_{L^{p_1} L^{q_1}} \| \, |\nabla_t|^\beta |\nabla_x|^\alpha g\|_{L^{p_2} L^{q_2}} \\ + \| \, |\nabla_t|^\beta f\|_{L^{p_1} L^{q_1}} \| |\nabla_x|^\alpha g\|_{L^{p_2} L^{q_2}} \\ + \| \, |\nabla_x|^\alpha f\|_{L^{p_1} L^{q_1}} \| |\nabla_t|^\beta g\|_{L^{p_2} L^{q_2}} \\ + \| \, |\nabla_t|^\beta |\nabla_x|^\alpha f\|_{L^{p_1} L^{q_1}} \|g\|_{L^{p_2} L^{q_2}}.\end{aligned}$$

Finally in another very recent preprint Hart-Torres-Wu [26, Corollary 4.5] improved on the version in Theorem 5.3 for full fractional derivatives obtaining the following.

Theorem 5.5. *Let $1 < p_1, p_2, q_1, q_2 < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, for $s \in 2\mathbb{N}$ or $s > \max(0, \frac{n}{p} - n, \frac{n}{q} - n)$ and all $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$,*

$$\||\nabla|^s (fg)\|_{L^p L^q} \lesssim \||\nabla|^s f\|_{L^{p_1} L^{q_1}} \|g\|_{L^{p_2} L^{q_2}} + \|f\|_{L^{p_1} L^{q_1}} \||\nabla|^s g\|_{L^{p_2} L^{q_2}}.$$

We observe that the last two theorems were proved with different methods and neither one seems to imply the other. We also refer the reader to [26] for other Leibniz-type rules where the product of two functions is replaced by some bilinear multiplier operators with limited regularity.

§ 6. Reproducing formulas

We conclude this survey by listing three reproducing formulas of great value in analysis which turned out to also hold for $L^p L^q$ spaces.

As it is well-known band-limited signals can be recovered from their samples via the *Shannon sampling theorem*. For example, a band-limited signal in $\mathbb{R}^m$ with Fourier transform supported on, say, $(-\pi, \pi)^m$ can be recovered from its values on the integer lattice $\mathbb{Z}^m$ via the reproducing formula

$$f(x) = \sum_{k \in \mathbb{Z}^m} \left(f(k) \prod_{i=1}^m \frac{\sin \pi (x_i - k_i)}{\pi (x_i - k_i)} \right).$$

One can then look at reading other properties of a function (in terms of functional norms) from those of the samples. That is we look for a characterization

$$\|f\|_{X(\mathbb{R}^m)} \approx \|Sf\|_{X_d(\mathbb{Z}^m)},$$

where Sf is the sequence of the samples of f on $\mathbb{Z}^m$ and X_d is a discrete version of the space of functions X.

For example, the case $X = L^p$ is the classical *Plancherel-Polya inequality* [35] (see also Boas [12] and Frazier-Jawerth [21]) where $X_d = l^p$. One has

$$\left(\int_{\mathbb{R}^m} |f(x)|^p \, dx\right)^{1/p} \approx \left(\sum_{\mathbb{Z}^m} |f(k)|^p\right)^{1/p}.$$

Another example was studied in [40] for the Besov spaces $X = \dot{B}_p^{\alpha,q}$ and appropriate discrete spaces of sequences X_d (defined through Littlewood-Paley theory).

The tools available now in the mixed norm context, in particular the boundedness of maximal operators and smooth multipliers, permit to consider a version of Plancherel-Polya inequality in L^pL^q. We have the following result from [41, Thereom 4.4].

Theorem 6.1. *Let $f \in \mathcal{S}'(\mathbb{R}^{n+1})$. If* supp $\hat{f} \subset \overline{B(0,\pi)}$*, then for* $1 < p, q < \infty$,

$$\|\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}\|_{l_j^p l_k^q(\mathbb{Z}^{n+1})} \le c_{p,q}\|f\|_{L_t^p L_x^q(\mathbb{R}^{n+1})}. \tag{6.1}$$

Moreover, if supp $\hat{f} \subset B(0,(1-\varepsilon)\pi)$*, $\varepsilon > 0$,then for* $1 < p, q < \infty$,

$$\|f\|_{L_t^p L_x^q(\mathbb{R}^{n+1})} \le c_{p,q,\epsilon}\|\{f(j,k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}\|_{l_j^p l_k^q(\mathbb{Z}^{n+1})} \tag{6.2}$$

(Actually in (6.1) we can consider $0 < p, q < \infty$ but we do not know if (6.2) holds in the range $0 < p, q \le 1$.)

Next we state *Calderón's reproducing formula.* The formula first discovered by Calderón in [13] has a very rich history involving L^p and other spaces of functions. See, for example, [10] for a detailed account of it. The result for L^pL^q below is taken from [41, Proposition 5.1].

Theorem 6.2. *Let ψ be a function in $\mathcal{S}(\mathbb{R}^{n+1})$ real valued, radial, with all moments equal to zero and such that*

$$\int_0^\infty \widehat{\psi}(s\zeta)^2 \frac{ds}{s} = 1.$$

Then, for $f \in L_t^p L_x^q(\mathbb{R}^{n+1})$, $1 < p, q < \infty$,

$$f(t,x) = \int_0^\infty f * \psi_s * \psi_s(t,x)\, \frac{ds}{s}, \tag{6.3}$$

in the sense that

$$f(t,x) = \lim_{\epsilon\to 0} f^\epsilon(t,x) \equiv \lim_{\epsilon\to 0} \int_\epsilon^{1/\epsilon} f * \psi_s * \psi_s(t,x)\, \frac{ds}{s}$$

in the $L_t^p L_x^q$ norm.

Finally, we state the wavelet characterization of the $L_t^p L_x^q(\mathbb{R}^{n+1})$ spaces, which can be seen as a discretization of (6.3). As in [41, Theorem 6.4], for simplicity, we state the last theorem for the space dimension $n = 1$ and we refer to that work for full details. It is interesting to observe that smooth band-limited wavelets (obtained for example by tensor products) for $L^2(\mathbb{R}^2)$ also provide a basis of wavelets for all $L^p L^q$ spaces. In particular, we can select three functions ψ_1, ψ_2, ψ_3 so that each ψ_i are functions in $\mathcal{S}$ and such that $\{\psi_{1Q}, \psi_{2Q}, \psi_{3Q} : \nu, j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$ where $\psi_{iQ} = 2^\nu \psi_i(2^\nu t - k, 2^\nu x - j)$ and Q is now the dyadic cube $Q = Q_{\nu,k,j} = I_{\nu k} \times I_{\nu j} = [2^{-\nu}k, 2^{-\nu}(k+1)) \times [2^{-\nu}j, 2^{-\nu}(j+1))$.

Theorem 6.3. *Let ψ_1, ψ_2, ψ_3 generate a complete orthonormal family of smooth band-limited wavelets for $L^2(\mathbb{R}^2)$. Then, for all $1 < p, q < \infty$,*

$$\left\| \left(\sum_{i=1}^{3} \sum_{\nu,j,k\in\mathbb{Z}} |\langle f|\psi_{iQ}\rangle|^2 2^{2\nu} \chi_Q(t,x) \right)^{1/2} \right\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})} \approx \|f\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R})}.$$

Once again, the harmonic analysts tools developed can be used to follow the approach for L^p spaces in the original work of Frazier-Jawerth [22] or the book of Hernandez-Weiss [27]. Such approach relies on the use of the Petree maximal function (which we do not need to define here) and the Fefferman-Stein theorem, which can be adapted to the case of the $L_t^p L_x^q$ spaces.

We conclude by mentioning that, after the initial submission of this article, we learned of recent wavelet-type characterization of certain Triebel-Lizorkin spaces defined on mixed norms by Georgiadis-Johnsen-Nielsen [23]. We refer the reader to that work for a detailed presentation.

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