数理解析研究所講究録 1345 巻 2003 年 189-202

CLASSIFYING SPACES OF DEGENERATING POLARIZED HODGE STRUCTURES

KAZUYA KATO, SAMPEI USUI

Suugaku wa mugen enten kou kokoro
Koute kogarete harukana tabiji

by Kazuya Kato and Sampei Usui,
which was translated by Luc Illusie as

L'impossible voyage aux points à l'infini
N'a pas fait battre en vain le coeur du géomètre.

CONTENTS

§0. Introduction
§0.1. A dream of Griffiths
§0.2. Polarized logarithmic Hodge structures
§0.3. Moduli of PLH
§0.4. Complements
§0.5. Notation
§1. Spaces of nilpotent orbits and spaces of nilpotent i-orbits
§1.1. Hodge structures and variations of Hodge structures
§1.2. Classifying spaces of Hodge structures
§1.3. Extended classifying spaces
§2. Logarithmic Hodge structures
§2.1. Logarithmic structures
§2.2. Ringed spaces $(X^{\log}, \mathcal{O}_X^{\log})$
§2.3. Polarized logarithmic Hodge structures
§2.4. Polarized logarithmic Hodge structures and nilpotent orbits
§2.5. Period maps
§2.6. Logarithmic Hodge structures
§3. Strong topology and logarithmic manifolds
§3.1. Strong topology
§3.2. Generalizations of analytic spaces
§3.3. Sets E_σ and $E_\sigma^\sharp$
§3.4. Spaces E_σ, $\Gamma\backslash D_\Sigma$, $E_\sigma^\sharp$ and $D_\Sigma^\sharp$
§3.5. Infinitesimal calculus and logarithmic manifolds
§4 The main results

2000 *Mathematics Subject Classification.* Primary 14C30; Secondary 14D07, 32G20. Partly supported by the Grants-in-Aid for Scientific Research (B) (2) No.11440003 and (A) (1) No.11304001, the Ministry of Education, Science, Sports and Culture, Japan

§4.1. Theorem A: The spaces E_σ, $\Gamma\backslash D_\Sigma$ and $\Gamma\backslash D_\Sigma^\sharp$
§4.2. Theorem B: The functor $\underline{\mathrm{PLH}}_\Phi$
§4.3. Infinitesimal period maps
§5. The fundamental diagram
§5.1. Borel-Serre spaces (review)
§5.2. Spaces of SL(2)-orbits (review)
§5.3. Spaces of valuative nilpotent orbits
§5.4. Valuative nilpotent i-orbits and SL(2)-orbits
§6. The map $D_{\mathrm{val}}^\sharp \to D_{\mathrm{SL}(2)}$
§6.1. Review of [CKS] and some related results
§6.2. Proof of Theorem 5.4.2
§6.3. Proof of Theorem 5.4.3 (i)
§6.4. Proofs of Theorem 5.4.3 (ii) and Theorem 5.4.4
§7. Proof of Theorem A
§7.1. Proof of Theorem A (i)
§7.2. The action of $\sigma_{\mathbf{C}}$ on E_σ
§7.3. Proof of Theorem A for $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$
§7.4. Proof of Theorem A for $\Gamma\backslash D_\Sigma$
§8. Proof of Theorem B
§8.1. Logarithmic local systems
§8.2. Proof of Theorem B
§8.3. Relationship of categories of generalized analytic spaces
§9. $\flat$-spaces
§9.1. Definitions and main properties
§9.2. Proofs of Theorem 9.1.4 for $\Gamma\backslash \mathcal{X}_{\mathrm{BS}}^\flat$, $\Gamma\backslash D_{\mathrm{BS}}^\flat$, $\Gamma\backslash D_{\mathrm{BS,val}}^\flat$
§9.3. Proof of Theorem 9.1.4 for $\Gamma\backslash D_{\mathrm{SL}(2),\le 1}^\flat$
§9.4. Extended period maps
§10. Local structures of $D_{\mathrm{SL}(2)}$ and $\Gamma\backslash D_{\mathrm{SL}(2),\le 1}^\flat$
§10.1. Local structures of $D_{\mathrm{SL}(2)}$
§10.2. A special open neighborhood $U(p)$
§10.3. Proof of Theorem 10.1.3
§10.4. Local structures of $D_{\mathrm{SL}(2),\le 1}$ and $\Gamma\backslash D_{\mathrm{SL}(2),\le 1}^\flat$
§11. Moduli of PLH with coefficients
§11.1. The space $\Gamma\backslash D_\Sigma^A$
§11.2. PLH with coefficients
§11.3. The moduli
§12. Examples and problems
§12.1. Siegel upper half spaces
§12.2. Case $G_{\mathbf{R}} \simeq O(1, n-1, \mathbf{R})$
§12.3. Example of weight 3 (A)
§12.4. Example of weight 3 (B)
§12.5. Relationship with [U2]
§12.6. Problem of complete fans
§12.7. Other problems

§0. INTRODUCTION

This rather long Introduction is a survey of this paper. We describe the main subject of this paper in §0.1, a key idea in §0.2, the main results in §0.3, and the plan of the paper in §0.4. We fix general notation in §0.5.

§0.1. A DREAM OF GRIFFITHS

0.1.1. Let D be the classifying space of polarized Hodge structures of fixed weight w and fixed Hodge numbers $(h^{p,q})$ defined by Griffiths [G1]. For example, in the case

$$(1) \qquad w=1,\ \ h^{1,0}=h^{0,1}=g,\ \ h^{p,q}=0 \text{ if } (p,q)\neq(1,0),(0,1),$$

D is Siegel's upper half space of degree g. In [G3, §9], Griffiths considered a general D "aux points à l'infini", and described a dream to enlarge D to classifying spaces of degenerating polarized Hodge structures. The subject of this paper is to realize his dream by defining eight enlargements of D with maps between them which form the following diagram (2), and to show that the quotient space $\Gamma\backslash D_\Sigma$ of the enlargement D_Σ, defined by a fan (= cone decomposition) Σ and a discrete subgroup Γ of $\mathrm{Aut}(D)$ satisfying certain conditions, is the fine moduli space of polarized logarithmic Hodge structures (= logarithmic degenerations of polarized Hodge structures).

$$(2) \qquad \begin{tikzcd} & & D_{\mathrm{SL}(2),\mathrm{val}} \arrow[r, hook] \arrow[d] & D_{\mathrm{BS},\mathrm{val}} \arrow[d] \\ D_{\Sigma,\mathrm{val}} \arrow[d] & D^\sharp_{\Sigma,\mathrm{val}} \arrow[l] \arrow[r] \arrow[d] & D_{\mathrm{SL}(2)} & D_{\mathrm{BS}} \\ D_\Sigma & D^\sharp_\Sigma \arrow[l] & & \end{tikzcd}$$

In the above classical case (1), for a subgroup Γ of $\mathrm{Sp}(g,\mathbf{Z})$ of finite index, $\Gamma\backslash D_{\mathrm{BS}}$ is the Borel-Serre compactification ([BS]) of $\Gamma\backslash D$, and $\Gamma\backslash D_\Sigma$ is a Mumford compactification (i.e., a toroidal compactification) ([AMRT]) of $\Gamma\backslash D$ if Σ is chosen suitably.

In general,

$$\begin{aligned} D_\Sigma &= (\text{the space of nilpotent orbits}),\\ D^\sharp_\Sigma &= (\text{the space of nilpotent } i\text{-orbits}),\\ D_{\mathrm{SL}(2)} &= (\text{the space of SL(2)-orbits}),\\ D_{\mathrm{BS}} &= (\text{the space of Borel-Serre orbits}),\\ D_{\Sigma,\mathrm{val}} &= (\text{the space of valuative nilpotent orbits}),\\ D^\sharp_{\Sigma,\mathrm{val}} &= (\text{the space of valuative nilpotent } i\text{-orbits}),\\ D_{\mathrm{SL}(2),\mathrm{val}} &= (\text{the space of valuative SL(2)-orbits}),\\ D_{\mathrm{BS},\mathrm{val}} &= (\text{the space of valuative Borel-Serre orbits}). \end{aligned}$$

Here an orbit is either a point of D or a family which runs on D to a direction of degeneration outside D. Nilpotent orbits and SL(2)-orbits often appear in the theory of degeneration of polarized Hodge structures ([Sc], [CKS]). Borel-Serre orbits are defined by the method of Borel-Serre [BS]. Nilpotent i-orbits are as follows. Whereas a nilpotent orbit is an orbit under the group $\exp(\sigma_{\mathbf{C}})$ where σ is a cone consisting of mutually commutative nilpotent operators and $\sigma_{\mathbf{C}}$ denotes the $\mathbf{C}$-vector space spanned by σ, a nilpotent i-orbit is an orbit under the group $\exp(i\sigma_{\mathbf{R}})$.

A valuative orbit which appears in the last four spaces runs to an "infinitely narrow direction" of degeneration, and the upper space (space of valuative orbits) of a vertical arrow in the above diagram (2) is obtained from the lower space as the limit by dividing the direction of degeneration into narrower and narrower directions. We can say also that vertical arrows in (2) are projective limits of kinds of blowing-ups.

0.1.2. The space $\Gamma\backslash D_\Sigma$ is of primary interest in this paper. A main result of this paper is, roughly speaking, the following (for the precise statement, see Theorem 0.3.8, Theorem 0.3.11 below).

Theorem. *$\Gamma\backslash D_\Sigma$ is the fine moduli space of "polarized logarithmic Hodge structures" with a "Γ-level structure" whose "local monodromies are in the directions in Σ".*

Roughly speaking,

$$\Gamma\backslash D = (\text{polarized Hodge structures with a "}\Gamma\text{-level structure"}) \subset$$

$$\Gamma\backslash D_\Sigma = \begin{pmatrix}\text{"polarized logarithmic Hodge structures" with a "}\Gamma\text{-level struc-}\\ \text{ture" whose "local monodromies are in the directions in }\Sigma\text{"}\end{pmatrix}.$$

The space $\Gamma\backslash D_\Sigma$ is not necessarily a complex analytic space, but has a kind of complex strucuture, and infinitesimal calculus can be performed on $\Gamma\backslash D_\Sigma$ nicely. In the terminology of this paper, $\Gamma\backslash D_\Sigma$ is a "logarithmic manifold", as explained in 0.3.9 below. Classically variations of polarized Hodge structures give period maps into $\Gamma\backslash D$. Polarized logarithmic Hodge structures give extended period maps into $\Gamma\backslash D_\Sigma$ defined also on the boundary, and infinitesimal calculus on $\Gamma\backslash D_\Sigma$ enables us to consider the differentials of the extended period maps.

To prove the above theorem and to prove that $\Gamma\backslash D_\Sigma$ has good properties such as Hausdorff property, nice infinitesimal calculus etc., we need to consider other spaces in the diagram (2); we discuss from the right to the left in the diagram (2) to deduce nice properties of $\Gamma\backslash D_\Sigma$ starting from nice properties of the Borel-Serre compactifications in [BS].

0.1.3. Enlargements of D such as the Satake-Baily-Borel and as toroidal have been constructed only when, like in the case 0.1.1 (1), D is a Hermitian symmetric domain and the tangent bundle of D coincides with the horizontal tangent bundle. That is, when one of the following (1), (2) is satisfied.

(1) $w = 2t+1$, $h^{p,w-p} = 0$ if $p \neq t, t+1$.

(2) $w = 2t$, $h^{t+1,t-1} \leq 1$, and $h^{p,w-p} = 0$ if $p \neq t-1, t, t+1$.

The only successful attempts to go beyond these cases are the work [CK] of Cattani-Kaplan for the Satake-Baily-Borel type and the work [U2] for troidal. ([CK] has no differential of period maps.) By the construction of the present article, we can talk about the extended period maps associated to degenerations of surfaces of general type, of Calabi-Yau manifolds, or of other higher dimensional algebraic manifolds, and about their differentials.

§0.2. Polarized logarithmic Hodge structures

Here we describe the theory of "polarized logarithmic Hodge structure" (PLH, for short) and the relationship with the space of nilpotent orbits. To avoid to enter too technical explanations, we often omit the details refering the reader to the contents of this paper.

0.2.1. *Fs logarithmic analytic spaces.* The subject of the logarithmic geometry in the sense of Fontaine-Illusie ([Kk1]) is a "logarithmic local ringed space", that is, a local ringed space $(X, \mathcal{O}_X)$ endowed with a *logarithmic structure*, which means a sheaf of commutative monoids M_X endowed with a homomorphism $\alpha : M_X \to \mathcal{O}_X$, where we regard $\mathcal{O}_X$ as a sheaf of monoids with respect to the multiplication, satisfying $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$.

We will consider "fs logarithmic structures" defined as follows.

A commutative monoid $\mathcal{S}$ is called an *fs monoid* if it is finitely generated, if $ab = ac$ implies $b = c$ in $\mathcal{S}$ (hence $\mathcal{S}$ is embedded in the group $\mathcal{S}^{\mathrm{gp}} = \{\frac{a}{b} \mid a, b \in \mathcal{S}\}$), and if $a \in \mathcal{S}^{\mathrm{gp}}$ with $a^n \in \mathcal{S}$ for some $n \geq 1$ implies $a \in \mathcal{S}$.

For a local ringed space X and an fs monoid $\mathcal{S}$, and for a homomorphism $h : \mathcal{S} \to \mathcal{O}_X$ (for the multiplication in $\mathcal{O}_X$), we have a logarithmic structure on X associated to h, which is defined to be the pushout of $\mathcal{S} \leftarrow h^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ in the category of sheaves of commutative monoids. A logarithmic structure is called an *fs logarithmic structure* if it is locally obtained in this way. An analytic space endowed with an fs logarithmic structure is called an *fs logarithmic analytic space.*

We denote by

$$(1) \qquad \mathcal{A}, \quad \mathcal{A}(\log),$$

the category of analytic spaces and the category of fs logarithmic analytic spaces, respectively.

0.2.2. We discuss PLH on an fs logarithmic analytic space X. We have a certain ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ over $(X, \mathcal{O}_X)$ ([KkNc]). A PLH on X is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a local system $H_{\mathbf{Z}}$ on $X^{\log}$, an intersection pairing $\langle\ ,\ \rangle$ on $H_{\mathbf{Q}} = \mathbf{Q} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, and a Hodge filtration F on $\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, satisfying certain conditions.

In this Introduction, not to enter complicated arguments but to have a clear picture, we consider only PLH on $X := \Delta^n$ with $\Delta := \{z \in \mathbf{C} \mid |z| < 1\}$ and $n \geq 0$ which is endowed with the logarithmic structure M_X associated to $\mathbf{N}^r \to \mathcal{O}_X$, $a \mapsto \prod_{1 \leq j \leq r} z_j^{a(j)}$, for some r such that $0 \leq r \leq n$. Here z_j, $(1 \leq j \leq n)$ denote the coordinate functions. Let $U := (\Delta^*)^r \times \Delta^{n-r}$ where $\Delta^* := \Delta - \{0\}$. Then,

$$M_X = \{f \in \mathcal{O}_X \mid f \text{ is invertible on } U\}.$$

In this case, $(X^{\log}, \mathcal{O}_X^{\log})$ is described as follows. First,

$$X^{\log} := |\Delta|^r \times (\mathbf{S}^1)^r \times \Delta^{n-r}$$

where $|\Delta| := \{r \in \mathbf{R} \mid 0 \leq r < 1\}$, $\mathbf{S}^1 := \{u \in \mathbf{C}^\times \mid |u| = 1\}$, and the canonical map $\tau : X^{\log} \to X$ is given by

$$\big((r_j)_{1 \leq j \leq r}, (u_j)_{1 \leq j \leq r}, (z_j)_{r+1 \leq j \leq n}\big) \mapsto \big((r_j u_j)_{1 \leq j \leq r}, (z_j)_{r+1 \leq j \leq n}\big).$$

Next

$$\mathcal{O}_X^{\log} := \tau^{-1}(\mathcal{O}_X)[\log(z_j)\ (1 \leq j \leq r)] \subset (j^{\log})_*(\mathcal{O}_U),$$

where $j^{\log} : U \hookrightarrow X^{\log}$ is the evident extension of the inclusion map $j : U \hookrightarrow X$.

A PLH on X is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ where $H_{\mathbf{Z}}$ is a local system on $X^{\log}$ of free $\mathbf{Z}$-modules of finite rank, $\langle\ ,\ \rangle$ is a non-degenerate $\mathbf{Q}$-bilinear form $H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$, and F is a decreasing filtration on $\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, which satisfy certain condition (see §2.3 for the precise definition). If $r = 0$, then $X = U$, $X^{\log} = X$ and $\mathcal{O}_X^{\log} = \mathcal{O}_X$, and this condition is that the fiber $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x, F(x))$ at each $x \in X$ is a polarized Hodge structure. In general, at each $x \in U$, we require that the fiber is a polarized Hodge structure, but at each $x \in X - U$, we require a delicate condition. Note that for $x \in X - U$ and $y \in X^{\log}$, lying over x, $\mathcal{O}_{X,y}^{\log}$ is not a local ring. For example, if $x = 0 \in X$, then $\mathcal{O}_{X,y}^{\log}$ is the polynomial ring $\tau^{-1}(\mathcal{O}_{X,0})[\log(z_1), \ldots, \log(z_r)]$ in r-variables over $\tau^{-1}(\mathcal{O}_{X,0})$. Hence there are many homomorphisms $\mathcal{O}_{X,y}^{\log} \to \mathbf{C}$ extending $\mathcal{O}_{X,x} \to \mathbf{C}, f \mapsto f(x)$. The specializations of $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ by these homomorphisms form an "orbit", and the condition at x is that this orbit is a nilpotent orbit (cf. §2.4). Thus roughly speaking,

(1) (PLH on X) = (analytic family of nilpotent orbits parametrized by X).

0.2.3. *How a PLH arises.* Let X and U be as in 0.2.2, and let $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a variation of polarized Hodge structure on U having unipotent local monodromy around $X - U$. Then, the nilpotent orbit theorem of Schmid ([Sc]) can be interpreted as a statement that $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ extends uniquely to a PLH on X.

PLH arises in a geometric way as follows. Let Y be a complex analytic manifold and let $f : Y \to X$ be a projective morphism such that the restricion $f' : V \to U$, where $V := f^{-1}(U)$, is smooth and that $Y - V$ is a divisor with normal crossings on Y. Assume that f is a "logarithmically smooth morphism" (2.3.10), and assume, moreover, that $\operatorname{Coker}\left(f^{-1}(M_{X,f(y)}^{\mathrm{gp}}/\mathcal{O}_{X,f(y)}^{\times}) \to M_{Y,y}^{\mathrm{gp}}/\mathcal{O}_{Y,y}^{\times}\right)$ is torsion free for any $y \in Y$. Then, by a generalization of Steenbrink [St], we have the following PLH $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ on X satisfying the Griffiths transversality (2.3.5) for each $m \geq 0$:

$$H_{\mathbf{Z}} = R^m(f^{\log})_*\mathbf{Z}/(\text{torsion}),$$
$$F^p = \mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} R^m f_*(\omega_{Y/X}^{\geq p}).$$

Here $f^{\log} : Y^{\log} \to X^{\log}$ is the map induced from f, $\omega_{Y/X}^{\geq p}$ is the degree $\geq p$ part of the relative de Rham complex $\omega_{Y/X}^{\bullet}$ with logarithmic poles, and F^p is regarded as an $\mathcal{O}_X^{\log}$-submodule of $\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$ via a certain canonical isomorphism

$$\mathcal{O}_X^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} R^m f_*(\omega_{Y/X}^{\bullet}).$$

By using the Lefschetz decomposition, a polarization of Y over X defines a polarization $\langle\ ,\ \rangle : H_{\mathbf{Q}} \times H_{\mathbf{Q}} \to \mathbf{Q}$ in the usual way. This PLH is the unique extension of the classical variation of polarized Hodge structure $(H'_{\mathbf{Z}}, \langle\ ,\ \rangle', F')$

$$H'_{\mathbf{Z}} = R^m(f')_*\mathbf{Z}/(\text{torsion}),$$
$$(F')^p = R^m(f')_*(\Omega_{V/U}^{\geq p}),$$

arising from $f' : V \to U$. (More generally, see 2.3.10.)

0.2.4. *Extended period maps.* Let X and U be as above. For a given PLH on X, by associating to each $x \in X$ the nilpotent orbit which appears at x, we have a map $X \to \Gamma\backslash D_\Sigma$ where Γ is the group of global monodromy of $H_{\mathbf{Z}}$, and Σ is the fan consisting of local monodromy cones at points of X. This extends the classical period map $U \to \Gamma\backslash D$ as in the diagram

$$\begin{array}{ccc} U & \to & \Gamma\backslash D \\ \cap & & \cap \\ X & \to & \Gamma\backslash D_\Sigma. \end{array}$$

In the geometric situation of $f : Y \to X$ described in 0.2.3, the differential of the corresponding extended period map is related to the (degenerating) Kodaira-Spencer map of $f : Y \to X$ just as in the classical case $r = 0$ (see 4.3.8).

§0.3. Moduli of PLH

We describe our main results in this paper. In 0.3.1–0.3.4, we fix our situation and notation precisely.

0.3.1. *Griffiths domains D.* We use the Notation in §0.5 below:

$$\Phi_0 = \big(w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0\big)$$

is a 4-tuple consisitng of an integer w, called a weight, a set of non-negative integers $(h^{p,q})_{p,q\in\mathbf{Z}}$ satisfying certain conditon, called Hodge numbers, a free $\mathbf{Z}$-module H_0 of rank $\sum_{p,q} h^{p,q}$, called a lattice, and of a non-degenerate $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ which is symmetric if w is even and anti-symmetric if w is odd, called a polarization. Here $H_{0,\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} H_0$. Let $G_R := \mathrm{Aut}(H_{0,R}, \langle\ ,\ \rangle_0)$ for $R = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

As in §1.2, the classyfying space of polarized Hodge structures (= Griffiths domain) D of type Φ_0 and its "compact dual" $\check{D}$ is defined as follows.

$$\begin{aligned} \mathcal{F} &:= \mathrm{Flag}(H_{0,\mathbf{C}}, (h^{p,q})) := \{F \text{ decreasing filtration} \mid \dim_{\mathbf{C}} F^p = \textstyle\sum_{p'\geq p} h^{p',w-p'}\} \\ &\supset \check{D} := \{F \in \mathcal{F} \mid \langle F^p, F^q\rangle_0 = 0\ (p+q > w)\} \\ &\supset D := \{F \in \check{D} \mid i^{p-q}\langle x, \overline{x}\rangle_0 > 0 \text{ if } x \in H^{p,q} - \{0\}, p+q = w\} \\ &\quad = \{F \in \mathcal{F} \mid (H_0, \langle\ ,\ \rangle_0, F) \text{ is a polarized Hodge structure}\}. \end{aligned}$$

Here $H^{p,q} := F^p \cap \overline{F}^q$. Then D is open in $\check{D}$, and D (resp. $\check{D}$) is homogeneous under $G_{\mathbf{R}}$ (resp. $G_{\mathbf{C}}$).

0.3.2. *Fans Σ.* As in Notation §0.5, let $\mathfrak{g}_R := \mathrm{Lie}\, G_R$ for $R = \mathbf{Q}, \mathbf{R}, \mathbf{C}$. We use the terminology in §1.3. A *nilpotent cone* is a cone σ in $\mathfrak{g}_{\mathbf{R}}$ which is non-degenerate (i.e., $\sigma \cap (-\sigma) = \{0\}$) and is generated over $\mathbf{R}_{\geq 0}$ by a finite number of mutually commutative nilpotent elements. A *rational nilpotent cone* is a nilpotent cone which is generated over $\mathbf{R}_{\geq 0}$ by a finite number of elements of $\mathfrak{g}_{\mathbf{Q}}$. A *fan Σ in $\mathfrak{g}_{\mathbf{Q}}$* is a set of rational nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ which satisfies

$$\begin{cases} \sigma \in \Sigma,\ \tau \text{ is a face of } \sigma \Longrightarrow \tau \in \Sigma, \\ \sigma, \sigma' \in \Sigma \Longrightarrow \sigma \cap \sigma' \text{ is a face of } \sigma \text{ and of } \sigma'. \end{cases}$$

0.3.3. *Nilpotent orbits, nilpotent i-orbits.* We use also the following terminology and notation in §1.3. A *nilpotent orbit* (resp. *i-orbit*) is a pair (σ, Z) consists of a nilpotent cone $\sigma = \sum_{1\le j\le r} \mathbf{R}_{\ge 0}N_j$ and a subset $Z \subset \check{D}$ which satisfy, for some $F \in Z$,

$$\begin{cases} Z = \exp(\sigma_{\mathbf{C}})F \ \ (\text{resp. } \exp(i\sigma_{\mathbf{R}})F), \\ NF^p \subset F^{p-1} \ \ (\forall p, \forall N \in \sigma) \ \ (\text{Griffiths transversality}), \\ \exp(\sum_{1\le j\le r} iy_j N_j)F \in D \ \ (\forall y_j \gg 0) \ \ (\text{positivity}). \end{cases}$$

As sets, we define

$$D_\Sigma \ (\text{resp. } D_\Sigma^\sharp) := \{(\sigma, Z) \text{ nilpotent orbit (resp. } i\text{-orbit}) \mid \sigma \in \Sigma, \ Z \subset \check{D}\}.$$

There is the canonical surjection $D_\sigma^\sharp \to D_\sigma$, $(\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z)$.

0.3.4. As in 1.3.11 (ii), we say a subgroup Γ of $G_{\mathbf{Z}}$ is *strongly compatible with a fan* Σ if

$$\begin{cases} \gamma \in \Gamma, \sigma \in \Sigma \Longrightarrow \mathrm{Ad}(\gamma)(\sigma) \in \Sigma, \\ \Gamma(\sigma) := \Gamma \cap \exp(\sigma) \Longrightarrow \log \Gamma(\sigma) \text{ generates } \sigma. \end{cases}$$

We say a subgroup Γ of $G_{\mathbf{Z}}$ is *neat* if the subgroup of $\mathbf{C}^\times$, generated by all the eigenvalues of all $\gamma \in \Gamma$, is torsion free. It is known that there exists a neat subgroup of $G_{\mathbf{Z}}$ of finite index.

0.3.5. Let Σ be a fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ. As in 2.5.2, we denote by

$$(1) \qquad \Phi := \big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0, \Gamma, \Sigma\big)$$

the 6-tuple consisting of the 4-tuple $\big(w, (h^{p,q})_{p+q=w}, H_0, \langle\ ,\ \rangle_0\big)$ as in 0.3.1 and of the above Σ and Γ.

For an fs logarithmic analytic space X, we have a notion "PLH on X of type Φ". It is a PLH on X of weight w and of Hodge numbers $(h^{p,q})$ endowed with a "Γ-level structure" on local isomorphisms between $(H_{\mathbf{Z}}, \langle\ ,\ \rangle)$ and $(H_0, \langle\ ,\ \rangle_0)$ whose "local monodromies are in the directions in Σ" (see 2.5.2, for the precise definition in a more general setting).

Now our first main result is the following.

Theorem 0.3.6. *Assume Γ is neat. Define the functor $\underline{\mathrm{PLH}}_{\Phi,\mathcal{A}(\log)}$ from the category $\mathcal{A}(\log)$ in* 0.2.1 (1) *to the category of sets by*

$$\underline{\mathrm{PLH}}_{\Phi,\mathcal{A}(\log)}(X) := (\textit{isomorphism classes of PLH on } X \textit{ of type } \Phi).$$

Then there exists a structure of a logarithmic local ringed space over $\mathbf{C}$ *on the set* $\Gamma\backslash D_\Sigma$ *having the following properties* (i), (ii).

(i) *There is an isomorphism of functors*

$$\psi : \underline{\mathrm{PLH}}_{\Phi,\mathcal{A}(\log)} \xrightarrow{\sim} \mathrm{Mor}(\ \ , \Gamma\backslash D_\Sigma)|_{\mathcal{A}(\log)}.$$

(ii) *For a logarithmic local ringed space Z over $\mathbf{C}$ and for a morphism of functors $h : \underline{\mathrm{PLH}}_{\Phi,\mathcal{A}(\log)} \to \mathrm{Mor}(\ \ ,Z)|_{\mathcal{A}(\log)}$, there exists a unique morphism $f : \Gamma\backslash D_\Sigma \to Z$ such that $h = f \circ \psi$.*

The part (ii) of this theorem says that $\Gamma\backslash D_\Sigma$ is the universal object for period maps from objects of $\mathcal{A}(\log)$ into logarithmic local ringed spaces. However the logarithmic local ringed space $\Gamma\backslash D_\Sigma$ over $\mathbf{C}$ itself does not belong to $\mathcal{A}(\log)$ in general. It is a very special object of a category $\mathcal{B}(\log) \supset \mathcal{A}(\log)$. The category $\mathcal{B}(\log)$ is explained in 0.3.7 below, and how "special" is explained in 0.3.9 below.

0.3.7. *The strong topology, the categories $\mathcal{B}$, $\mathcal{B}(\log)$.* The underlying local ringed space over $\mathbf{C}$ of $\Gamma\backslash D_\Sigma$ is not necessarily an analytic space in general. Sometimes, it can be something like

$$S := \{(x,y) \in \mathbf{C}^2 \mid x \neq 0\} \cup \{(0,0)\} \tag{1}$$

endowed with a topology which is stronger than the topology as a subspace of $\mathbf{C}^2$, called the "strong topology".

Let Z be an analytic space and S be a subset of Z. A subset U of S is open in the *strong topology of S in Z* if and only if, for any analytic space Y and any morphism $\lambda : Y \to Z$ such that $\lambda(Y) \subset S$, $\lambda^{-1}(U)$ is open on Y.

If S is a locally closed analytic subspace of Z, the strong topology coincides with the topology as a subspace of Z. (Cf. 3.1.3 to see how the strong topology of the set S in (1) is stronger than the topology as a subspace of $\mathbf{C}^2$.)

Define the categories

$$\mathcal{B} \supset \mathcal{A}, \quad \mathcal{B}(\log) \supset \mathcal{A}(\log) \tag{2}$$

as follows (cf. 3.2.4).

Let $\mathcal{B}$ be the full subcategory of the category of local ringed spaces over $\mathbf{C}$ consisting of objects X having the following property: X has an open covering $(U_\lambda)_\lambda$ such that, for each λ, there exists an isomorphism of local ringed spaces over $\mathbf{C}$ between U_λ and a subset S_λ of an analytic space Z_λ, where S_λ is endowed with the strong topology in Z_λ and with the inverse image of $\mathcal{O}_{Z_\lambda}$.

Let $\mathcal{B}(\log)$ be the full subcategory of logarithmic local ringed spaces over $\mathbf{C}$ consisting of objects of $\mathcal{B}$ endowed with an fs logarithmic structure.

Our second main result is the following theorem which generalizes the part (i) of Theorem 0.3.6. We can define PLH on objects of $\mathcal{B}(\log)$.

Theorem 0.3.8 (cf. Theorem B in §4.2). *Assume Γ is neat. Define the functor $\underline{\mathrm{PLH}}_{\Phi,\mathcal{B}(\log)}$ from the category $\mathcal{B}(\log)$ in 0.3.7 (2) to the category of sets in the same way as $\underline{\mathrm{PLH}}_{\Phi,\mathcal{A}(\log)}$. Then, $\Gamma\backslash D_\Sigma$ belongs to $\mathcal{B}(\log)$ and represents the functor $\underline{\mathrm{PLH}}_{\Phi,\mathcal{B}(\log)}$. Thus we have*

$$\underline{\mathrm{PLH}}_{\Phi,\mathcal{B}(\log)}(X) = (\textit{isomorphism classes of PLH on } X \textit{ of type } \Phi) = \mathrm{Mor}(X, \Gamma\backslash D_\Sigma)$$

for $X \in \mathcal{B}(\log)$.

For an PLH of type Φ on X ($X \in \mathcal{B}(\log)$), the corresponding morphism $X \to \Gamma\backslash D_\Sigma$ is the associated *period map*.

0.3.9. *Logarithmic manifolds.* Our space $\Gamma\backslash D_\Sigma$ is a very special object in $\mathcal{B}(\log)$, called a "logarithmic manifold" (cf. §3.5).

We first describe the idea of logarithmic manifold by using the example $S \subset \mathbf{C}^2$ in 0.3.7 (1). Let $Z = \mathbf{C}^2$ with coordinate functions x, y, and endow Z with the logarithmic structure associated to $\mathbf{N} \to \mathcal{O}_Z$, $n \mapsto x^n$. Then, the sheaf ω^1_Z of logarithmic differential forms on Z (= the sheaf of differential forms with logarithmic poles along $x = 0$) is the free $\mathcal{O}_Z$-module with basis $(d\log(x), dy)$. For each $z \in Z$, let ω^1_z be the module of logarithmic differential forms on the point z which is regarded as an fs logarithmic analytic space endowed with the ring $\mathbf{C}$ and with the inverse image of M_Z (cf. 2.1.7). Then, if z does not belong to the part $x = 0$ of Z, z is just a usual point $\mathrm{Spec}(\mathbf{C})$ with the trivial logarithmic structure, and $\omega^1_z = 0$. If z is in the part $x = 0$, z is a point $\mathrm{Spec}(\mathbf{C})$ with the induced logarithmic structure $M_z = \bigsqcup_{n \geq 0} \mathbf{C}^\times x^n \simeq \mathbf{C}^\times \times \mathbf{N}$, and hence ω^1_z is a one dimensional $\mathbf{C}$-vector space generated by $d\log(x)$. Thus ω^1_z is not equal to the fiber of ω^1_Z at z which is a 2-dimensional $\mathbf{C}$-vector space with basis $(d\log(x), dy)$. Now the the above set S has a presentation

$$S = \{z \in Z \mid \text{the image of } y\,d\log(x) \text{ in } \omega^1_z \text{ is zero}\}. \tag{1}$$

Recall that zeros of a holomorphic function on Z form a closed analytic subset of Z. Here we discovered that S is the set of "zeros" of the differential form $y\,d\log(x)$ on Z, but the meaning of "zero" is not that the image of $y\,d\log(x)$ in the fiber of ω^1_Z is zero (the latter "zeros" form the closed analytic subset $y = 0$ of Z). The "zeros in the new sense" of a differential form with logarithmic poles is the idea of a "logarithmic manifold".

The precise definition is as follows (cf. 3.5.7). In [KkNc], we have a notion of "logarithmically smooth" fs logarithmic analytic spaces. An fs logarithmic analytic space is *logarithmically smooth* if and only if it has an open covering whose each member is isomorphic to an open set Z of a toric variety $\mathrm{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ ($\mathcal{S}$ an fs monoid) whose logarithmic structure is associated to the inclusion homomorphism $\mathcal{S} \hookrightarrow \mathbf{C}[\mathcal{S}]$. For example, $Z = \mathbf{C}^2$ with the above logarithmic structure is logarithmically smooth.

By a *logarithmic manifold*, we mean a logarithmic local ringed space over $\mathbf{C}$ which has an open covering $(U_\lambda)_\lambda$ with the following property: For each λ, there exist a logarithmically smooth fs logarithmic analytic space Z_λ, a finite subset I_λ of $\Gamma(Z_\lambda, \omega^1_{Z_\lambda})$, and an isomorphism of logarithmic local ringed spaces over $\mathbf{C}$ between U_λ and an open subset of

$$S_\lambda = \{z \in Z_\lambda \mid \text{the image of } I_\lambda \text{ in } \omega^1_z \text{ is zero}\}, \tag{2}$$

where S_λ is endowed with the strong topology in Z_λ and with the inverse images of $\mathcal{O}_{Z_\lambda}$ and M_{Z_λ}.

0.3.10. The reason why our space $\Gamma\backslash D_\Sigma$ is not necessarily an analytic space but a logarithmic manifold is as follows.

For each $\sigma \in \Sigma$, there are a logarithmic version $\check{E}_\sigma$ of $\check{D}$, which is a logarithmically smooth fs logarithmic analytic space, and a subset E_σ of $\check{E}_\sigma$, which is a logarithmic version of the subset D of $\check{D}$ and is closely related to $\Gamma\backslash D_\Sigma$. See §3.3 for the definitions of $\check{E}_\sigma$ and E_σ.

Points of $\check{E}_\sigma$ yield "orbits". As in 0.3.3 or 1.3.6, there are two conditions, the Griffiths transversality and the positivity, for an "orbit" to be a nilpotent orbit. Points of $\check{E}_\sigma$

which yield orbits satisfying the Griffiths transversality form a subset $\tilde{E}_\sigma$ of $\check{E}_\sigma$, points of $\tilde{E}_\sigma$ which yield orbits satisfying the positivity form the subset E_σ of $\tilde{E}_\sigma$, and thus E_σ consists of all points of $\check{E}_\sigma$ which yield nilpotent orbits. In $\check{E}_\sigma$, $\tilde{E}_\sigma$ is the set of common zeros in the above "new sense (0.3.9)" of logarithmic differential forms (3.5.10), E_σ is an open set of $\tilde{E}_\sigma$ in the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$, and thus E_σ is a logarithmic manifold (Theorem A (i)). Here it is essential that we work with the strong topology to have this openness (see 12.3.10).

By taking the corresponding orbits, we have a map $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ where $D_\sigma := D_{\{\text{face of } \sigma\}}$ ($\Gamma(\sigma)^{\mathrm{gp}}$ becomes strongly compatible with the fan {face of σ}). Furthermore, E_σ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ in the category $\mathcal{B}(\log)$, $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ is a local isomorphism of logarithmic local ringed spaces over $\mathbf{C}$, and the images of the last maps for $\sigma \in \Sigma$ form an open covering of $\Gamma\backslash D_\Sigma$. From the fact that E_σ are logarithmic manifolds, we can deduce that $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ and $\Gamma\backslash D_\Sigma$ are also logarithmic manifolds as well. We have

Theorem 0.3.11 (cf. Theorem A in §4.1). *Assume Γ is neat.*

(i) *E_σ and $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ for $\sigma \in \Sigma$, and $\Gamma\backslash D_\Sigma$ are logarithmic manifolds. They are Hausdorff spaces.*

(ii) *For $\sigma \in \Sigma$, E_σ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ in the category of logarithmic manioflds.*

(iii) *For $\sigma \in \Sigma$, $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ is a local isomorphism of logarithmic manifolds. When $\sigma \in \Sigma$ varies, the images of these maps form an open covering of $\Gamma\backslash D_\Sigma$.*

(iv) *There is a canonical homeomorphism $\Gamma\backslash D_\Sigma^\sharp \simeq (\Gamma\backslash D_\Sigma)^{\log}$.*

0.3.12. As in §11, we can generalize the above theorems to the moduli of PLH with coefficients. Let A be a finite dimensional semi-simple $\mathbf{Q}$-algebra endowed with a map $A \to A$, $a \mapsto a^\circ$, satisfying

$$(a+b)^\circ = a^\circ + b^\circ, \quad (ab)^\circ = b^\circ a^\circ \quad (\forall a, \forall b \in A).$$

By a PLH with coefficients in A, or by an A-PLH, we mean a PLH $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ endowed with a ring homomorphism $A \to \mathrm{End}_{\mathbf{Q}}(H_{\mathbf{Q}})$ satisfying

$$\langle ax, y\rangle = \langle x, a^\circ y\rangle \quad (\forall a \in A, \forall x, \forall y \in H_{\mathbf{Q}}).$$

The theorems 0.3.6, 0.3.8 and 0.3.11 can be generalized to the moduli of A-PLH.

To prove the main theorems 0.3.6, 0.3.8 and 0.3.11, as already mentioned in §0.1, we need to study other spaces in the diagram (2) in 0.1.1. Concerning those spaces, we have the following result.

Theorem 0.3.13. (i) *The topological spaces $D_\Sigma^\sharp$, $D_{\Sigma,\mathrm{val}}^\sharp$, $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, D_{BS}, $D_{\mathrm{BS},\mathrm{val}}$ are Hausdorff.*

(ii) *Let Γ be a subgroup of $G_{\mathbf{Z}}$* (resp. *a subgroup of $G_{\mathbf{Z}}$ which is compatible with Σ*). *Then, the quotient spaces $\Gamma\backslash D_{\mathrm{SL}(2)}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS},\mathrm{val}}$* (resp. *$\Gamma\backslash D_\Sigma^\sharp$ and $\Gamma\backslash D_{\Sigma,\mathrm{val}}^\sharp$*) *are Hausdorff.*

If Γ is neat, the projections $D_\Sigma^\sharp \to \Gamma\backslash D_\Sigma^\sharp$, $D_{\Sigma,\mathrm{val}}^\sharp \to \Gamma\backslash D_{\Sigma,\mathrm{val}}^\sharp$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS},\mathrm{val}} \to \Gamma\backslash D_{\mathrm{BS},\mathrm{val}}$ are local homeomorphisms.

In Theorem 0.3.13, the results concerning $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, D_{BS}, $D_{\mathrm{BS,val}}$ are already proved in [KU2]. The space D_{BS} was constructed also by Borel and Ji in [BJ] independently.

§0.4. Complements

0.4.1. One of the remaining open problems is to find "big fans" Σ so that the spaces $\Gamma\backslash D_\Sigma$ contain all possible boundary points. See §12.6.

0.4.2. The plan of this paper is as follows.

In §1–§3 are preliminaries to state the main results of the present paper, Theorem A and Theorem B, in §4. In §1, we define the sets D_Σ and $D_\Sigma^\sharp$. In §2, we describe the theory of polarized logarithmic Hodge structures. In §3, we discuss the strong topology, logarithmic manifolds, the spaces E_σ, $\tilde{E}_\sigma$, $\check{E}_\sigma$, the categories $\mathcal{B}$, $\mathcal{B}(\log)$, and other enlargements of the category of analytic spaces. In §4, we state Therem A and Theorem B without proofs. The above Theorem 0.3.8 (and hence Theorem 0.3.6 (i)) is contained in Theorem B, Theorem 0.3.11 is contained in Theorem A, and Theorem 0.3.6 (ii) is contained in Thoerem A, Theorem B, and Theorem 3.2.5 as explained in 4.2.2. We also discuss, in §4, infinitesimal properties of the extended period maps.

In §5–§8, we prove Theorem A and Theorem B by moving from the right to the left in the diagram 0.1.1 (2). In §5, we review the spaces $D_{\mathrm{SL}(2)}$, D_{BS}, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS,val}}$ defined in [KU2], and then we define $D_{\Sigma,\mathrm{val}}$ and $D^\sharp_{\Sigma,\mathrm{val}}$. By using the work [CKS] of Cattani-Kaplan-Schmid on SL(2)-orbits in several variables, we connect, in §5–§6, the spaces $D^\sharp_{\Sigma,\mathrm{val}}$ and $D_{\mathrm{SL}(2)}$ as in the diagram 0.1.1 (2). In §7, we prove Theorem A, and in §8, we prove Theorem B.

In §9–§12, we give complements, examples, genealizations, and open problems. In §9, we consider the relationship of the present work with the enlargements of D in Cattani-Kaplan [CK]. In §10, we describe local structures of $D_{\mathrm{SL}(2)}$. In §11, we consider the moduli of PLH with coefficients. Though the case with coefficients is more general than the case without coefficients, we have chosen the formulation of this paper not to put coefficients everywhere (then the description of the paper would become too complicated), but to describe the theory without coefficients except in §11 where we show that the results with coefficients can be simply deduced from those without coefficients. In §12, we give examples and discuss open problems.

0.4.3. Corrections to previous works.

We indicate three mistakes in our previous works [UK1], [UK2].

(i) In [UK1, (5.2)], there is a mistake in the definition of the notion of polarized logarithmic Hodge structures of type Φ. This mistake and its correction are explained in 2.5.6.

(ii) In [KU2, Lemma 4.7], the definition of $B(U, U', U'')$ is written as $\{g\tilde{\rho}(t)k\mathbf{r} \mid \cdots\}$, which is wrong. It should be corrected as $\{\tilde{\rho}(t)gk\mathbf{r} \mid \cdots\}$. This point will be explained in 5.2.17. This mistyping happened after the referees finished their careful readings and corrections of mistakes.

(iii) In [UK2, Remarks 3.15, 3.16], we announced that we would consider a space $D^\flat_{\mathrm{SL}(2)}$ in this paper. However, we consider actually only a part $D^\flat_{\mathrm{SL}(2),\le 1}$ of $D^\flat_{\mathrm{SL}(2)}$ in this paper (§9). The reason is that the authors realized that $D^\flat_{\mathrm{SL}(2)}$ is not Hausdorff and seems not to be a good object to consider, but that the part $D^\flat_{\mathrm{SL}(2),\le 1}$ is Haussdorff and is certainly a nice object.

(iv) The present work was announced in [KU1] under the title "Logarithmic Hodge structures and classifying spaces" and in [KU2] under the title "Logarithmic Hodge structures and their moduli", but we have changed the title as of the present work.

0.4.4. As the first trial of geometric application of the present paper, S. Saito [Ss] proved a logarithmic version of infinitesimal Torelli theorem for certain degenerating hypersurfaces in a projective space.

More precisely, he considered a general degeneration of hypersurfaces $X \subset \mathbf{P}^{m+1} \times \Delta$ in a projective space over a disc Δ with X having only one singular point, which has a semi-stable reduction over Δ. When the dimension of the fiber X_t is 2 and the multiplicity of the singularity of the central fiber X_0 is 2, the above considered family reduces to a degeneration with a rational double point of type A_{2s-1} ($s \in \mathbf{Z}_{>0}$).

The main result [Ss, Theorem(2-1)] asserts that, under suitable assumptions, the infinitesimal logarithmic Torelli theorem (i.e., the injectivity of the right vertical map in Theorem 4.3.8 below) holds for the above degeneration. To prove this, he generalized the method of Jacobi rings in the classical case [G2].

0.4.5. The authors are grateful to Professors Kazuhiro Fujiwara, Chikara Nakayama for stimulating discussions and advices. A part of this work was done when the first author was a visitor of Institut Henri Poincaré whose hospitality is gratefully appreciated.

The first line of the Japanese poem (5-7-5 syllables), at the beginning of this paper, was made by Kato and then, following a Japanese tradition of collaboration, the second line (7-7 syllables) was made by Usui. We are very grateful to Professor Luc Illusie for his beautiful translation.

§0.5. Notation

Throughout this paper, we use the following notation.

We denote $\mathbf{N} := \mathbf{Z}_{\geq 0}$.

Let H be a $\mathbf{Z}$-module. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we denote $H_A := A \otimes_{\mathbf{Z}} H$.

We fix a 4-tuple

$$\Phi_0 = \left(w, (h^{p,q})_{p,q \in \mathbf{Z}}, H_0, \langle\ ,\ \rangle_0\right)$$

where w is an integer, $(h^{p,q})_{p,q \in \mathbf{Z}}$ is a set of non-negative integers satisfying

$$\begin{cases} h^{p,q} = 0 \text{ for almost all } p,\ q, \\ h^{p,q} = 0 \text{ if } p + q \neq w, \\ h^{p,q} = h^{q,p} \text{ for any } p,\ q, \end{cases}$$

H_0 is a free $\mathbf{Z}$-module of rank $\sum_{p,q} h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form on $H_{0,\mathbf{C}}$ which is symmetric if w is even and anti-symmetric if w is odd. In the case w is even, say $w = 2t$, we assume that the signature (a, b) of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ satisfies

$$a \text{ (resp. } b) = \textstyle\sum_j h^{t+j,t-j},$$

where j ranges over all even (resp. odd) integers. (By this assumption, the classifying space D of Griffiths (1.2.1) becomes non-empty.) Let

$$G_{\mathbf{Z}} := \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0),$$

and for $R = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let

$$\begin{aligned} G_R &:= \mathrm{Aut}(H_{0,R}, \langle\ ,\ \rangle_0), \\ \mathfrak{g}_R &:= \mathrm{Lie}\, G_R \\ &= \{N \in \mathrm{End}_A(H_{0,R}) \mid \langle Nx, y\rangle_0 + \langle x, Ny\rangle_0 = 0\ (\forall x, \forall y \in H_{0,R})\}. \end{aligned}$$

Following [BS], a *parabolic* subgroup of $G_{\mathbf{R}}$ means a parabolic subgroup of $(G^\circ)_{\mathbf{R}}$, where G° denotes the connected component of G in the Zariski topology containing the unity. (Note that $G^\circ = G$ if w is odd, and $G^\circ = \{g \in G \mid \det(g) = 1\}$ if w is even.)

We denote by

$$\mathcal{A}, \quad \mathcal{A}(\log),$$

the category of analytic spaces, and the category of fs logarithmic analytic spaces, i.e., analytic spaces endowed with an fs logarithmic structure, respectively. Here "analytic spaces" is in the sense of Grothendieck, i.e., we allow nilpotent elements in the structure sheaf.