Persistence and extinction threshold for homogeneous dynamical models with continuous time and its applications

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概 要

In this paper, we develop a stability theory of the zero solution for the continuous-time homogeneous semilinear dynamical system. For the discrete-time homogeneous dynamical system, Thieme and Jin [6, 7, 8] show that the cone spectral radius of a homogeneous operator gives the threshold value for the stability of the zero solution. We apply this idea to the continuous-time dynamical system under appropriate conditions commonly used in population dynamics. Using this theory, we investigate a two-sex structured population model to find the threshold value for population extinction and persistence.

1 Introduction

In structured population dynamics, the basic system is usually formulated by the semilinear Cauchy problem in a state space X:

$$\frac{du}{dt} = -Au + B(u), \tag{1}$$

where $-A$ is a linear operator (generator for the survival process) and B is a nonlinear operator describing the birth process of new individuals such that $B(0) = 0$ and it has the Fréchet derivative $B'[0]$ at the origin. Then the linearized system $du/dt = (-A + B'[0])u$ describes the growth of a small population.

As is well known in epidemic models [1], we can define the next generation operator (NGO) to compute the basic reproduction number $\mathcal{R}_0$. Assume that $-A$ is quasi-positive, it has positive inverse and $B'[0]$ is a positive operator. Then the NGO, denoted by K, is calculated as $K = B'[0]A^{-1}$ and the basic reproduction number is calculated by the spectral radius of NGO: $\mathcal{R}_0 = \rho(K)$ [5]. In fact, the dominant exponential solution $e^{\lambda t}\phi$ of the linearized equation at the zero equilibrium satisfeis

$$B'[0]\phi = B'[0] \int_0^{\infty} e^{-(\lambda+A)s} B'[0]\phi ds,$$

where $B'[0]\phi$ denotes the density of newly produced individuals. Then we know that the spectral radius of $B'[0] \int_0^{\infty} e^{-As} ds = B'[0]A^{-1}$ becomes the threshold value whether λ is positive or negative. Based on the principle of linearized stability, we know thet the zero solution of (1) is locally stable if $\mathcal{R}_0 < 1$, while it is unstable if $\mathcal{R}_0 > 1$. Since $r(B'[0]A^{-1}) = r(A^{-1}B'[0])$, some authors use $K = A^{-1}B'[0]$ as the *next infection operator* (NIO) in epidemic models. In the following, for calculation purpose, we use the NIO-like operator.

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If B has the homogeneous nonlinearity, it is not differentiable at the origin, so we can not define the basic reproduction number and cannot use the linearized stability principle to examine the stability of the zero solution. For example, let us consider the two-sex model. The mating and reproduction can be described by a homogeneous function of degree one [5]. This type of models with discrete time were studied by Jin et al. [6, 7, 8] and Thieme [13, 14, 15, 16]. In those papers, the basic population dynamics is described as $x_n = F(x_{n-1})$, $n \in \mathbb{N}$, $x_0 \in X_+$, where the population structure is encoded in the cone X_+ of ordered normed vector space X. They assume that the function F has the first order approximation $B : X_+ \to X_+$ at the zero vector and B is homogeneous of degree one. Since the spectral radius of the linearized operator does not work, they used the *cone spectral radius* to obtain a threshold value for population persistence and extinction. This is seen as an extension of the linearized stability principle.

Here we apply the above idea to equation 1. Different from the discrete-time models, B is not necessarily positive in many applications. So we can not apply the Jin and Thieme's method for discrete time directly. Instead we assume that there exists some $\epsilon > 0$ such that $I + \epsilon B$ is positive and order-preserving, because we can rewrite (1) as

$$\frac{du}{dt} = -(\frac{1}{\epsilon} + A)u + \frac{1}{\epsilon}(u + \epsilon B(u)). \tag{2}$$

For this modeified system, the NIO-like operator is calculated as

$$(\frac{1}{\epsilon} + A)^{-1}\frac{1}{\epsilon}(I + \epsilon B) = (I + \epsilon A)^{-1}(I + \epsilon B).$$

Then we can expect that its cone spectral radius $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B))$ is a threshold value for the stability of the zero solution.

Here we introduce some basic definitions and propositions, although we skip their proofs. Let X, Y be ordered vector spaces with cones X_+ and Y_+, respectively.

Definition 1. *$B : X_+ \to Y$ is called (positively) homogeneous (of degree one), if $B(\alpha x) = \alpha B(x)$ for all $\alpha \in \mathbb{R}_+, x \in X_+$.*

By definition for a homogeneous map B, $B(0) = 0$. For a positive homogeneous operator $B : X_+ \to X_+$, we define the *cone operator norm* by $||B||_+ := \sup\{||B(x)||; x \in X_+, ||x|| \leq 1\}$. If this supremum exists, we call B is bounded. It is easy to show that $||B(x)|| \leq ||B||_+||x||, \quad x \in X_+$. Let $H(X_+, Y)$ denote the set of bounded homogeneous maps $B : X_+ \to Y$ and $H(X_+, Y)_+$ denote the set of bounded homogeneous maps $B : X_+ \to Y_+$ and $HM(X_+, Y_+)$ the set of those maps in $H(X_+, Y_+)$ that are also order-preserving. Then $H(X_+, Y)$ is a real vector space and $|| \cdot ||_+$ is a norm on $H(X_+, Y)$. It follows for $B \in H(X_+, Y_+)$ and $C \in H(Y_+, Z_+)$ that $CB \in H(X_+, Z_+)$ and $||CB||_+ \leq ||C||_+||B||_+$. For a homogeneous operator $B : X_+ \to X_+$, define the cone spectral radius of B by

$$\mathbf{r}_+(B) := \inf_{n \in \mathbb{N}} ||B^n||_+^{1/n} = \lim_{n \to \infty} ||B^n||_+^{1/n}. \tag{3}$$

Definition 2. *Let X be a normed real vector space and X_+ be an positive closed cone in X. Let $x, y \in X$ and denote $x \geq y$ when $x - y \in X_+$. X is called an ordered normed vector space.*

Definition 3. *Let Y and Z be ordered vector spaces with cones Y_+ and Z_+ and $U \subset Y$. A map $B : U \to Z$ is called positive if $B(U \cap Y_+) \subset Z_+$ and order-preserving if $B(x) \geq B(y)$ for all $x, y \in U$ and $x \geq y$.*

Definition 4. *Let $x \in X$ and $u \in X_+$. Then x is called u-bounded if there exists some $c > 0$ such that $-cu \leq x \leq cu$. The set of u-bounded elements in X is denoted by X_u, $\dot{X}_u = X_u \backslash \{0\}$ and $X_{u+} = X_u \cap X_+$.*

Definition 5. *Let $B : X_+ \to X_+$ and $u \in X_+$.*

(1) *B is called pointwise u-bounded, if for any $x \in X_+$, there exists some $n \in \mathbb{N}$ such that $B^n x \in X_u$. The point u is called a pointwise order bound of B. If B is pointwise u-bounded for some $u \in X_+$, then B is also called pointwise order bounded.*

(2) *B is called uniformly u-bounded if there exists some $c > 0$ such that $B(x) \leq c||x||u$ for all $x \in X_+$. The element u is called a uniform order bound of B. An operator B is called uniformly order bounded if B is uniformly u-bounded for some $u \in X_+$.*

Pointwise order boundedness implies uniformly order boundedness under some conditions. The following proposition is given in Thieme [16].

Proposition 1. *Let $u \in X_+$ and $B : X_+ \to X_+$ be order-preserving and homogeneous. Assume that X_+ is complete and B is pointwise u-bounded and continuous, then some powers of B is uniformly u-bounded.*

Corollary 1. *Let $B : X_+ \to X_+$ be a homogeneous and continuous operator. Assume that X_+ is solid and B is bounded. Then B is uniformly u-bounded for any interior point $u \in X_+$.*

Let $B : X_+ \to X_+$ be a homogeneous and continuous operator.

Definition 6. *B is called pseudo-compact if $\mathbf{r}_+(B) > 0$ and the following holds: If (x_n) is a sequence in $X_+ \cap X_u$ and (λ_n) is a sequence in $[\mathbf{r}_+(B), \infty)$ such that (x_n) is bounded with respect to the u-norm and $\lambda_n \to \mathbf{r}_+(B)$ and $||(\lambda_n - B)x_n|| \to 0$ and $(\lambda_n - B)x_n \in X_+$ for all $n \in \mathbb{N}$, then (x_n) has a convergent subsequence.*

Under some conditions, Thieme shows the existence of eigenvector for pseudo-compact operator [16].

Theorem 1. *Assume that X_+ contains a normal point $u \neq 0$. Let $B : X_+ \to X_+$ be a homogeneous operator. If B is order-preserving, uniformly order bounded and pseudo-compact, then there exists $v \in \dot{X}_+$ such that $B(v) = \mathbf{r}_+(B)v$.*

Let X be an ordered Banach space with a positive cone X_+. We introduce the measure of noncompactness in Kuratowski [9] and Nussbaum [10, 11]. For a bounded subset $S \subset X$, define $d(S)$ as the diameter of S and $\alpha(S)$ as the *measure of noncompactness* of S as

$$d(S) := \inf\{d > 0 : \text{there exists } x \in X \text{ such that } S \subset U_d(x)\},$$

$$\alpha(S) := \inf\{d > 0 : S = \bigcup_{i=1}^{n} S_i, n < \infty \text{ and } d(S_i) \leq d \text{ for } 1 \leq i \leq n\}.$$

In this definition, $U_d(x)$ is an open ball centered at x with diameter d. Generally, we suppose that β is a map which assigns to each bounded subset S of X a nonnegative real number $\beta(S)$. We will call β a generalized measure of noncompactness if β satisfies the following properties:

(1) $\beta(S) = 0$ if and only if the closure of S is compact.

(2) $\beta(\overline{\text{co}}(S)) = \beta(S)$ for every bounded set S in X, where $\overline{\text{co}}(S)$ denotes the convex full of S.

(3) $\beta(S + T) \leq \beta(S) + \beta(T)$ for all bounded sets S and T, where $S + T = \{s + t; s \in S, t \in T\}$.

(4) $\beta(S \cup T) = \max(\beta(S), \beta(T))$.

It is well-known that α satisfies these four properties.
If D is a subset of X, β is the generalized measure of noncompactness, and $f : D \to X$ a continuous map, f is called k-set-contraction with respect to β if $\beta(f(S)) \leq k\beta(S)$ holds for every bounded subset S in D. If $\beta = \alpha$, we shall simply say that f is a k-set-contraction.

2 Existence of a positive eigenvector

Throughout this section, $A : X \to X$ is a bounded linear operator and $B : X_+ \to X$ be a bounded, continuous and homogeneous operator. For a sufficiently small $\epsilon > 0$, if the operator $I + \epsilon B$ is positive and order-preserving, we call the operator B the *semi order-preserving operator.* Although we skip the proof, we have

Lemma 1. *Let $B : X_+ \to X_+$ be a bounded and homogeneous operator and $L : X \to X$ be a bounded linear operator. Assume that B is compact and that $r = \mathbf{r}_+(L + B) > 0$ and $r \in \rho(L)$. Then $L + B$ is pseudo-compact.*

Theorem 2. *Let X_+ be solid. Assume that for any sufficiently small $\epsilon > 0$, $(I + \epsilon A)^{-1}$ is positive and B is semi order-preserving and compact. Further assume that $r = \mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) \in \rho((I + \epsilon A)^{-1})$. Then for a small ϵ such that $I + \epsilon B$ is positive and order-preserving, there exists $v \in X_+$ such that $v \neq 0$ and $(I + \epsilon A)^{-1}(I + \epsilon B)(v) = rv$.*

Proof. It is sufficient to see that the assumptions in Theorem 1 are satisfied. Since X_+ is solid, by Corollary 1, $I + \epsilon B$ is uniformly u-bounded for any interior point $u \in X_+$. Since $(I + \epsilon A)^{-1}$ is positive, $(I + \epsilon A)^{-1}(I + \epsilon B)$ is uniformly $(I + \epsilon A)^{-1}u$-bounded. This implies $(I + \epsilon A)^{-1}(I + \epsilon B)$ is uniformly order-bounded. Next let us show the pseudo-compactness of $(I + \epsilon A)^{-1}(I + \epsilon B)$. Observe that

$$(I + \epsilon A)^{-1}(I + \epsilon B) = (I + \epsilon A)^{-1} + \epsilon(I + \epsilon A)^{-1}B.$$

As $(I + \epsilon A)^{-1}$ is a bounded linear operator and $\epsilon(I + \epsilon A)^{-1}B$ is compact, it follows from Lemma 1 that $(I + \epsilon A)^{-1}(I + \epsilon B)$ is pseudo-compact. Therefore the assumptions in Theorem 1 are all satisfied. □

Let X_+ be solid. Assume that for any sufficiently small $\epsilon > 0$, $(I + \epsilon A)^{-1}$ is positive and $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) \in \rho((I + \epsilon A)^{-1})$ and that B is semi order-preserving. Choose small ϵ and $\tilde{\epsilon}$ such that $I + \epsilon B$ and $I + \tilde{\epsilon} B$ are positive and order-preserving. Let $0 < \tilde{\epsilon} < \epsilon$ and define $r := \mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B))$ and $\tilde{r} := \mathbf{r}_+((I + \tilde{\epsilon} A)^{-1}(I + \tilde{\epsilon} B))$. After long calculations, we can show the followings:

Lemma 2. *It holds that $r \geq 1 \Longleftrightarrow \tilde{r} \geq 1$ and $\tilde{r} > 1 \Longrightarrow r \geq \tilde{r} > 1$.*

Lemma 3. *Let $D := \{\epsilon > 0; I + \epsilon B \text{ is positive and order-preserving}\}$. Assume that ϵ is not the supremum of D. Then $r > 1 \Longrightarrow \tilde{r} > 1$.*

Remark 1. *If A is positive, then $r > 1$ always implies $\tilde{r} > 1$.*

By Lemma 2 and Remark 1, the sign of $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) - 1$ is independent from the choice of ϵ under some conditions.

Proposition 2. *Let A be a positive bounded linear operator. Let B be a semi order-preserving bounded homogeneous operator. Assume that $(I + \epsilon A)^{-1}$ is positive for all $\epsilon > 0$ and that $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) \in \rho((I + \epsilon A)^{-1})$. Then the one of these three properties holds:*

(1) $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) > 1$ *for all* $\epsilon \in D$,

(2) $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) = 1$ *for all* $\epsilon \in D$,

(3) $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) < 1$ *for all* $\epsilon \in D$.

3 Persistence and extinction

Let X be an ordered Banach space with a positive solid closed cone X_+. Let $T(t), t \geq 0$ be the semigroup generated by $-A$. Throughout this section, we assume that $T(t)$ is positive, i.e. $T(t)(X_+) \subset X_+$ and $||T(t)|| \to 0$ as $t \to \infty$. By introducing an equivalent norm, we can assume that there exists some $\theta > 0$ such that $||T(t)|| \leq e^{-\theta t}$ [2].

In the following, we consider the semilinear model (1). For the uniqueness and the existence of solution, we assume B is Lipschitz continuous, bounded and nonlinear. By the same kind of arguments as in Theorem 10.19 of Smith and Thieme [12], we can show that the following property holds:

Proposition 3. *Assume that B is positive and order-preserving. Then the solution semiflow Φ is order-preserving: If $x, y \in X_+$ satisfy $x \geq y$, then $\Phi(t,x) \geq \Phi(t,y)$.*

Proof. Let $\epsilon > 0$. Let $\tau > 0$ be determined after. Let $x, y \in X_+$ satisfy $x \geq y$. Define the map $w : \mathbb{R}_+ \to X$ as $w(t) := \Phi(t,x) - \Phi(t,y)$. We rewrite the solution as the mild solution satisfying

$$\Phi(t,x) = T(t)x + \int_0^t T(t-s)B(\Phi(s,x))ds. \tag{4}$$

We get

$$\begin{aligned} w(t) &= T(t)(x-y) + \int_0^t T(t-s)\left(B(\Phi(s,x)) - B(\Phi(s,y))\right)ds, \\ &= T(t)(x-y) + \int_0^t T(t-s)\left(B(w(s)+\Phi(s,y)) - B(\Phi(s,y))\right)ds. \end{aligned}$$

Let $\tilde{G}(w)(t)$ denote the right-hand side of this equation. Define the complete metric space $(K_\tau, ||\cdot||_\infty)$, where $K_\tau := \{w \in C([0,\tau], X_+) : ||w(t) - (x-y)|| \leq \epsilon, 0 \leq t \leq \tau\}$ and $||w||_\infty = \max_{t\in[0,\tau]} ||w(t)||$. Let $\Lambda > 0$ be the Lipschitz constant of B and $w \in K_\tau$. Then by the triangle inequality,

$$\begin{aligned} ||\tilde{G}(w)(t) - (x-y)|| \leq &||T(t)(x-y) - (x-y)|| \\ &+ \int_0^t ||T(t-s)|| \times ||B(w(s)+\Phi(s,y)) - B(\Phi(s,y))||ds. \end{aligned} \tag{5}$$

By the Lipschitz continuity of B and the boundedness of $T(t)$,

$$||\tilde{G}(w)(t) - (x-y)|| \leq ||T(t)(x-y) - (x-y)|| + \int_0^t e^{-\theta(t-s)}\Lambda||w(s)||ds. \tag{6}$$

The definition of K_τ implies $||w(t)|| \leq ||x-y|| + \epsilon$. By using this inequality, we obtain the estimates:

$$\begin{aligned} ||\tilde{G}(w)(t) - (x-y)|| \leq &||T(t)(x-y) - (x-y)|| \\ &+ \frac{1-e^{-\theta t}}{\theta}\Lambda(||x-y|| + \epsilon). \end{aligned} \tag{7}$$

Since the right-hand side of this inequality goes to 0 as $t \to 0$, we can choose τ such that $||\tilde{G}(w)(t) - (x-y)|| \leq \epsilon$ for all $t \in [0,\tau]$. Obviously, $\tilde{G}$ is a map from K_τ into $C([0,\tau], X_+)$. Hence $\tilde{G}$ is a map from K_τ into K_τ. Next we show that the map $\tilde{G}$ is a strict contraction on K_τ for sufficiently small τ. Let $w_1, w_2 \in K_\tau$. For $t \in [0,\tau]$,

$$\begin{aligned} &||\tilde{G}(w_1)(t) - \tilde{G}(w_2)(t)|| \\ &\leq \int_0^t ||T(t-s)||||B(w_1(s)+\Phi(s,y)) - B(w_2(s)+\Phi(s,y))||ds, \\ &\leq \int_0^t e^{-\theta(t-s)}\Lambda||w_1(s) - w_2(s)||ds, \end{aligned} \tag{8}$$

Hence we get

$$||\tilde{G}(w_1)(t) - \tilde{G}(w_2)(t)|| \le \frac{1-e^{-\theta t}}{\theta}\Lambda \sup_{0\le s\le \tau} ||w_1(s) - w_2(s)||. \tag{9}$$

We take the supremum over t,

$$\sup_{0\le s\le \tau} ||\tilde{G}(w_1)(s) - \tilde{G}(w_2)(s)|| \le \frac{1-e^{-\theta\tau}}{\theta}\Lambda \sup_{0\le s\le \tau} ||w_1(s) - w_2(s)||. \tag{10}$$

We can choose τ so small that $\tilde{G}$ becomes a strict contraction. Thus $\tilde{G}$ has a fixed point in K_τ. Since the mild solution exists globally, this implies $\Phi(t,x) - \Phi(t,y) \ge 0$ □

Theorem 3. *If* $\mathbf{r}_+((I+\epsilon A)^{-1}(I+\epsilon B)) < 1$, *then for any solution* $u(t)$ *with initial data* $u_0 \in X_+$, $||u(t)|| \to 0$ *as* $t \to \infty$

Proof. Suppose $\mathbf{r}_+((I+\epsilon A)^{-1}(I+\epsilon B)) < 1$. Define S as the semigroup generated by $-\epsilon^{-1}(I+\epsilon A)$ and define $B' := \epsilon^{-1}(I+\epsilon B)$. For $u_0 \in X_+$ and an integer $n \in \mathbb{N}$, define the homogeneous operator $D_n : X_+ \to X_+$ as

$$\begin{aligned} D_n u_0 &:= S(\frac{1}{n})u_0 + \int_0^{\frac{1}{n}} S(\frac{1}{n} - s)B'(u(s))ds, &(11)\\ &= T(\frac{1}{n}, u_0) + \int_0^{\frac{1}{n}} T(\frac{1}{n} - s, B(u(s)))ds, &(12)\end{aligned}$$

where $u(s)$ is the mild solution with initial data u_0. The first equation (11) shows the operator D_n is order-preserving by Proposition 3 and the second equation (12) shows the pseudo-compactness of the operator D_n. Denote $r_n := \mathbf{r}_+(D_n)$. To show by contradiction, we assume $r_1 \ge 1$. Let Φ be the solution semiflow. Let u is an interior point of X_+, there exists some $c > 0$ such that for any $x \in X_+$, $x \le c||x||u$. Hence there exists some c' such that $\Phi(t,x) \le \Phi(t, c||x||w) \le c'||x||w$ for any $x \in X_+$. Thus D_n is uniformly w-bounded. Next we show that D_n is pseudo-compact. Let $\{x_m\}$ be a sequence in $X_+ \cap X_w$ such that $\{x_m\}$ is bounded with respect to w-norm and let $\{\lambda_m\}$ be a sequence in $[r_n, \infty)$ such that $\lambda_m \to r_n$ as $m \to \infty$. Furthermore, assume $||(\lambda_m - D_n)x_m|| \to 0$ and $(\lambda_m - D_n)x_m \in X_+$. Then

$$\begin{aligned} x_m &= (\lambda_m - T(\frac{1}{n}))^{-1}\left((\lambda_m - D_n)x_m + \int_0^{\frac{1}{n}} T(\frac{1}{n} - s)B(\Phi(s, x_m))ds\right),\\ &:= (\lambda_m - T(\frac{1}{n}))^{-1}\left((\lambda_m - D_n)x_m + E_n(x_m)\right). &(13)\end{aligned}$$

Since E_n is a compact operator as shown by Smith and Thieme [12], $\{x_m\}$ has a convergent subsequence and D_n is pseudo-compact. By Theorem 1, there exists some $v_n \in X_+$ such that $||v_n|| = 1$ and $D_n v_n = r_n v_n$. We can easily know $r_n^n = r_1$. Then $D_1 v_n = D_n^n v_n = r_n^n v_n = r_1 v_n$. Since D_1 is pseudo-compact, $\{v_n\}$ and $\{r_1\}$ satisfies the condition for pseudo-compact and thus v_n has a convergent subsequence. Choose a convergent subsequence of $\{v_n\}$ and define the limit as v_∞. Let $t = l_k \times \frac{1}{k} + \epsilon_k$, where $t \in \mathbb{R}_+, l_k \in \mathbb{N}$ and ϵ_k satisfies $0 \le \epsilon_k < \frac{1}{k}$. Then

$$\begin{aligned}\Phi(t, v_\infty) &= \lim_{k\to\infty} \Phi(t, v_k) = \lim_{k\to\infty} \Phi(\epsilon_k, \Phi(l_k \times \frac{1}{k}, v_k)),\\ &= \lim_{k\to\infty} \Phi(\epsilon_k, r_k^{l_k} v_k) = \lim_{k\to\infty} {r_1}^t \times r_1^{-\epsilon_k}\Phi(\epsilon_k, v_k) = r_1^t v_\infty.\end{aligned}$$

Thus $v(t) := {r_1}^t v_\infty$ is the solution with initial data $v(0) = v_\infty$. It is easy to see that $v'(0) \ge 0$. Hence $-(I+\epsilon A)v_\infty + (I+\epsilon B)v_\infty \ge 0$ and we get $\mathbf{r}_+((I+\epsilon A)^{-1}(I+\epsilon B)) \ge 1$. This is a contradiction. Therefore we can show $r_1 < 1$ and $||D_1^m|| \to 0$ as $m \to \infty$. Let $x \in X_+$ and $t > 0$. Define $\delta := \max_{0\le s\le 1} ||\Phi(s,x)||$. Choose $m \in \mathbb{N}$ such that $0 \le t - m < 1$. Then it follows that $||\Phi(t,x)|| = ||\Phi(m+t-m, x)|| \le ||D_1^m|| \times \delta \to 0$ as $t \to \infty$. □

Next we generalize the principle of linearized stability to the case that first order approximation at the origin is not linear but only homogeneous. Let us consider the equation,

$$\frac{du}{dt} = -Au + F(u). \tag{14}$$

$F : X_+ \to X_+$ is nonlinear operator. We assume that the solution of the equation (14) with an initial data in X_+ exists uniquely and globally. Then we can show the stability at zero point.

Theorem 4. *Let $F, B : X_+ \to X, F(0) = 0$ and let B be a homogeneous compact uniformly u-bounded operator. Assume that we can choose $\epsilon > 0$ such that $I + \epsilon B$ is positive order-preserving map and $r := \mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) < 1$. Further assume that for any $\eta > 0$ there exists some $\delta > 0$ such that $(I+\epsilon F)(x) \leq (1+\eta)(I+\epsilon B)(x)$ for all $x \in X_+$ with $||x|| < \delta$. Then the zero point is locally asymptotically stable.*

Proof. Define $D_1 : X_+ \to X_+$ as

$$D_t(x) := T(t)x + \int_0^t T(t-s)(1+\eta)B(u(s))ds \tag{15}$$

where T is the semigroup induced by $-\frac{\eta}{\epsilon} - A$ and $u(s)$ is the mild solution of $u' = -\epsilon^{-1}(\eta I + \epsilon A)u + (1+\eta)B(u)$ with initial data $u(0) = x$. We can show $\mathbf{r}_+(D_1) < 1$ by the same proof as Theorem 3. Hence there exists some $N \in \mathbb{N}$ such that $||D_N|| = ||{D_1}^N|| < 1$ and $D = \max_{0 \leq s \leq N} ||D_s||$. Let $x \in X_+$ satisfy $||x|| \leq \min\{1, D^{-1}\}\delta := \tilde{\delta}$ and let $u(t)$ be the mild solution of $u' = -\epsilon^{-1}(\eta + \epsilon A)u + (1+\eta)B(u)$ with initial data $u(0) = x$. Since $||u(t)|| \leq \delta$ for any $t \in [0, N]$, $||D_N(x)|| \leq \tilde{\delta}$, we get $u(t) \to 0$ as $t \to \infty$. □

Theorem 5. *Let $\rho : X_+ \to \mathbb{R}$ be a continuous function. Assume that X_+ is normal and that there exists some homogeneous semi order-preserving operator B such that for some small $\epsilon > 0$ and any $\alpha \in (0, 1)$, there exists some $\delta > 0$ such that $(1-\alpha)(I+\epsilon B)(x) \leq (I+\epsilon F)(x)$ for all $x \in X_+$ with $||x|| \leq \delta$. Further assume that*

1. *If $\rho(x) > 0$, then $\rho(\Phi(t, x)) > 0$ for all $t > 0$, where Φ is the solution semiflow.*

2. *There exists some $r > 1$ and $v \in \dot{X}_+$ such that $(I + \epsilon A)^{-1}(I + \epsilon B)(v) \geq rv$.*

3. *For any $x \in X_+$ with $\rho(x) > 0$, there exists some $t > 0$ and $\xi > 0$ such that $\Phi(t, x) \geq \xi v$.*

Then there exists some $\eta > 0$ such that $\limsup_{t\to\infty} ||\Phi(t, x)|| \geq \eta$ for any $x \in X_+$ with $\rho(x) > 0$.

Proof. Choose $\alpha > 0$ such that $(1-\alpha)r > 1$. Suppose the assertion does not hold. There exists some $x \in X_+$ such that $\rho(x) > 0$ and $\limsup_{t\to\infty} ||\Phi(t, x)|| < \frac{\delta}{2}$, where Φ is the solution semiflow of $u' = -Au + F(u)$. Then by the shift of time, we can assume that $||\Phi(t, x)|| \leq \delta$ for all $t \geq 0$ and there exists some $\xi > 0$ such that $x \geq \xi v$. Define Ψ as the solution semiflow of $u' = -\epsilon^{-1}(I + \epsilon A)u + \epsilon^{-1}(1-\alpha)(I + \epsilon B)(u)$. Then $\Phi(t, x) \geq \Psi(t, x) \geq \Psi(t, \xi v) \geq \exp(((1-\alpha)r - 1)t)\, v$. Since X_+ is normal, there exists some $\tilde{M} > 0$ such that $||x|| \leq \tilde{M}||y||$ for all $x \in X, y \in X_+$ with $-y \leq x \leq y$, and it follows that

$$\tilde{M}||\Phi(t, x)|| \geq \exp(((1-\alpha)r - 1)t)\, ||v|| \to \infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction. □

4 Application to two-sex population dynamics

As a demographic example, let us consider the following two-sex age-structured population model[1]:

$$\begin{cases} \frac{dm_1}{dt} = -(\mu_m + \eta_m(x))m_1 + \beta(x)\gamma_m\phi(m_2, f_2), \\ \frac{dm_2}{dt} = -\mu_m m_2 + \eta_m(x)m_1, \\ \frac{df_1}{dt} = -(\mu_f + \eta_f(x))f_1 + \beta(x)\gamma_f\phi(m_2, f_2), \\ \frac{df_2}{dt} = -\mu_f f_2 + \eta_f(x)f_1. \end{cases} \tag{16}$$

Here the state space is $X_+ := \mathbb{R}^4_+$ and $x = (m_1, m_2, f_1, f_2)^{\mathrm{T}}$ (where T denotes the transpose of the vector). $\eta_m, \eta_f, \beta : X_+ \to \mathbb{R}_+$ are functions of $x \in X_+$. The numbers m_1 and f_1 denote the population size of male and female children, respectively. The numbers m_2 and f_2 denote the population size of male and female adults, respectively. Male and female individuals die at per capita rate μ_m and μ_f, respectively. Male [female] children grow up to adult per capita rate $\eta_m(x)$ [$\eta_f(x)$]. The function $\beta(x)$ is the density-dependent birth rate. The numbers γ_m, γ_f denote the sex ratio at birth, so $\gamma_m + \gamma_f = 1$. Finally the function $\phi : X_+ \times X_+ \to \mathbb{R}_+$ is a mating or pair formation function. We assume that ϕ has the following properties; (1) ϕ is order-preserving; (2) ϕ is homogeneous; (3) $\phi(m, 0) = \phi(0, f) = 0$. We assume that $\phi, \eta_m, \eta_f, \beta$ are positive Lipschitz continuous functions. Under this assumption, the solution of the equation (16) with initial data in X_+ exists uniquely and globally. Define the operator A and F by

$$A := \begin{pmatrix} \mu_m & 0 & 0 & 0 \\ 0 & \mu_m & 0 & 0 \\ 0 & 0 & \mu_f & 0 \\ 0 & 0 & 0 & \mu_f \end{pmatrix}, \quad F(x) := \begin{pmatrix} -\eta_m(x)m_1 + \beta(x)\gamma_m\phi(m_2, f_2) \\ \eta_m(x)m_1 \\ -\eta_f(x)f_1 + \beta(x)\gamma_f\phi(m_2, f_2) \\ \eta_f(x)f_1 \end{pmatrix}, \tag{17}$$

where $x = (m_1, m_2, f_1, f_2)^T$.

Define a homogeneous operator B by the map F with $\eta_m(x), \eta_f(x)$ being replaced by $\eta_m(0), \eta_f(0)$. Then $-A$ is resolvent positive and A is positive. Further, $I + \epsilon B$ is positive and order-preserving for any ϵ with $\epsilon \le \max\{\eta_m(0)^{-1}, \eta_f(0)^{-1}\}$. Hence by Proposition 2, the sign of $\mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) - 1$ is definite for all such ϵ.

Let us find conditions under which $r := \mathbf{r}_+((I + \epsilon A)^{-1}(I + \epsilon B)) > 1$. Suppose $r > 1$. Then by Theorem 2, there exists some $v := \dot{X}_+$ such that

$$(I + \epsilon A)^{-1}(I + \epsilon B)v = rv.$$

Let $v = (m_1, m_2, f_1, f_2)^{\mathrm{T}}$. Then the components satisfy the following equations:

$$m_2 = \frac{\epsilon\eta_m m_1}{r(1 + \epsilon\mu_m) - 1}, \quad f_2 = \frac{\epsilon\eta_f f_1}{r(1 + \epsilon\mu_f) - 1}. \tag{18}$$

Hence we get

$$(1 + \epsilon\mu_m)^{-1}\left(m_1 - \epsilon\eta_m m_1 + \epsilon^2\beta\gamma_m\phi\left(\frac{\eta_m}{r(1 + \epsilon\mu_m) - 1}, \frac{\eta_f}{r(1 + \epsilon\mu_f) - 1}\right)\right) = rm_1 > m_1. \tag{19}$$

Since ϕ is order-preserving, the left-hand side of inequality (19) is decreasing function of r. Thus the left-hand side with r replaced by 1 is greater than m_1. Then we obtain

$$(1 + \epsilon\mu_m)^{-1}\left(m_1 - \epsilon\eta_m m_1 + \epsilon^2\beta\gamma_m\phi\left(\tfrac{\eta_m}{1\times(1+\epsilon\mu_m)-1}, \tfrac{\eta_f}{1\times(1+\epsilon\mu_f)-1}\right)\right) > m_1. \tag{20}$$

$$\left(m_1 - \epsilon\eta_m m_1 + \epsilon^2\beta\gamma_m\phi\left(\tfrac{\eta_m}{\epsilon\mu_m}, \tfrac{\eta_f}{\epsilon\mu_f}\right)\right) > (1 + \epsilon\mu_m)m_1. \tag{21}$$

$$\phi\left(\tfrac{\eta_m m_1}{\mu_m}, \tfrac{\eta_f f_1}{\mu_f}\right) > \tfrac{(\mu_m+\eta_m)m_1}{\beta\gamma_m}. \tag{22}$$

[1]For demographic two-sex problems, the reader may refer to [3], [4] and [5].

By the same way, we can get

$$\phi\left(\frac{\eta_m m_1}{\mu_m}, \frac{\eta_f f_1}{\mu_f}\right) > \frac{(\mu_f+\eta_f)f_1}{\beta\gamma_f}. \tag{23}$$

Hence if $r > 1$, then there exists some $m_1, f_1 > 0$ such that

$$\phi\left(\frac{\eta_m m_1}{\mu_m}, \frac{\eta_f f_1}{\mu_f}\right) > \max\left\{\frac{(\mu_m+\eta_m)m_1}{\beta\gamma_m}, \frac{(\mu_m+\eta_f)f_1}{\beta\gamma_f}\right\}. \tag{24}$$

Conversely, m_1, and f_1 exist such that they satisfy the inequality (24). Define $m_2, f_2 > 0$ by the equation (18). Then there exists some $r > 1$ such that $(I+\epsilon A)^{-1}(I+\epsilon B)v \geq rv$. It implies $\mathbf{r}_+((I+\epsilon A)^{-1}(I+\epsilon B)) > 1$. Similarly, we can know that $r \geq 1$ holds if and only if there exists some $m_1, f_1 > 0$ such that

$$\phi\left(\frac{\eta_m m_1}{\mu_m}, \frac{\eta_f f_1}{\mu_f}\right) \geq \max\left\{\frac{(\mu_m+\eta_m)m_1}{\beta\gamma_m}, \frac{(\mu_m+\eta_f)f_1}{\beta\gamma_f}\right\}.$$

Theorem 6. *Assume that for any $m_1, f_1 > 0$, it holds that*

$$\phi\left(\frac{\eta_m m_1}{\mu_m}, \frac{\eta_f f_1}{\mu_f}\right) < \max\left\{\frac{(\mu_m+\eta_m)m_1}{\beta\gamma_m}, \frac{(\mu_f+\eta_f)f_1}{\beta\gamma_f}\right\}.$$

Then the zero point is asymptotically stable.

Theorem 7. *Assume that there exist some $m_1, f_1 > 0$ such that*

$$\phi\left(\frac{\eta_m m_1}{\mu_m}, \frac{\eta_f f_1}{\mu_f}\right) > \max\left\{\frac{(\mu_m+\eta_m)m_1}{\beta\gamma_m}, \frac{(\mu_f+\eta_f)f_1}{\beta\gamma_f}\right\}.$$

Thus the population weakly persists. More precisely, there exists some $\delta > 0$ such that for any initial data x with $m_1 + m_2 > 0$ and $f_1 + f_2 > 0$, the solution $u(t)$ satisfies $\limsup_{t\to\infty} ||u(t)|| \geq \delta$.

Proof. For $x = (m_1, m_2, f_1, f_2)^T$, define the function ρ as $\rho(x) := \min\{m_1 + m_2, f_1 + f_2\}$. It is sufficient to show that the condition 3 in Theorem 5 holds. Assume $\rho(x) > 0$. Then $\Phi(t, x)$ is an interior point in $\mathbb{R}^4_+$ for any $t > 0$, where Φ is the solution semiflow. As the condition 2 is satisfied by the above argument, the condition 3 follows. □

We can define an index like as the basic reproduction number by the same way in Thieme [15]. Define the reproduction number for two-sex population by

$$\mathcal{R}_0 := \phi\left(\frac{\beta\eta_m\gamma_m}{\mu_m(\mu_m+\eta_m)}, \frac{\beta\eta_f\gamma_f}{\mu_f(\mu_f+\eta_f)}\right). \tag{25}$$

Theorem 8. *The sign relation* $\mathrm{sign}(r-1) = \mathrm{sign}(\mathcal{R}_0 - 1)$ *holds.*

If we assume that β is a decreasing function and that there exists some $\alpha > 0$ such that $\phi(m_2, f_2) \geq \alpha \min\{m_2, f_2\}$ for any $m_2, f_2 \geq 0$, point-dissipativeness and eventually boundedness on every bounded sets hold. Thus the solution semiflow has a compact attractor of neighborhoods of compact sets in X_+. Then a positive equilibrium exists if $r > 1$.

Theorem 9. *Assume that β is decreasing function and $\beta(x) \to 0$ as $x \to \infty$ and that there exists some $\alpha > 0$ such that $\phi(m_2, f_2) \geq \alpha \min\{m_2, f_2\}$. Then the solution semiflow has a compact attractor of neighborhood of compact sets in X_+.*

By Theorem 6.2. in Thieme [12], we can prove that positive equilibrium exists.

Theorem 10. *Assume that $r > 1$, β is decreasing function, $\beta(x) \to 0$ as $x \to \infty$, and there exists some $\alpha > 0$ such that $\phi(m_2, f_2) \geq \alpha \min\{m_2, f_2\}$. Then there exists equilibrium $x \in X_+$ with $||x|| > 0$.*

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