数理解析研究所講究録 第2098巻 2018年 57-59

APPLICATION OF FRAGMENTATION NORMS TO TRANSPORTED POINTS BY HAMILTONIAN ISOTOPIES

MORIMICHI KAWASAKI AND RYUMA ORITA

ABSTRACT. We prove that a certain C^0-robust condition of a Hamiltonian function H induces the existence of a point transported ε out of the original point by the Hamiltonian diffeomorphism φ_H. Related to our observation, we provide a problem on Hamiltonian pseudo-rotations.

1. MAIN RESULT

Let (M,ω) be a symplectic manifold. For a Hamiltonian $H\colon S^1\times M\to\mathbb{R}$ with compact support, we set $H_t=H(t,\cdot)$ for $t\in S^1=\mathbb{R}/\mathbb{Z}$. The *Hamiltonian vector field* X_{H_t} associated with H_t is a time-dependent vector field defined by the formula

$$\omega(X_{H_t},\cdot)=-dH_t.$$

The *Hamiltonian isotopy* $\{\varphi_H^t\}_{t\in\mathbb{R}}$ associated with H is defined by

$$\begin{cases}\varphi_H^0=\mathrm{id}_M,\\ \frac{d}{dt}\varphi_H^t=X_{H_t}\circ\varphi_H^t \quad \text{for all } t\in\mathbb{R},\end{cases}$$

and its time-one map $\varphi_H=\varphi_H^1$ is referred to as the *Hamiltonian diffeomorphism* (*with compact support*) generated by H. We denote by $\mathrm{Ham}(M)$ the group of Hamiltonian diffeomorphisms of M with compact support.

In his famous work [B], Banyaga proved the simplicity of $\mathrm{Ham}(M)$ when M is closed. The key ingredient was the proof of the fragmentation lemma for this group, which enables us to define fragmentation norms on $\mathrm{Ham}(M)$ as follows. Let U be an open subset of M. The fragmentation lemma implies that for every $\phi\in\mathrm{Ham}(M)$ there exists a positive integer n such that ϕ can be represented as a product of n diffeomorphisms $\theta_i\in\mathrm{Ham}\big(\psi_i(U)\big)$, where $\psi_i\in\mathrm{Ham}(M)$ and $1\le i\le n$. For $\phi\neq\mathrm{id}_M$, its *fragmentation norm* $\|\phi\|_U$ with respect to the open subset U is defined to be the minimal number of factors in such a product. We set $\|\phi\|_U=0$ when $\phi=\mathrm{id}_M$.

Definition 1.1. Let U be an open subset of M. The fragmentation norm $\|\cdot\|_U$ is *controlled by the C^0-topology* if there exist an C^0-open neighborhood $\mathcal{U}\subset\mathrm{Ham}(M)$ of the identity and a positive number $C>0$ such that $\|\phi\|_U<C$ for any $\phi\in\mathcal{U}$.

Entov, Polterovich and Py proved the following interesting theorem.

Theorem 1.2 ([EPP, Theorem 4])**.** *Let Σ be a compact Riemann surface* (*possibly with boundary*) *equipped with an area form. Let $U\subset\Sigma$ be an open subset diffeomorphic to an open disc. Then the fragmentation norm $\|\cdot\|_U$ is controlled by the C^0-topology.*

Our main result is the following theorem which is an application of Theorem 1.2.

Date: August 31, 2018.
This work is supported by JSPS KAKENHI Grant Numbers JP18J00765, JP18J00335.

Theorem 1.3. *Let Σ_g be a closed Riemann surface of genus $g \geq 1$ equipped with an area form and a Riemannian metric. Let L_1 and L_2 be non-contractible simple closed curves in Σ_g. Then there exist positive numbers $\varepsilon_0 > 0$ and $C > 0$ such that for any Hamiltonian $H\colon S^1 \times \Sigma_g \to \mathbb{R}$ satisfying $H|_{S^1 \times L_1} > C$ and $H|_{S^1 \times L_2} < 0$, there exists $x \in \Sigma_g$ such that $d\big(x, \varphi_H(x)\big) > \varepsilon_0$.*

Here d is the distance induced by the Riemannian metric on Σ_g.

Remark 1.4. Since the Hamiltonians kH, $k \in \mathbb{Z}_{>0}$, also satisfy $kH|_{S^1 \times L_1} > C$ and $kH|_{S^1 \times L_2} < 0$, Theorem 1.3 yields that $\varphi_H^{k_i}$ never converges to the identity with respect to the C^0-topology for any sequence $k_i \to \infty$.

2. Proof of Theorem 1.3

To prove Theorem 1.3, we use the following theorem which is an application of Lagrangian spectral invariants [LZ, K].

Theorem 2.1 ([KO])**.** *Let Σ_g be a closed Riemann surface of genus $g \geq 1$ equipped with an area form and U a contractible open subset of Σ_g. Then, there exists a positive number r satisfying the following condition. Let L_1 and L_2 be non-contractible simple closed curves in Σ_g. For any $C > 0$ and any Hamiltonian $H\colon S^1 \times \Sigma_g \to \mathbb{R}$ satisfying $H|_{S^1 \times L_1} > C$ and $H|_{S^1 \times L_2} < 0$,*

$$\|\varphi_H\|_U > r \cdot C.$$

Proof of Theorem 1.3. Fix a Riemannian metric on Σ_g. For a positive number ε, we define a subset $\mathcal{U}_\varepsilon$ of $\mathrm{Ham}(\Sigma_g)$ by

$$\mathcal{U}_\varepsilon = \left\{ \phi \in \mathrm{Ham}(\Sigma_g) \;\middle|\; d\big(x, \phi(x)\big) \leq \varepsilon \text{ for any } x \in \Sigma_g \right\}.$$

By Theorem 1.2, there exist positive numbers C and ε_0 such that $\|\phi\|_U < C$ holds for any $\phi \in \mathcal{U}_{\varepsilon_0}$.

By Theorem 2.1, for any Hamiltonian $H\colon S^1 \times \Sigma_g \to \mathbb{R}$ satisfying $H|_{S^1 \times L_1} > C/r$ and $H|_{S^1 \times L_2} < 0$,

$$\|\varphi_H\|_U > r \cdot C/r = C.$$

Then $\varphi_H \notin \mathcal{U}_{\varepsilon_0}$ and hence, there exists a point x in Σ_g such that $d\big(x, \phi(x)\big) > \varepsilon_0$. □

3. A problem on Hamiltonian pseudo-rotations

Let $\mathbb{C}P^n$ be the n-dimensional complex projective space equipped with the Fubini–Study form ω_{FS}. A *Hamiltonian pseudo-rotation* of $\mathbb{C}P^n$ is a Hamiltonian diffeomorphism with exactly $n+1$ fixed points. Observe that this is the minimal possible number of fixed points of Hamiltonian diffeomorphisms of $\mathbb{C}P^n$. Ginzburg and Gürel proved the following crucial theorem.

Theorem 3.1 ([GG, Theorem 5.13])**.** *Let φ be a Hamiltonian pseudo-rotation of $\mathbb{C}P^n$ with exponentially Liouville mean index vector $\vec{\Delta}$ (see* [GG, Definition 5.11] *for the definition). Then there exists a sequence $k_i \to \infty$ such that*

$$\varphi^{k_i} \xrightarrow{C^0} \mathrm{id}.$$

We consider $(\mathbb{C}P^2, \omega_{\mathrm{FS}})$. The real projective space $\mathbb{R}P^2$ is naturally embedded in $(\mathbb{C}P^2, \omega_{\mathrm{FS}})$ as a Lagrangian submanifold. There is another Lagrangian submanifold L_W called the Chekanov torus which is disjoint from $\mathbb{R}P^2$. Then the authors proved the following theorem.

Theorem 3.2 ([KO])**.** *Let U be a displaceable open subset of $\mathbb{C}P^2$. Then, there exists a positive number r satisfying the following condition. For any $C > 0$ and any Hamiltonian $H\colon S^1 \times \mathbb{C}P^2 \to \mathbb{R}$ satisfying $H|_{S^1\times\mathbb{R}P^2} > C$ and $H|_{S^1\times L_W} < 0$,*

$$\|\varphi_H\|_U > r \cdot C.$$

Combining with Theorems 3.1 and 3.2, an argument similar to the proof of Theorem 1.3 yields the following corollary.

Corollary 3.3. *Assume that the fragmentation norm $\|\cdot\|_U$ is controlled by the C^0-topology. Let U be a displaceable open subset of $\mathbb{C}P^2$ and $H\colon S^1 \times \mathbb{C}P^2 \to \mathbb{R}$ a Hamiltonian satisfying $H|_{S^1\times\mathbb{R}P^2} > C$ and $H|_{S^1\times L_W} < 0$. Then φ_H is not a Hamiltonian pseudo-rotation of $\mathbb{C}P^2$ with exponentially Liouville mean index vector $\vec{\Delta}$.*

Thus, we pose the following problem.

Problem 3.4. *Does there exist a positive number $C > 0$ such that for any Hamiltonian $H\colon S^1 \times \mathbb{C}P^2 \to \mathbb{R}$ satisfying $H|_{S^1\times\mathbb{R}P^2} > C$ and $H|_{S^1\times L_W} < 0$, the Hamiltonian diffeomorphism φ_H is not a Hamiltonian pseudo-rotation of $\mathbb{C}P^2$ (i.e., φ_H has more than 3 fixed points)?*

References

[B] A. Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comment. Math. Helv.* **53** (1978) no. 2 174–227.

[EPP] M. Entov, L. Polterovich and P. Py, *On continuity of quasimorphisms for symplectic maps*, With an appendix by Michael Khanevsky. Progr. Math., 296, Perspectives in analysis, geometry, and topology, 169–197, Birkhäuser/Springer, New York (2012).

[GG] V. Ginzburg and B. Gürel, *Hamiltonian Pseudo-rotations of Projective Spaces*, to appear in Invent. Math., arXiv:1712.09766v1.

[K] M. Kawasaki, Function theoretical applications of Lagrangian spectral invariants, in preparation.

[KO] M. Kawasaki and R. Orita, *Disjoint superheavy subsets and fragmentation norms*, https://cgp.ibs.re.kr/archive/preprints/2018, Preprint (2018).

[LZ] R. Leclercq and F. Zapolsky, Spectral invariants for monotone Lagrangians, *J. Topol. Anal.*, Online Ready (17 May 2017), https://doi.org/10.1142/S1793525318500267.

(Morimichi Kawasaki) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: kawasaki@kurims.kyoto-u.ac.jp

(Ryuma Orita) Department of Mathematical Sciences, Tokyo Metropolitan University, Tokyo 192-0397, Japan

E-mail address: ryuma.orita@gmail.com

URL: https://ryuma-orita.github.io/