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Note on the spaces of real resultants with bounded multiplicity

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Abstract

For positive integers $d, m, n \geq 1$ with $(m, n) \neq (1, 1)$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{Q}_n^{d,m}(\mathbb{K})$ denote the space of m-tuples $(f_1(z), \cdots, f_m(z)) \in \mathbb{K}[z]^m$ of $\mathbb{K}$-coefficients monic polynomials of the same degree d such that polynomials $\{f_k(z)\}_{k=1}^m$ have no common *real* root of multiplicity $\geq n$ (but may have complex common root of any multiplicity). These spaces can be regarded as one of generalizations of the spaces defined and studied by Arnold and Vassiliev [16]. In this paper, we study the homotopy types of $\mathrm{Q}_n^{d,m}(\mathbb{K})$ and announce the results obtained in [12].

1 Introduction

The basic notations. For connected spaces X and Y, let $\mathrm{Map}(X, Y)$ (resp. $\mathrm{Map}^*(X, Y)$) denote the space consisting of all continuous maps (resp. base point preserving continuous maps) from X to Y with the compact-open topology, and let $\mathbb{R}\mathrm{P}^N$ (resp. $\mathbb{C}\mathrm{P}^N$) denote the N-dimensional real projective (resp. complex projective) space. Note that $\mathrm{Map}(S^1, \mathbb{R}\mathrm{P}^N)$ has two path-components $\mathrm{Map}_\epsilon(S^1, \mathbb{R}\mathrm{P}^N)$ for $\epsilon \in \{0, 1\}$ when $N \geq 2$. It is well-known that any map $f \in \mathrm{Map}_\epsilon(S^1, \mathbb{R}\mathrm{P}^N)$ lifts to the map $F \in \mathrm{Map}(S^1, S^N)$ such that $F(-x) = (-1)^\epsilon F(x)$ for any $x \in S^1$. For each $\epsilon \in \{0, 1\}$, let $\Omega_\epsilon \mathbb{R}\mathrm{P}^N$ denote the path component given by $\Omega_\epsilon \mathbb{R}\mathrm{P}^N = \mathrm{Map}_\epsilon(S^1, \mathbb{R}\mathrm{P}^N) \cap \mathrm{Map}^*(S^1, \mathbb{R}\mathrm{P}^N)$.

The motivation. The principal motivation of this research derived from the results obtained by Vassiliev [16]. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{P}_n^d(\mathbb{K})$ denote the space of all $\mathbb{K}$-coefficients monic polynomials $f(z) \in \mathbb{K}[z]$ of degree d which have no *real* root of multiplicity $\geq n$ (but may have complex ones of arbitrary multiplicity). By identifying $S^1 = \mathbb{R} \cup \{\infty\}$ and $\mathbb{C} = \mathbb{R}^2$, we have *the jet map*

$$j_{n,\mathbb{K}}^{d,1} : \mathrm{P}_n^d(\mathbb{K}) \to \Omega_{[d]_2} \mathbb{R}\mathrm{P}^{d(\mathbb{K})n-1} \simeq \Omega S^{d(\mathbb{K})n-1} \tag{1.1}$$

defined by

$$j_{n,\mathbb{K}}^{d,1}(f(z))(\alpha) = \begin{cases} [f(\alpha) : f(\alpha) + f'(\alpha) : \cdots : f(\alpha) + f^{(n-1)}(\alpha)] & \text{if } \alpha \in \mathbb{R} \\ [1 : 1 : 1 : \cdots : 1] & \text{if } \alpha = \infty \end{cases}$$

for $(f(z),\alpha)\in \mathrm{P}^d_n(\mathbb{K})\times S^1$, where $[d]_2\in\{0,1\}$ and $d(\mathbb{K})$ denote the integers defined by

$$(1.2)\qquad [d]_2=\begin{cases}1 & \text{if } d\equiv 1 \pmod 2\\ 0 & \text{if } d\equiv 0 \pmod 2\end{cases} \quad\text{and}\quad d(\mathbb{K})=\dim_{\mathbb{R}}\mathbb{K}=\begin{cases}1 & \text{if } \mathbb{K}=\mathbb{R}\\ 2 & \text{if } \mathbb{K}=\mathbb{C}\end{cases}$$

For $\mathbb{K}=\mathbb{R}$, Vassiliev obtained the following result:

Theorem 1.1 ([16] (cf. [7], [9])). *The jet map* $j^{d,1}_{n,\mathbb{R}}:\mathrm{P}^d_n(\mathbb{R})\to\Omega_{[d]_2}\mathbb{R}\mathrm{P}^{n-1}\simeq\Omega S^{n-1}$ *is a homotopy equivalence through dimension* $(\lfloor\frac{d}{n}\rfloor+1)(n-2)-1$ *for* $n\geq 4$ *and a homology equivalence through dimension* $\lfloor\frac{d}{3}\rfloor$ *for* $n=3$*, where* $\lfloor x\rfloor$ *denotes the integer part of a real number* x*.* □

Remark 1.2. Remark that a map $f:X\to Y$ is called *a homotopy equivalence* (resp. *a homology equivalence*) *through dimension* N if the induced homomorphism

$$f_*:\pi_k(X)\to\pi_k(Y)\quad(\text{resp. } f_*:H_k(X,\mathbb{Z})\to H_k(Y,\mathbb{Z}))$$

is an isomorphism for any integer $k\leq N$. Similarly, when G is a group and $f:X\to Y$ is a G-equivariant map between G-spaces X and Y, the map f is called a *G-equivariant homotopy equivalence through dimension* N (resp. a *G-equivariant homology equivalence through dimension* N) if the restriction map $f^H=f|X^H:X^H\to Y^H$ is a homotopy equivalence through dimension N (resp. a homology equivalence through dimension N) for any subgroup $H\subset G$, where W^H denote the H-fixed subspace of a G-space W given by $W^H=\{x\in W: h\cdot x=x \text{ for any } h\in H\}$. □

The main purpose of this note is to generalize this result given in [9] for the space $\mathrm{Q}^{d,m}_n(\mathbb{K})$.

Basic definitions. From now on, let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, let $d,m,n\geq 1$ be positive integers such that $(m,n)\neq(1,1)$, and we always assume that z is a variable.

Definition 1.3. (i) Let $\mathrm{Q}^{d,m}_n(\mathbb{K})$ denote the space of m-tuples $(f_1(z),\cdots,f_m(z))\in\mathrm{P}^d(\mathbb{K})^m$ of $\mathbb{K}$-coefficients monic polynomials of the same degree d such that $f_1(z),\cdots,f_m(z)$ have no common *real* root of multiplicity $\geq n$ (but they may have a common *complex* root of any multiplicity).

(ii) Let $(f_1(z),\cdots,f_m(z))\in\mathrm{P}^d(\mathbb{K})^m$ be an m-tuple of monic polynomials of the same degree d. Then it is easy to see that $(f_1(z),\cdots,f_m(z))\in\mathrm{Q}^{d,m}_n(\mathbb{K})$ iff the derivative polynomials $\{f^{(k)}_j(z):1\leq j\leq m,\ 0\leq k<n\}$ have no common real root. Thus, by identifying $S^1=\mathbb{R}\cup\infty$, one can define define *the jet map*

$$(1.3)\qquad j^{d,m}_{n,\mathbb{K}}:\mathrm{Q}^{d,m}_n(\mathbb{K})\to\Omega_{[d]_2}\mathbb{R}\mathrm{P}^{d(\mathbb{K})mn-1}\simeq\Omega S^{d(\mathbb{K})mn-1}\quad\text{by}$$

$$(1.4)\qquad j^{d,m}_{n,\mathbb{K}}(f_1(z),\cdots,f_m(z))(\alpha)=\begin{cases}[\boldsymbol{f}_1(\alpha):\cdots:\boldsymbol{f}_m(\alpha)] & \text{if } \alpha\in\mathbb{R}\\ [1:1:\cdots:1] & \text{if } \alpha=\infty\end{cases}$$

for $(f_1(z),\cdots,f_m(z))\in \mathrm{Q}^{d,m}_n(\mathbb{K})$, where we identify $\mathbb{C}=\mathbb{R}^2$ in (1.4) if $\mathbb{K}=\mathbb{C}$, and $\boldsymbol{f}_k(z)$ $(k=1,\cdots,m)$ is the n-tuple of monic polynomials of the same degree d defined by

$$\text{(1.5)}\qquad \boldsymbol{f}_k(z)=(f_k(z),f_k(z)+f_k'(z),f_k(z)+f_k''(z),\cdots,f_k(z)+f_k^{(n-1)}(z)).$$

Note that $\mathrm{P}^d_n(\mathbb{K})=\mathrm{Q}^{d,1}_n(\mathbb{K})$, and that the map $j^{d,m}_{n,\mathbb{K}}$ coincides the map $j^{d,1}_{n,\mathbb{K}}$ given in (1.1) for $m=1$. Similarly, one can define a natural map

$$\text{(1.6)}\qquad i^{d,m}_{n,\mathbb{K}}:\mathrm{Q}^{d,m}_n(\mathbb{K})\to \mathrm{Q}^{d,mn}_1(\mathbb{K})\qquad \text{by}$$

$$\text{(1.7)}\qquad i^{d,m}_{n,\mathbb{K}}(f_1(z),\cdots,f_m(z))=\big(\boldsymbol{f}_1(z),\cdots,\boldsymbol{f}_m(z)\big).$$

It is well-known that there is a homotopy equivalence

$$\text{(1.8)}\qquad \Omega S^{N+1}\simeq S^N\cup e^{2N}\cup e^{3N}\cup\cdots\cup e^{kN}\cup e^{(k+1)N}\cup\cdots.$$

We will denote the kN-skeleton of ΩS^{N+1} by $J_k(\Omega S^{N+1})$, i.e.

$$\text{(1.9)}\qquad J_k(\Omega S^{N+1})\simeq S^N\cup e^{2N}\cup e^{3N}\cup\cdots\cup e^{(k-1)N}\cup e^{kN}.$$

This space is usually called the *k-stage James filtration* of ΩS^{N+1}. □

Previous results. Let $D(d;m,n,\mathbb{K})$ denote the positive integer defined by

$$\text{(1.10)}\qquad D(d;m,n,\mathbb{K})=(d(\mathbb{K})mn-2)(\Big\lfloor\frac{d}{n}\Big\rfloor+1)-1 =\begin{cases}(2mn-2)(\lfloor\frac{d}{n}\rfloor+1)-1 & \text{if } \mathbb{K}=\mathbb{C},\\ (mn-2)(\lfloor\frac{d}{n}\rfloor+1)-1 & \text{if } \mathbb{K}=\mathbb{R}.\end{cases}$$

Recall the following known results for the case $m=1$ or $n=1$.

Theorem 1.4 ([9], [13], [16], [17]). (i) *If $d(\mathbb{K})m\geq 4$ and $n=1$, the jet map*

$$j^{d,m}_{1,\mathbb{K}}:\mathrm{Q}^{d,m}_1(\mathbb{K})\to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{d(\mathbb{K})m-1}\simeq \Omega S^{d(\mathbb{K})m-1}$$

is a homotopy equivalence through dimension $D(d;m,1,\mathbb{K})$.

(ii) *If $d(\mathbb{K})n\geq 4$ and $m=1$, the jet map*

$$j^{d,1}_{n,\mathbb{K}}:\mathrm{Q}^{d,1}_n(\mathbb{K})=\mathrm{P}^d_n(\mathbb{K})\to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{d(\mathbb{K})n-1}\simeq \Omega S^{d(\mathbb{K})n-1}$$

is a homotopy equivalence through dimension $D(d;1,n,\mathbb{K})$.

(iii) *If $d(\mathbb{K})m\geq 4$ and $d(\mathbb{K})n\geq 1$, there are homotopy equivalences*

$$\mathrm{Q}^{d,1}_n(\mathbb{K})=\mathrm{P}^d_n(\mathbb{K})\simeq J_{\lfloor\frac{d}{n}\rfloor}(\Omega S^{d(\mathbb{K})n-1})\quad\text{and}\quad \mathrm{Q}^{d,m}_1(\mathbb{K})\simeq J_d(\Omega S^{d(\mathbb{K})m-1}).$$

Thus, there is a homotopy equivalence $\mathrm{Q}^{d,1}_n(\mathbb{K})=\mathrm{P}^d_n(\mathbb{K})\simeq \mathrm{Q}^{\lfloor\frac{d}{n}\rfloor,n}_1(\mathbb{K})$.

(iv) *In particular, if $(\mathbb{K},m)=(\mathbb{R},3)$ and $d\geq 1$ is an odd integer, there is a homotopy equivalence $\mathrm{Q}^{d,3}_1(\mathbb{R})\simeq J_d(\Omega S^2)$.* □

Note that the conjugation on $\mathbb{C}$ naturally induces a $\mathbb{Z}/2$-action on the space $\mathrm{Q}_n^{d,m}(\mathbb{C})$. From now on, we regard $\mathbb{R}\mathrm{P}^N$ as the $\mathbb{Z}/2$-space with trivial $\mathbb{Z}/2$-action, and recall the following result given in [9].

Theorem 1.5 ([9])**.** (i) *If $m \geq 4$, then the jet map*

$$j_{1,\mathbb{C}}^{d,m} : \mathrm{Q}_1^{d,m}(\mathbb{C}) \to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{2m-1} \simeq \Omega S^{2m-1}$$

is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D(d; m, 1, \mathbb{R})$.

(ii) *If $n \geq 4$, then the jet map*

$$j_{n,\mathbb{C}}^{d,1} : \mathrm{Q}_n^{d,1}(\mathbb{C}) = \mathrm{P}_n^d(\mathbb{C}) \to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{2n-1} \simeq \Omega S^{2n-1}$$

is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D(d; 1, n, \mathbb{R})$. □

2 The main results

The main purpose of this paper is to study the homotopy type of the space $\mathrm{Q}_n^{d,m}(\mathbb{K})$ and report about the generalizations of the above two theorems (Theorems 1.4 and 1.5) for the case $m \geq 2$ and the case $n \geq 2$. Note that the following results may be regarded as one of real analogues of the result obtained in [11] (cf. [5]). More precisely, the main results are stated as follows.

Theorem 2.1. *If $d(\mathbb{K})mn \geq 4$, the jet map*

$$j_{n,\mathbb{K}}^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{K}) \to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{d(\mathbb{K})mn-1} \simeq \Omega S^{d(\mathbb{K})mn-1}$$

is a homotopy equivalence through dimension $D(d; m, n, \mathbb{K})$. □

Note that the conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$-action on the space $\mathrm{Q}_n^{d,m}(\mathbb{C})$. Since the map $j_{n,\mathbb{C}}^{d,m}$ is a $\mathbb{Z}/2$-equivariant map and $(j_{n,\mathbb{C}}^{d,m})^{\mathbb{Z}/2} = j_{n,\mathbb{R}}^{d,m}$, we also obtain the following result.

Corollary 2.2. *If $mn \geq 4$, the jet map*

$$j_{n,\mathbb{C}}^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{C}) \to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{2mn-1} \simeq \Omega S^{2mn-1}$$

is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D(d; m, n, \mathbb{R})$. □

Corollary 2.3. *If $d(\mathbb{K})mn \geq 4$, the jet embedding*

$$i_{n,\mathbb{K}}^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{K}) \to \mathrm{Q}_1^{d,mn}(\mathbb{K})$$

is a homotopy equivalence through dimension $D(d; m, n, \mathbb{K})$. □

Theorem 2.4. *If $d(\mathbb{K})mn \geq 4$, there is a homotopy equivalence*

$$\mathrm{Q}_n^{d,m}(\mathbb{K}) \simeq J_{\lfloor \frac{d}{n} \rfloor}(\Omega S^{d(\mathbb{K})mn-1}).$$

Thus, there are homotopy equivalences $\mathrm{Q}_n^{d,m}(\mathbb{K}) \simeq \mathrm{Q}_{mn}^{d,1}(\mathbb{K}) \simeq \mathrm{Q}_1^{\lfloor \frac{d}{n} \rfloor, mn}(\mathbb{K})$. □

Remark 2.5. (i) The above results can be proved by using the Vassiliev spectral sequence ([1], [14], [16]) and the scanning maps ([6], [7], [8], [15]). The detail of their proofs are omitted and see [12] in detail.

(ii) For positive integers $d, m, n \geq 1$ with $(m, n) \neq (1, 1)$ and a field $\mathbb{F}$ with its algebraic closure $\overline{\mathbb{F}}$, let $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ denote the space of all m-tuples $(f_1(z), \cdots, f_m(z)) \in \mathbb{F}[z]^m$ of monic $\mathbb{F}$-coefficients polynomials of the same degree d such that polynomials $\{f_k(z)\}_{k=1}^m$ have no common root in $\overline{\mathbb{F}}$ of multiplicity $\geq n$. The space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ is first defined and studied by B. Farb and J. Wolfson [5] for investigation the homological density of algebraic cycles in a closed manifold. By the classical theory of resultants, the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$ is an affine variety defined by systems of polynomial equations $\{F_k\}_{k=1}^N$ with integer coefficients. Thus both varieties given by this system of equations can be defined over $\mathbb{Z}$ and (by extension of scalars or reduction modulo a prime number) over any field $\mathbb{F}$. So $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ is an affine variety for any field $\mathbb{F}$.

(iii) Since this system of equations can be obtained by using the generalized resultants, we shall call the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$ as the space of resultants with bounded multiplicity. Note that the space $\mathrm{Q}_n^{d,m}(\mathbb{K})$ can be regarded as one of generalizations of real analogues of the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$, Because of this reason, we shall call the space $\mathrm{Q}_n^{d,m}(\mathbb{K})$ as the space of *real* resultants of bounded multiplicity although it is not an affine variety.

(iv) The homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$ was already well investigated in [11] (cf. [3], [4], [7], [15]). □

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