A Brief Overview of the S-transforms

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1. Introduction: The S-transform

The traditional spectrum analysis, built on the theory of Fourier analysis, relies on the assumption that signals are stationary; that is, their statistical properties are time invariant. In reality, signals generated from real-world applications are finite-duration and non-stationary. Time-frequency analysis offers a variety of techniques that map a one-dimensional temporal signal into a function of both time and frequency. Such a function is called a time-frequency representation or spectrum describing the temporal variation of frequency content within the signal. Time-frequency analysis techniques are effective in detecting local signal structure and processing non-stationary signals. It have been applied successfully in a wide range of fields including geophysics, speech recognition, music analysis, oceanology and bio-medicine. Comprehensive reviews on the related theory and applications can be found in [1, 2, 3].

The Stockwell transform or the S transform (ST), proposed by Stockwell in 1996 [4], is a linear time-frequency analysis method. Let $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $\int_{-\infty}^{\infty} \psi(t)dt = 1$, the ST of a signal $x(t)$ in $L^2(\mathbb{R})$ with respect to the window function $\psi(t)$ is defined by

$$ST_x(t,f) = |f| \int_{-\infty}^{\infty} x(\tau)\overline{\psi(|f|(\tau - t))}e^{-j2\pi\tau f}d\tau, \quad t \in \mathbb{R}, \tag{1.1}$$

where $f \in \mathbb{R}/\{0\}$. At zero frequency $f = 0$, the ST is equal to the average of the signal, *i.e.*,

$$ST_x(t,0) = \int_{-\infty}^{\infty} x(\tau)d\tau. \tag{1.2}$$

The ST can also be defined in the frequency domain, *i.e.*,

$$ST_x(t,f) = \int_{-\infty}^{\infty} X(\alpha + f)\overline{\Psi\left(\frac{\alpha}{|f|}\right)}e^{j2\pi\alpha t}d\alpha, \tag{1.3}$$

where $t \in \mathbb{R}$ and $f \in \mathbb{R}/\{0\}$. $\Psi(f)$ are the Fourier spectrum of the signal x and the window function ψ, respectively. The discrete analog of Equation (1.3) is often used to compute the ST by taking advantage of the efficiency of the fast Fourier transform (FFT) algorithm. The original Stockwell transform was proposed with the Gaussian window, $\psi(t) = \frac{1}{\sqrt{2\pi}}e^{t^2/2}$. The window width is proportional to the inverse of the frequency variable. There are many functions eligible to be a localizing window function.The choice of the Gaussian window function is due to the fact that the joint time and frequency resolution reaches the lower bound of the uncertainty principle [2].

The ST was first derived as the "phase correction" of the continuous wavelet transform [4, 5] and thus it inherits the multi-scale resolution feature from the wavelet transform. In particular, the ST and the continuous Morlet wavelet is apart by a frequency dependent phase correction [4, 6, 7, 8]. But unlike the wavelet transform, the ST has the absolutely referenced phase information, *i.e.,* the phase information at any time given by the ST is always referenced to the Fourier phase of the signal at zero time. This absolutely referenced phase ensures that the time average of the ST spectrum returns the Fourier spectrum of the signal. Thus the ST has a closed relationship to the well-understood Fourier spectrum. The ST can also be interpreted as a modification of the short-time Fourier transform with a frequency-dependent window width. Such interpretation makes the ST a well-received tool for signal processing. Its underlying mathematics has been rigorously investigated in papers [9, 10, 7, 11].

Similar to the two-dimensional Fourier transform (2D FT), an extension of the ST to a two-dimensional signal is simply a product of two one-dimensional S-transforms (i.e., one along each dimension) [12]. However such a localizing window is rotational-sensitive and hence generates different weights for different spatial directions. This is not desirable when rotational difference is not considered. Zhou [13] then proposed the polar form of the 2D ST that is rotationally invariant and used it to differentiate texture in geophysics. The PST provides space-frequency information with a resolution inversely proportionally to the spatial inverse of frequencies. In addition, the PST is rotation-invariant. That is, rotating a local region of an image does not affect its local spectrum. The PST has been successfully used to establish image biomakers for multiple sclerosis and brain tumor [14, 15, 16].

2. On the Use of the S-Transforms

As a hybrid of short-time Fourier transform and wavelet transform, the ST therefore has quickly gained popularity in the signal processing community. Howerver, when using the conventional ST for practical applications there are still some challenges in terms of the clarify of the spectrum and computational intensity.

Since the precision of the joint time-frequency resolution given by the ST is limited by the uncertainty principle, the inverse frequency window width may not separate different signal components well especially when they are spread over a wide range of frequencies. Pinnegar[17] introduced variations of the ST with arbitrary and varying shape in order to determine the P-wave arrival time in a noisy seismogram. Guo *et al.* [18] proposed a modified version of the ST by multiplying the S-spectrum by a parametrized dilatation factor in such a way that low frequencies are amplified and high frequencies reduced. Sejdić *et al.* [19] considered the generalized ST (GST) with a choice of the parametrized window width given as $\frac{1}{|f|^q}$ while Pei and Wang [20] proposed a version of the GST with arbitrary linear scaling window width $\frac{p}{|f|}$. In their approaches, the concentration measure has been applied on the normalization form of the GST-spectrogram instead of energy distributions. Since the GST-spectrogram is not a $L^2(\mathbb{R})$ function, such a normalization will generate some bias in the representation. Combining the GSTs of [19] and [20] and implementing an auto-optimization selection of the parameters p and q to minimize the energy concentration of the GST, Liu [21] developed adaptive S-transforms (AST) to maximize the separation of different frequency components for studying the brain functions using magnetoencephalography (MEG) signals.

It is important to obtain well separated multiple components of the signal in order to accurately estimate their frequencies related to physiological activities or natural vibrations of objects. To further improve the readability of the ST spectrogram, Liu and Zhu [22] introduced the reassigned spectrogram of Stockwell transform by remapping the surface of the spectrogram of Stockwell transform. The reassigned spectrogram of Stockwell transform therefore has signal energy highly concentrated at the instantaneous frequency/group delay curves and greatly increases the resolution and readability of the time-frequency structure of the underlying signal.

One major limitation of using the ST for practical applications is its intensive computation. For a signal of length N, the full ST requires to compute N^2 ST coefficients and the total computational complexity is $O(N^3)$. This limits its use for large size or higher dimensional signals. This problem was initially addressed by Brown *et al.* [23] combining parallel and vector computations to provide a 25-fold reduction in computation time. The S-spectrum contains high amount of information redundancy. To improve its computational efficiency, the discrete orthonormal Stockwell transforms (DOST) are proposed [24, 25, 26]. The DOST is based on a set of orthonormal basis functions that localize the Fourier spectrum of the signal. It samples N ST coefficients resulting a time-frequency representation with zero information redundancy while retaining the advantageous phase properties of the ST. The fast DOST algorithms developed by [25, 26] reduce the computational complexity from $O(N^2 \cdot \log(N))$ to $O(N \cdot \log(N))$. The fast 2D DOSTs developed by [26, 29] make it possible to analyze image texture and compress images using the 2D ST. The development of the DOST releases the potential of the ST for more practical applications. However, due to its non-redundancy, the DOST provides a rather coarse time-frequency representation with its frequency resolution proportionally scaled to the logarithm of the frequency. Such a representation may not be always easy to interpret and be sufficient to reveal all the details within a specific signal. Certain amount of information redundancy producing a finer representation is sometimes preferable when analyzing a signal. More versatile algorithms [27, 28] have been developed using the framework of frames.

The ST has being successfully used in a wide range of applications including geophysical signal analysis [30, 13, 31, 32], power system analysis [33, 34, 35], image compression [39], bio-medical signal processing [36, 21, 22, 37, 16, 38], and ocean wave analysis [40]. Note that due to the large body of work related to the ST, this overview does not include all the relevant references.

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