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Time-localized solutions for some soliton equations

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1 Introduction

"Rogue wave" is one of the recent topics in nonlinear water waves, which describes sudden excitation of big amplitude in calm or moderate state of waves. The important characteristic of rogue waves is that they appear from nowhere and disappear without a trace [1]. Such type of solutions have been constructed for various soliton equations among which the focusing nonlinear Schrödinger (NLS) equation,

$$iu_t = u_{xx} + \frac{1}{2}|u|^2 u, \tag{1}$$

is the most famous and fundamental example. There are two types of rogue wave solutions for NLS eq. (1). One is the Akhmediev breather given by rational function of exponentials, and the other is the Peregrine rogue wave and its higher order ones which are rational solutions multiplied by a gauge factor [1]-[3]. Both of these solutions tend to the plane wave with finite amplitude in the limit $t \to \pm\infty$, and nontrivial wave structures appear only in a limited time region. In this sense the rogue waves are the solutions which are localized in time.

The rogue wave solutions have been studied mostly for soliton equations of complex field, such as the NLS equation, vector NLS equation, Yajima-Oikawa equation, Davey-Stewartson equation and so on. Especially the algebraic structures of rational rogue wave solutions have been revealed in detail [4],[5]. On the other hand, rogue waves for equations of real dependent variables are not well-studied relatively. In this paper we consider solutions localized in time for equations of real variables. The Boussinesq equation is one of the real variable equations which admit time-localized solutions. It should be noted that the rational rogue wave solutions for the Boussinesq equation were constructed in [6]-[8]. In section 2 we first classify the Boussinesq equation into several types according to the signs of coefficients and next consider the lowest order exponential rational solutions and rational solutions for all types. In some cases we obtain the Akhmediev breathers and rational rogue waves, and in some other cases singular solutions are derived. In section 3 the determinant expressions of multi-breather and higher order rational solutions are studied. Finally we demonstrate a simple way of constructing equations and their time-localized solutions by using the bilinear method in section 4.

2 Boussinesq equations and their basic solutions

The Boussinesq equation is written as

$$\pm u_{tt} = u_{xxxx} + 3(u^2)_{xx} + \sigma u_{xx}, \tag{2}$$

where σ is a constant. We can not change the sign $\pm$ in left-hand side by scaling of variables. We call eq. (2) the Boussinesq I equation if the sign is $+$, and we call it the

Boussinesq II equation if the sign is $-$, because they are derived from the KPI and KPII equations,

$$(u_t + u_{xxx} + 6uu_x)_x = \pm u_{yy},$$

respectively, by the reduction $u_t = \sigma u_x$ and rewriting y by t. We can change σ in eq. (2) by constant shift $u \to u+\text{const}$, thus the value of σ is linked with the boundary condition of u. Let us normalize the boundary value of u by

$$u \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{3}$$

Then the absolute value of σ can be changed by scaling of variables but the sign of σ is unchangeable. So we can take $\sigma = \pm 1$ or 0 without loss of generality.

The Boussinesq I equation is transformed into the bilinear form,

$$(D_x^4 + \sigma D_x^2 - D_t^2) f \cdot f = 0, \tag{4}$$

through the dependent variable transformation,

$$u = (2 \log f)_{xx}. \tag{5}$$

Substituting the perturbative form,

$$f = 1 + e^{ikx+\omega t} + e^{-ikx+\omega t} + Ae^{2\omega t}, \tag{6}$$

into the bilinear eq. (4) and determining ω and A, we obtain the Akhmediev breather solution,

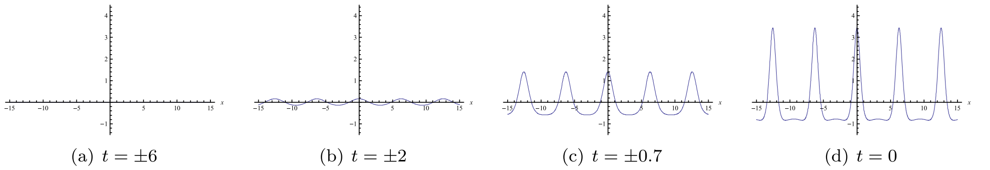
$$f \Bumpeq a \cosh \omega t - \cos kx, \tag{7}$$

$$\omega = k\sqrt{k^2 - \sigma}, \quad a = \sqrt{A} = \sqrt{\frac{4k^2 - \sigma}{k^2 - \sigma}},$$

where k is an arbitrary constant satisfying $k^2 > \sigma$. Here $\Bumpeq$ means equivalence by multiplication of an exponential factor and constant shifts of independent variables. From (5) we get

$$u = 2k^2 \frac{a \cosh \omega t \cos kx - 1}{(a \cosh \omega t - \cos kx)^2}, \tag{8}$$

which is regular and localized in time. Therefore the Boussinesq I equation admits the Akhmediev breather solution for any σ. Time evolution of the solution (8) is shown in Fig. 1. We note that the solution is symmetric in time reverse.



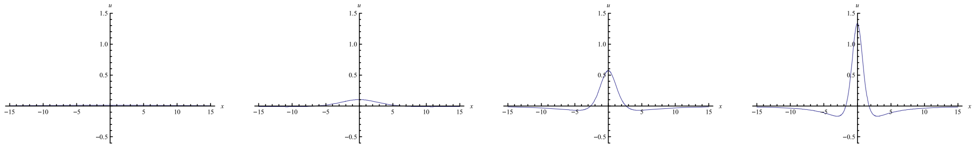
(a) $t = \pm 6$ (b) $t = \pm 2$ (c) $t = \pm 0.7$ (d) $t = 0$

Figure 1: (x, u)-plots of Akhmediev breather solution (8) with $k = 1$ for Boussinesq I equation with $\sigma = -1$.

The fundamental rational rogue wave solution is obtained by the limit $k \to 0$ in the above solution. The limit can be taken for $\sigma = -1$ and we get

$$f \Bumpeq t^2 + x^2 + 3, \quad u = 4\frac{t^2 - x^2 + 3}{(t^2 + x^2 + 3)^2}.$$

This is a regular and time-localized solution of Boussinesq I equation with $\sigma = -1$. Fig. 2 shows time evolution of the above rogue wave solution which has time reversal symmetry. In the case of $\sigma = +1$, we can not take the limit $k \to 0$ because of the condition $k^2 > \sigma$,



(a) $t = \pm 20$ (b) $t = \pm 6$ (c) $t = \pm 2$ (d) $t = 0$

Figure 2: (x, u)-plots of rational rogue wave solution for Boussinesq I equation with $\sigma = -1$.

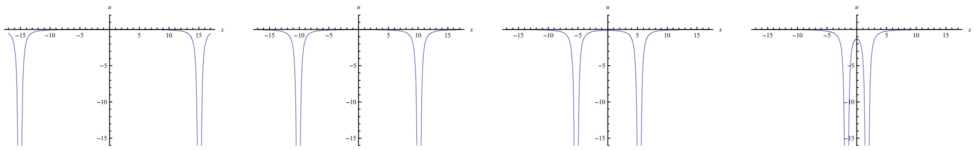
however we can start from the ansatz,

$$f = t^2 + Ax^2 + B, \tag{9}$$

and determine constants A and B from the bilinear eq. (4). Then we obtain a singular rational solution for $\sigma = +1$,

$$f = t^2 - x^2 + 3, \quad u = -4\frac{t^2 + x^2 + 3}{(t^2 - x^2 + 3)^2},$$

which describes repulsive interaction of a pair of singularities. Time evolution of this solution is shown in Fig. 3. For $\sigma = 0$, it is easy to check that there is no rational solution



(a) $t = \pm 15$ (b) $t = \pm 10$ (c) $t = \pm 5$ (d) $t = 0$

Figure 3: (x, u)-plots of singular rational solution for Boussinesq I equation with $\sigma = +1$.

of the form of (9).

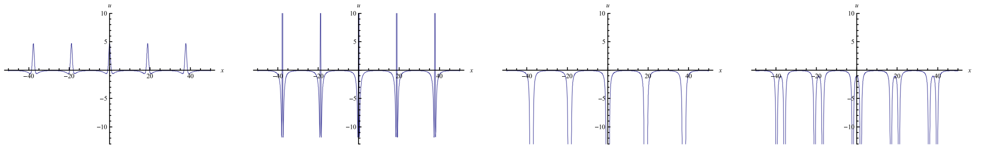
By using the variable transformation (5), the Boussinesq II equation is transformed into

$$(D_x^4 + \sigma D_x^2 + D_t^2) f \cdot f = 0. \tag{10}$$

Starting with the ansatz of perturbative form (6) and determining ω and A, we find that a blowing up breather solution exists for $\sigma > 0$ and there is no time-localized breather solution for $\sigma \leq 0$. For $\sigma > 0$, we have a singular breather solution (7) and (8) with

$$\omega = k\sqrt{\sigma - k^2}, \quad a = \sqrt{\frac{\sigma - 4k^2}{\sigma - k^2}},$$

where k is a constant satisfying $4k^2 < \sigma$. This breather is a finite-time blowing up solution but still localized in time, that is, singularities appear only in a finite interval around $t = 0$ and the vanishing condition (3) is satisfied. This solution is shown in Fig. 4.



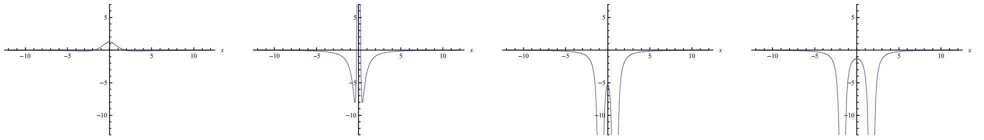
(a) $t = \pm 2.5$ (b) $t = \pm 2.28$ (c) $t = \pm 2.2$ (d) $t = 0$

Figure 4: (x, u)-plots of finite-time blowing up breather solution with $k = \frac{1}{3}$ for Boussinesq II equation with $\sigma = +1$.

For $\sigma = +1$, by taking the limit $k \to 0$ in the above blowing up breather, we get the finite-time blowing up rational rogue wave solution,

$$f \Bumpeq t^2 + x^2 - 3, \quad u = 4\frac{t^2 - x^2 - 3}{(t^2 + x^2 - 3)^2},$$

which is localized in time and shown in Fig. 5. For $\sigma = -1$, although the breather solution



(a) $t = \pm 2.5$ (b) $t = \pm 1.75$ (c) $t = \pm 1.5$ (d) $t = 0$

Figure 5: (x, u)-plots of finite-time blowing up rational rogue wave solution for Boussinesq II equation with $\sigma = +1$.

doesn't exist, we have a rational solution. In fact substituting (9) into the bilinear form of Boussinesq II equation (10) and determining A and B, we get a singular solution,

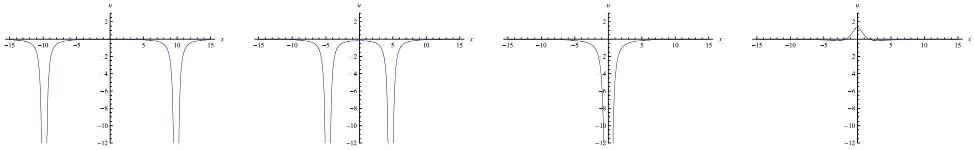
$$f = t^2 - x^2 - 3, \quad u = -4\frac{t^2 + x^2 - 3}{(t^2 - x^2 - 3)^2}.$$

This solution is not localized in time and describes the annihilation and creation of a pair of singularities (Fig. 6). There is no rational solution of the form of (9) for $\sigma = 0$.

3 Determinant structure of breather and rational solutions

For the Boussinesq I equation, the Akhmediev breathers can be superposed and the N-breather solution is given in terms of the Gram determinant,

$$\begin{aligned} f &= \det_{1\le i,j\le N}\left(\int_{-\infty}^{x}(e^{\xi_i} + e^{\eta_i})(e^{\xi_j^*} + e^{\eta_j^*})dx\right) \\ &= \det_{1\le i,j\le N}\left(\frac{e^{\xi_i+\xi_j^*}}{p_i + p_j^*} + \frac{e^{\xi_i+\eta_j^*}}{p_i + p_j} + \frac{e^{\eta_i+\xi_j^*}}{p_i^* + p_j^*} + \frac{e^{\eta_i+\eta_j^*}}{p_i^* + p_j}\right), \end{aligned}$$



(a) $t = \pm 10$ (b) $t = \pm 5$ (c) $t = \pm 1.75$ (d) $t = 0$

Figure 6: (x, u)-plots of singular rational solution for Boussinesq II equation with $\sigma = -1$.

$$\xi_j = p_j x + i\sqrt{3}p_j^2 t + \xi_{j0}, \quad \eta_j = p_j^* x + i\sqrt{3}p_j^{*2} t + \eta_{j0},$$

where p_j, ξ_{j0} and η_{j0} are arbitrary complex constants satisfying the reduction condition,

$$p_j^2 + p_j p_j^* + p_j^{*2} = -\frac{\sigma}{4},$$

and Re $p_j > 0$ for $j = 1, 2, \ldots, N$. Here * denotes complex conjugate. Since f is the determinant of positive definite Hermitian matrix, u in (5) gives regular solution. It is straightforward to prove that the above f actually satisfies the bilinear Boussinesq I equation (4) by using the Laplace expansion technique.

The higher order rational solutions are also expressed in the determinant form,

$$f = \begin{vmatrix} m_{10} & m_{11} & \cdots & m_{1,M-1} & m_{1,M+1} & m_{1,M+3} & \cdots & m_{1,2N-M-1} \\ m_{30} & m_{31} & \cdots & m_{3,M-1} & m_{3,M+1} & m_{3,M+3} & \cdots & m_{3,2N-M-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{2N-1,0} & m_{2N-1,1} & \cdots & m_{2N-1,M-1} & m_{2N-1,M+1} & m_{2N-1,M+3} & \cdots & m_{2N-1,2N-M-1} \end{vmatrix},$$

$$m_{ij} = A_i B_j \left. \frac{e^{(p+q)x + \sqrt{-3}\epsilon(p^2-q^2)t}}{p+q} \right|_{p=q=\sqrt{-\frac{\sigma}{12}}},$$

$$A_i = \sum_{k=0}^{i} \frac{a_{ik}}{(i-k)!}(p\partial_p)^{i-k}, \quad (i = 1, 3, \cdots, 2N-1),$$

$$B_j = \begin{cases} \displaystyle\sum_{l=0}^{j} \frac{b_{jl}}{(j-l)!}(q\partial_q)^{j-l}, & (j = M+1, M+3, \cdots, 2N-M-1), \\ \displaystyle\frac{1}{j!}(q\partial_q)^j, & (j = 0, 1, \cdots, M-1), \end{cases}$$

$$a_{ik} = \sum_{r=0}^{k} \frac{3^{r+1}-1}{(r+2)!} a_{i+2,k-r}, \quad (i = 1, 3, \cdots, 2N-3;\ 0 \le k \le i),$$

$$b_{jl} = \sum_{s=0}^{l} \frac{3^{s+1}-1}{(s+2)!} b_{j+2,l-s}, \quad (j = M+1, M+3, \cdots, 2N-M-3;\ 0 \le l \le j),$$

where ϵ is the sign ± 1 in the left-hand side of eq. (2) and $0 \le M \le N$. Here $a_{2N-1,k}$ $(0 \le k \le 2N-1)$ and $b_{2N-M-1,l}$ $(0 \le l \le 2N-M-1)$ are arbitrary constants, however they are not independent and we can take $a_{2N-1,0} = 1$, $a_{2N-1,2k} = 0$ $(1 \le k \le N-1)$,

$b_{2N-M-1,0} = 1$, $b_{2N-M-1,2l} = 0$ $(1 \le l \le \frac{2N-M-1}{2})$ without loss of generality. The coefficients a_{ik} $(1 \le i \le 2N-3)$ and b_{jl} $(M+1 \le j \le 2N-M-3)$ are recursively defined in the above way. The elements m_{ij} of determinant are given by the parameter derivatives. We can prove that the above f actually satisfies the bilinear form of Boussinesq equation in the same way with the case of NLS equation [5].

It should be pointed out that the above f is written in the form of two component determinant, that is, we have $N \times M$ matrix $(m_{2i-1,j-1})_{1\le i\le N, 1\le j\le M}$ on left, and $N \times (N-M)$ matrix $(m_{2i-1,2j-M-1})_{1\le i\le N, M+1\le j\le N}$ on right in the determinant. For soliton equations of complex variables such as NLS equation, the rational solutions are expressed by the single component Gram determinant which corresponds to the case of $M = 0$. This is because we need $M = 0$ to satisfy the complex conjugate condition. For real variable equations such as Boussinesq equation, the determinant solutions are not restricted by the conjugate condition and we have a free parameter M. Thus a wider class of rational solutions is obtained when the reality condition of f is satisfied. However it is still unclear whether those solutions are regular and whether they are localized in time.

4 Other equations

The bilinear method can be applied to nonintegrable equations also. In general the bilinear equation,

$$P(D)f \cdot f = 0,$$

where $P(D)$ is a polynomial of $D_x, D_y, D_t, \ldots$, admits at least 2-soliton solution and 1-breather solution of the form,

$$f = 1 + e^{\xi} + e^{\xi^*} + Ae^{\xi+\xi^*}, \quad \xi = px + qy + \omega t + \cdots .$$

Thus it might be possible to find regular and time-localized breather solutions among the above f. One simple example is the bilinear equation,

$$(D_x^4 + \sigma D_y^2 - D_t^2) f \cdot f = 0,$$

which is transformed to

$$u_{tt} = u_{xxxx} + 3(u^2)_{xx} + \sigma u_{yy},$$

through the variable transformation (5). This equation admits the breather solution,

$$f \backsimeq \sqrt{\frac{4k^4 - \sigma l^2}{k^4 - \sigma l^2}} \cosh(\sqrt{k^4 - \sigma l^2}\, t) - \cos(kx + ly),$$

where k and l are constants satisfying $k^4 > \sigma l^2$, and if $\sigma = -1$, we have the rational solution,

$$f \backsimeq t^2 + (kx + y)^2 + 3k^4.$$

Another example of bilinear equation,

$$(D_x^4 + D_x^2 D_y^2 - D_t^2) f \cdot f = 0,$$

is transformed to

$$\begin{aligned} &u_{tt} = u_{xxxx} + 3(u^2)_{xx} + u_{xxyy} + (2v_x^2 + uv_y)_{xx}, \\ &u_y = v_{xx}, \end{aligned}$$

by the variable transformations (5) and $v = (2\log f)_y$. This equation has the time-localized breather solution,

$$f \Bumpeq 2\cosh(k\sqrt{k^2+l^2}\,t) - \cos(kx+ly),$$

and we also find another breather type solution,

$$f \Bumpeq 2\cosh(\sqrt{2}ky)\cosh(k^2t) - \sinh(\sqrt{2}ky)\cos(kx),$$

by straightforward calculation. The bilinear formalism provides a simple and useful way to construct these kinds of solutions and the equations admitting such solutions simultaneously.

5 Concluding remarks

According to the signs of coefficients, there are various types of solutions of the Boussinesq eq. (2), such as the Akhmediev breather, rational rogue wave, finite-time blowing up breather, finite-time blowing up rational rogue wave and solutions of interacting singularities. Only some cases of the equations and solutions are relevant for the nonlinear water waves, but the Boussinesq equations may appear in some other contexts of physics. It might be interesting to study interpretations of those solutions in various physical systems.

The multi Akhmediev breathers and higher order rational solutions are presented in the determinant form. The multi Akhmediev breather solutions are regular and describe time-localized excitation of waves. Investigating properties of the higher order rational solutions including regularity and time-localization may be a future work.

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