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SEVERAL MODELS FOR WAVE-STRUCTURE INTERACTIONS

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ABSTRACT. We present in this note two mathematical problems related to the study of floating objects. In the first one, the fluid is described through the one-dimensional nonlinear shallow water equations; the main difficulty is to solve the free boundary problem that consists in finding the location of the contact points between the surface of the water and the floating object. In the second one, the lateral sides of the object are assumed to be vertical, so that that the coordinates of the contact points are known, but we consider a fluid described by a Boussinesq model. The problem can be reduced to a transmission problem and the presence of the dispersive terms requires the development of new tools. We also describe the main points of the analysis and refer to [9] and [4] respectively for full details.

1. INTRODUCTION

The floating body problem consists in studying the motion of a mechanical system formed by:

- A fluid delimited above by a free surface. Here, for simplicity, we shall also consider the case where the fluid domain is bounded below by a flat bottom located at $z = -h_0$, and we also assume the fluid to be incompressible and in irrotational motion;
- A partially immersed solid $\mathcal{C}$. For simplicity, we shall consider the case of an object $\mathcal{C}$ which is fixed. In the case of a freely floating object, the motion of the solid would be governed by Newton's laws in which the gravity force (and possibly other external forces) should be complemented by the force and torque exerted by the liquid on the solid.

In the absence of the immersed object, the problem reduces therefore to the standard water waves equations, so that the floating body problem can be viewed as a water waves/structure interaction problem. One of the first authors to consider it was Fritz John [10, 11], under several simplifying assumptions; in particular, he considered the linearized equations around the rest state. Indeed, the full problem is quite complex since two free boundary problems are involved. As said above, the first one is the standard water waves problem consisting in describing the evolution of the surface of the fluid when it is in contact with the air. The second free boundary problem comes from the fact that the *wetted surface* $\partial_{\mathrm{w}}\mathcal{C}(t)$, i.e. the portion of the boundary of the solid in contact with the fluid, depends on time. John's approach is still widely used, and is for instance the core of wave-structure simulation softwares.

Taking into account the nonlinear effect is now feasible, but at a considerable computational cost since the Laplace equation for the velocity potential must be

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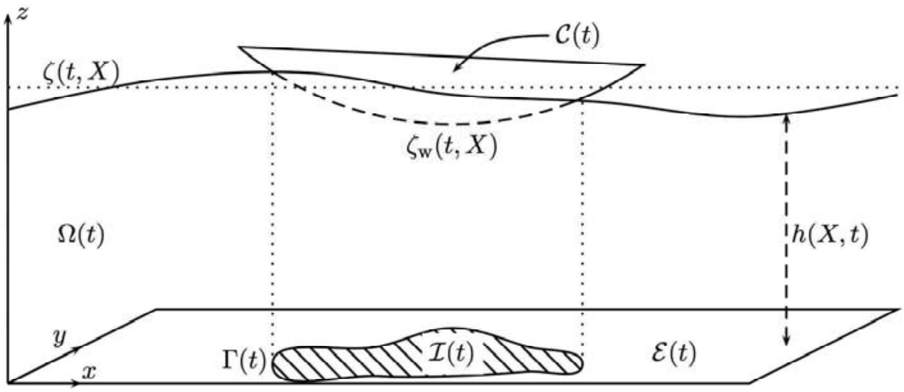

FIGURE 1. Notations

solved in a time dependant fluid domain that contains corners or wedges (at the contact line); we refer for instance to [6, 7, 8, 12]. Another possibility to be mentioned is a direct CFD approach based on the numerical resolution of the full Navier-Stokes equations (see for instance [15]).

Recently, a reduced formulation of the floating body problem was proposed in [14] in which a formulation of the water waves equation in terms of the surface elevation ζ above the rest state $z = 0$ and the horizontal discharge $Q(t,X) = \int_{-h_0}^{\zeta} V(t,X,z)\mathrm{d}z$ (V being the horizontal velocity in the fluid domain) is proposed, namely,

$$\begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ \partial_t Q + \nabla\cdot(\frac{1}{h}Q\otimes Q) + gh\nabla\zeta + \nabla\cdot\mathbf{R}(h,Q) + h\mathbf{a}_{\mathrm{NH}}(h,Q) = 0, \end{cases} \tag{1}$$

where ∇ denotes the gradient with respect to the horizontal coordinates, $h = h_0+\zeta$ is the water height, $\mathbf{R}(h,Q)$ measures the vertical variations of the horizontal velocity and $\mathbf{a}_{\mathrm{NH}}(h,Q)$ is the non-hydrostatic acceleration,

$$\mathbf{R}(h,Q) = \int_{-h_0}^{\zeta} (V-\overline{V})\otimes(V-\overline{V}), \tag{2}$$

$$\mathbf{a}_{\mathrm{NH}}(h,Q) = \frac{1}{h}\int_{-h_0}^{\zeta} \nabla\big[\int_z^{\zeta} \big(\partial_t V + V\cdot\nabla_{X,z}V\big)\cdot\mathbf{e}_z\big], \tag{3}$$

with $\mathbf{e}_z$ the vertical upward unit vector. It is shown in [14] that $\mathbf{R}$ and $\mathbf{a}_{\mathrm{NH}}$ are indeed functions of ζ and Q only (through the resolution of nonlocal elliptic equations).

Like the Zakharov-Craig-Sulem [16, 5] formulation, this formulation has the advantage of depenting only on the time and horizontal space variables, and are cast on the time independent horizontal plane $\mathbb{R}^d$. Another advantage is that it can be slightly modified to allow the presence of a floating object. In the configuration shown in Figure 1 where the wetted surface $\partial_{\mathrm{w}}\mathcal{C}(t)$ can be described as the graph of a function ζ_{w} over the interior region $\mathcal{I}(t)$ (the projection of the wetted surface on the horizontal coordinate plane), the equations (1) become, in the presence of an immersed object,

$$(4)\qquad \begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ \partial_t Q + \nabla\cdot(\frac{1}{h}Q\otimes Q) + gh\nabla\zeta + \nabla\cdot\mathbf{R}(h,Q) + h\mathbf{a}_{\mathrm{NH}}(h,Q) = -\frac{h}{\rho}\nabla\underline{P}, \end{cases}$$

where $\underline{P}$ stands for the pressure eveluated at the surface of the water. When this surface is in contact with the air (we denote by $\mathcal{E}(t)$ the horizontal projection of this region, and call it the *exterior region*), then $\underline{P}$ is assumed to be constant as in the water waves problem. Therefore, in the exterior region $\mathcal{E}(t)$, (4) coincides with the water waves equations (1). In the interior region $\mathcal{I}(t)$, things are different and we have $\underline{P} = \underline{P}_{\mathrm{i}}$, where $\underline{P}_{\mathrm{i}}$ is the *interior pressure*, i.e. the pressure exerted by the fluid on the bottom of the floating structure. This quantity is unknown and must be understood as the Lagrangian multiplier associated to the constraint

$$(5)\qquad \zeta(t,X) = \zeta_{\mathrm{w}}(X) \quad \text{on} \quad \mathcal{I}(t)$$

that is, the surface of the water coincides with the immersed bottom of the object (whose position is known here since we assumed that the object is fixed).

Of course, the above equations remain extremely complex. It was hinted in [14] that the same method could be adapted to simpler asymptotic models (instead of the full water waves equations). The aim of this note is to present and provide a mathematical study of two simple models in horizontal dimension $d = 1$. In this case, the interior region becomes an interval $\mathcal{I}(t) = (x_-(t), x_+(t))$ and the exterior region in $\mathcal{E}(t) = (-\infty, x_-(t)) \cup (x_+(t), +\infty)$ and $\Gamma(t) = \{x_-(t), x_+(t)\}$. These two models are the following:

- *The nonlinear shallow water equations with a floating object.* The problem is the following: find $x_-(t)$, $x_+(t)$, (ζ, q) and $(q_{\mathrm{i}}, \underline{P}_{\mathrm{i}})$ such that

$$(6)\qquad \begin{cases} \partial_t\zeta + \partial_x q = 0, \\ \partial_t q + \partial_x(\frac{1}{h}q^2) + gh\partial_x\zeta = 0 \end{cases} \quad \text{on} \quad (-\infty, x_-(t)) \cup (x_+(t), +\infty),$$

with

$$(7)\qquad \begin{cases} \partial_x q_{\mathrm{i}} = 0, \\ \partial_t q_{\mathrm{i}} + \partial_x(\frac{1}{h_{\mathrm{w}}}q_{\mathrm{i}}^2) + gh\partial_x\zeta_{\mathrm{w}} = -\frac{h}{\rho}\partial_x\underline{P}_{\mathrm{i}} \end{cases} \quad \text{on} \quad (x_-(t), x_+(t)),$$

and with the coupling conditions

$$(8)\qquad q(t, x_\pm(t)) = q_{\mathrm{i}}(t),$$

$$(9)\qquad \zeta(t, x_\pm(t)) = \zeta_{\mathrm{w}}(x_\pm(t)),$$

$$(10)\qquad \underline{P}_{\mathrm{i}}(t, x_\pm(t)) = P_{\mathrm{atm}},$$

the atmospheric pressure P_{atm} being assumed to be constant. We present in Section 2 a sketch of the proof of the results obtained in collaboration with T. Iguchi in [9] (where the case of freely floating objects is also considered).
- *The Boussinesq equations with a floating object with vertical sides.* We consider a more complex fluid model that, contrary to (6), includes dispersive terms (see [14, 3] for numerical simulations of wave structure interactions based on such models). On the other hand, we assume that the object is fixed and has vertical walls. A consequence of this assumption is that the coordinates $x_\pm(t)$ of the contact points are now time independent and located without loss of generality at $-R$ and R. Another consequence is

the coupling condition (9) must be removed and that one must subsitute (10) by another condition to be determined later.

In dimensionless variables (see details in Section 3), the problem is therefore the following: find (ζ, q) and $(q_{\rm i}, \underline{P}_{\rm i})$ such that

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{1}{3}\mu\partial_x^2)\partial_t q + \varepsilon \partial_x (q^2) + h\partial_x \zeta = 0 \end{cases} \quad \text{on} \quad (-\infty, -R) \cup (R, +\infty), \tag{11}$$

with ε and μ small dimensionless parameters and

$$\begin{cases} \partial_x q_{\rm i} = 0, \\ \partial_t q_{\rm i} + \partial_x (\frac{1}{h_{\rm w}} q_{\rm i}^2) + gh\partial_x \zeta_{\rm w} = -\frac{h}{\rho}\partial_x \underline{P}_{\rm i} \end{cases} \quad \text{on} \quad (-R, R), \tag{12}$$

and with the coupling conditions

$$q(t, \pm R) = q_{\rm i}(t), \tag{13}$$

$$\underline{P}_{\rm i}(t, \pm R) = P_{\rm atm} + \big(\zeta(t, \pm R) - \zeta_{\rm w}(\pm R)\big) + P_{\rm NH}, \tag{14}$$

where $P_{\rm NH}$ is a non-hydrostatic corrector that will be determined in Section 3 where we sketch some of the results obtained in collaboration with D. Bresch and G. Métivier in [4].

2. The nonlinear shallow water equations with a floating object

We recall that the problem consists in finding $x_-(t)$, $x_+(t)$, (ζ, q) and $(q_{\rm i}, \underline{P}_{\rm i})$ such that

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x (\frac{1}{h} q^2) + gh\partial_x \zeta = 0 \end{cases} \quad \text{on} \quad (-\infty, x_-(t)) \cup (x_+(t), +\infty), \tag{15}$$

with

$$\begin{cases} \partial_x q_{\rm i} = 0, \\ \partial_t q_{\rm i} + \partial_x (\frac{1}{h_{\rm w}} q_{\rm i}^2) + gh\partial_x \zeta_{\rm w} = -\frac{h}{\rho}\partial_x \underline{P}_{\rm i} \end{cases} \quad \text{on} \quad (x_-(t), x_+(t)), \tag{16}$$

and with the coupling conditions

$$q(t, x_\pm(t)) = q_{\rm i}(t), \tag{17}$$

$$\zeta(t, x_\pm(t)) = \zeta_{\rm w}(t, x_\pm(t)), \tag{18}$$

$$\underline{P}_{\rm i}(t, x_\pm(t)) = P_{\rm atm} \tag{19}$$

(see notations on Figure 2).

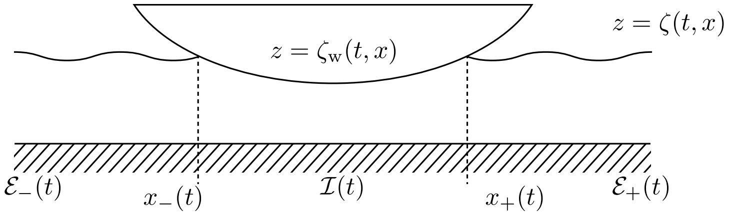

Figure 2. Waves interacting with a floating body

2.1. **Reformulation of the equations.** From the first equation of (16), the value if the discharge is independent of x in the interior region $(x_-(t), x_+(t))$; we therefore denote it $q_{\rm i}(t)$, while $\dot{q}_{\rm i}(t)$ stands for its time derivative. Integrating the second equation of (16) from $x_-(t)$ to $x_+(t)$ and using the boundary condition (19) yields

$$\dot{q}_{\rm i} = \frac{1}{\displaystyle\int_{x_-(t)}^{x_+(t)} \frac{1}{h_{\rm w}}} [\![\frac{1}{2}\frac{q_{\rm i}^2}{h_{\rm w}^2} + g h_{\rm w}]\!]$$

where for all function f, the notation $[\![f]\!]$ stands for $[\![f]\!] = f(x_+) - f(x_-)$.
The problem can therefore be reduced to an initial boundary value problem (with free boundary) cast in the exterior domain, namely,

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x (\frac{1}{h} q^2) + g h \partial_x \zeta = 0 \end{cases} \quad \text{on} \quad (-\infty, x_-(t)) \cup (x_+(t), +\infty), \tag{20}$$

with boundary conditions

$$q(t, x_\pm(t)) = q_{\rm i}(t) \quad \text{where} \quad \dot{q}_{\rm i} = \frac{1}{\displaystyle\int_{x_-(t)}^{x_+(t)} \frac{1}{h_{\rm w}}} [\![\frac{1}{2}\frac{q_{\rm i}^2}{h_{\rm w}^2} + g h_{\rm w}]\!] \tag{21}$$

and

$$\zeta(t, x_\pm(t)) = \zeta_{\rm w}(x_\pm(t)). \tag{22}$$

We also impose initial conditions of the form

$$(\zeta, q)_{|t=0} = (\zeta^{\rm in}, q^{\rm in}) \quad \text{in} \quad \underline{\mathcal{E}}, \qquad x_{\pm|t=0} = \underline{x}_\pm^{\rm in}, \qquad q_{{\rm i}|t=0} = q_{\rm i}^{\rm in}. \tag{23}$$

2.2. **Statement of the main result.** Usually, a 2×2 hyperbolic initial boundary value problem like (20) requires one boundary condition. The fact that we have here two boundary conditions makes the initial boundary value problem overdetermined, but this overdetermination is only apparent since we deal here with a *free boundary value problem*. Therefore, the second boundary condition (22) can be seen as the evolution equation for the free boundaries $x_\pm(t)$. The goal of this note is to explain simply the mechanisms at work, and we therefore only state a rough version of the result of [9] to which we refer for details, generalizations (to freely floating objects for instance) as well as for a general theory for hyperbolic free boundary problems.

Theorem 1. [9] *Assuming that the initial data are smooth enough and satisfy appropriate compatibility conditions, that the water depth at $t = 0$ is everywhere bounded from below by a positive constant, that the flow in the exterior domain is subcritical and that the following conditions hold at the initial contact points*

$$\sqrt{g(h_0 + \zeta^{\rm in})} - \Big| \frac{q^{\rm in}}{h_0 + \zeta^{\rm in} - \dot{x}_\pm(0)} \Big| > 0 \quad \textit{and} \quad \partial_x \zeta_{\rm w} \neq \partial_x \zeta^{\rm in},$$

there is a time $T > 0$ and a unique solution $(x_\pm, \zeta, q, q_{\rm i})$ solving (15)-(23) *on* $[0, T]$.

2.3. **Sketch of the proof.** We shall study here a general family of free boundary problems to which (20)-(22) can easily be related. We thus consider a 2×2 quasilinear hyperbolic system on a moving domain $(\underline{x}(t), \infty)$:

$$\partial_t U + A(U)\partial_x U = 0 \quad \text{in} \quad (\underline{x}(t), \infty) \tag{24}$$

with a boundary condition

$$U = U_{\mathrm{i}} \quad \text{on} \quad x = \underline{x}(t), \tag{25}$$

where $U_{\mathrm{i}} = U_{\mathrm{i}}(t, x)$ is a given $\mathbb{R}^2$-valued function, whereas $\underline{x}(t)$ is an unknown function. There are therefore two scalar boundary conditions; as previously explained, the problem is not overdetermined because one of these conditions must be understood as an evolution equation for the free boundary $\underline{x}$. Indeed, differentiating the boundary condition $U(t, \underline{x}(t)) = U_{\mathrm{i}}(t, \underline{x}(t))$ with respect to t and taking the Euclidean inner product of the resulting equation with $\partial_x U - \partial_x U_{\mathrm{i}}$, we obtain

$$\dot{\underline{x}} = \chi((\partial U)_{|x=\underline{x}}, (\partial U_{\mathrm{i}})_{|x=\underline{x}}), \tag{26}$$

where

$$\chi(\partial U, \partial U_{\mathrm{i}}) = -\frac{(\partial_x U - \partial_x U_{\mathrm{i}}) \cdot (\partial_t U - \partial_t U_{\mathrm{i}})}{|\partial_x U - \partial_x U_{\mathrm{i}}|^2}.$$

It has to be noted that this evolution equation is quite singular in the sense that it involves derivatives of U (while kinematic boundary conditions for instance would only involve U).

We assume throughout this section that there is $\mathcal{U}$ an open set in $\mathbb{R}^2$, that $A \in C^\infty(\mathcal{U})$ and that there exists $c_0 > 0$ such that for any $u \in \mathcal{U}$, the matrix $A(u)$ has eigenvalues $\lambda_+(u)$ and $-\lambda_-(u)$ satisfying $\lambda_\pm(u) \geq c_0$. This condition ensures that the system is strictly hyperbolic. We denote by $\mathbf{e}_\pm(u)$ normalized eigenvectors associated to the eigenvalues $\pm\lambda_\pm(u)$ of $A(u)$.
As shown by (26), a discontinuity of $\partial_x U$ at the free boundary is crucial so that we will work in a class of solutions satisfying

$$|(\partial_x U - \partial_x U_i)_{|x=x_\pm}| \geq c_0 \tag{27}$$

for some positive constant c_0 (in the context of the floating body problem, this is satisfied provided that the surface of the water makes a nonzero angle with the surface of the boat at the contact points).

2.3.1. *Fixing the boundary.* We use a diffeomorphism $\varphi(t, \cdot) : \mathbb{R}_+ \to (\underline{x}(t), \infty)$ (see [9] for the construction of an appropriate diffeomorphism) and put $u = U \circ \varphi$ and $u_{\mathrm{i}} = U_{\mathrm{i}} \circ \varphi$. We also write $\partial_t^\varphi u = (\partial_t U) \circ \varphi$ and $\partial_x^\varphi u = (\partial_x U) \circ \varphi$. Then, the free boundary problem (24)–(25) is recast as a problem on the fixed domain $\Omega_T = [0, T] \times \mathbb{R}_+$:

$$\begin{cases} \partial_t^\varphi u + A(u)\partial_x^\varphi u = 0 & \text{in} \quad \Omega_T, \\ u_{|x=0} = u_{\mathrm{i}|x=0} & \text{on} \quad (0, T). \end{cases} \tag{28}$$

We impose the initial conditions of the form

$$u_{|t=0} = u^{\mathrm{in}}(x) \quad \text{on} \quad \mathbb{R}_+, \qquad \underline{x}(0) = 0. \tag{29}$$

We also note that the equation (26) for the free boundary is then reduced to

$$\dot{\underline{x}} = \chi((\partial^\varphi u)_{|x=0}, (\partial^\varphi u_{\mathrm{i}})_{|x=0}). \tag{30}$$

The interior equation in (28) can be written as

$$\partial_t u + \mathcal{A}(u, \partial\varphi)\partial_x u = 0,$$

where $\mathcal{A}(u, \partial\varphi) = (\partial_x\varphi)^{-1}(A(u) - (\partial_t\varphi)\mathrm{Id})$. The eigenvalues of this matrix are $(\partial_x\varphi)^{-1}(\pm\lambda_\pm(u) - \partial_t\varphi)$, whereas the corresponding eigenvectors are $\mathbf{e}_\pm(u)$ which does not depend on $\partial\varphi$. To conserve the strict hyperbolicity of the initial problem, we assume further that

$$\lambda_\pm(u) \mp \partial_t\varphi \geq c_0 \quad \text{in} \quad (0,T)\times\mathbb{R}_+. \tag{31}$$

2.3.2. *Quasilinearization of the equations.* As usual with fully nonlinear problems, one is led to differentiate the equations in order to get a quasilinear structure. In the case of free boundary problems, an additional change of variable (the so-called Alinhac unknown [1]) is usually needed because of the dependence of the linearized equations on the diffeomorphism φ. Here, differentiating the equations once is not enough, so that we differenciate them twice to get a quasilinear structure, and we need to introduce a second order Alinhac unknown, namely, $u_{(2)} = \partial_t^\varphi \partial_t^\varphi u$.
- *Quasilinearization of the boundary condition.* We therefore differentiate twice the condition $u = u_{\mathrm{i}}$ on $x = 0$ and use the relation $\partial_t = \partial_t^\varphi + (\partial_t\varphi)\partial_x^\varphi$, to obtain after some computations

$$(\mathrm{Id} - \underline{\dot{x}}A(u)^{-1})^2 u_{(2)} + \underline{\ddot{x}}(\partial_x^\varphi u - \partial_x^\varphi u_{\mathrm{i}}) = g_1(\underline{\dot{x}}, u, \partial^\varphi u, \partial^\varphi\partial^\varphi u_{\mathrm{i}}),$$

where g_1 is a smooth function of its arguments. Decomposing this relation into the direction $\partial_x^\varphi u - \partial_x^\varphi u_{\mathrm{i}}$ and its perpendicular direction, we obtain an evolution equation for $\underline{x}$ as

$$\underline{\ddot{x}} = \chi_2(\underline{\dot{x}}, u, u_{(2)}, \partial^\varphi u, \partial^\varphi u_{\mathrm{i}}, \partial^\varphi\partial^\varphi u_{\mathrm{i}}),$$

where χ_2 is a smooth function of its arguments, and a boundary condition for $u_{(2)}$ as

$$\nu_{(2)} \cdot u_{(2)} = g_{(2)},$$

where $g_{(2)}$ is a smooth function of its arguments and

$$\nu_{(2)} = ((\mathrm{Id} - \underline{\dot{x}}A(u)^{-1})^2)^{\mathrm{T}}((\partial_x^\varphi u - \partial_x^\varphi u_{\mathrm{i}})^\perp).$$

-*Quasilinearization of the interior equation* Differentiating twice the equation satisfied by u one gets after some computations that

$$\partial_t u_{(2)} + \mathcal{A}(u, \partial\varphi)\partial_x u_{(2)} + B(u, \partial^\varphi u)u_{(2)} = f_{(2)}(u, \partial^\varphi u)$$

with B and $f_{(2)}$ depending smoothly on their arguments.
-*The quaslinear structure.* Summarizing the above arguments, the initial value problem (28)–(29) yields the following:

$$\begin{cases} \partial_t u + \mathcal{A}(u, \partial\varphi)\partial_x u = 0 & \text{in} \quad \Omega_T, \\ u_{|t=0} = u^{\mathrm{in}}(x) & \text{on} \quad \mathbb{R}_+, \\ \underline{\nu} \cdot u_{|x=0} = \underline{\nu} \cdot u_{\mathrm{i}|x=0} & \text{on} \quad (0,T), \end{cases} \tag{32}$$

(with $\underline{\nu}$ any vector non colinear wih the eigenvector associated to the negative eigenvalue of $\mathcal{A}$), together with

$$\begin{cases} \partial_t u_{(2)} + \mathcal{A}(u, \partial\varphi)\partial_x u_{(2)} + B(u, \partial^\varphi u)u_{(2)} = f_{(2)}(u, \partial^\varphi u) & \text{in} \quad \Omega_T, \\ u_{(2)|t=0} = u_{(2)}^{\mathrm{in}}(x) & \text{on} \quad \mathbb{R}_+, \\ \nu_{(2)} \cdot u_{(2)|x=0} = g_{(2)|x=0} & \text{on} \quad (0,T), \end{cases} \tag{33}$$

and an equation for the evolution of the free boundary given by

$$\begin{cases} \ddot{\underline{x}} = \chi(\dot{\underline{x}}, u, u_{(2)}, \partial^\varphi u, \partial^\varphi u_{\mathrm{i}}, \partial^\varphi \partial^\varphi u_{\mathrm{i}})_{|x=0} \quad \text{for} \quad t \in (0,T), \\ \underline{x}(0) = 0, \quad \dot{\underline{x}}(0) = x^{\mathrm{in}}_{(1)}, \end{cases} \tag{34}$$

where the initial data $u^{\mathrm{in}}_{(2)}$ and $x^{\mathrm{in}}_{(1)}$ should be chosen appropriately for the equivalence of (32)–(34) with (28)–(29).

2.3.3. *Conclusion.* The above system has a quasilinear structure and under the above assumption and appropriate compatibility conditions, one can construct a Kreiss symmetrizer (i.e. a symmetrizer for which the boundary condition is maximally dissipative), get energy estimates, and run a fixed point argument (we refer to [9] for full details).

3. The Boussinesq equations with a floating object with vertical sides

In this section, we consider the case a fluid described by the following Boussinesq model in the exterior domain,

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{1}{3}\mu\partial_x^2)\partial_t q + \varepsilon\partial_x(q^2) + h\partial_x\zeta = 0 \end{cases} \quad \text{on} \quad (-\infty, -R) \cup (R, +\infty); \tag{35}$$

the equations are here stated in dimensionless variables because the discussion on the parameters $\varepsilon = a/L$ and $\mu = H_0^2/L^2$ is important (a is the typical amplitude of the waves, H_0 the depth at rest, and L the typical horizontal scale for the waves). In the interior domain, since the object is fixed, we have

$$\begin{cases} \partial_x q_{\mathrm{i}} = 0, \\ \partial_t q_{\mathrm{i}} + h_{\mathrm{w}}\partial_x \zeta_{\mathrm{w}} = -\partial_x \underline{P}_{\mathrm{i}} \end{cases} \quad \text{on} \quad (-R, R), \tag{36}$$

(here $\underline{P}_{\mathrm{i}}$ is the dimensionless interior pressure) and we have the coupling conditions

$$q(t, \pm R) = q_{\mathrm{i}}(t, \pm R), \tag{37}$$

$$\underline{P}_{\mathrm{i}}(t, \pm R) = P_{\mathrm{atm}} + \big(\zeta(t, \pm R) - \zeta_{\mathrm{w}}(\pm R)\big) + P_{\mathrm{NH}}(\pm R), \tag{38}$$

where P_{NH} is a non-hydrostatic corrector that is determined below. We also impose initial conditions of the form

$$(\zeta, q)_{|t=0} = (\zeta^{\mathrm{in}}, q^{\mathrm{in}}) \quad \text{in} \quad \underline{\mathcal{E}}, \qquad q_{\mathrm{i}|t=0} = q_{\mathrm{i}}^{\mathrm{in}}. \tag{39}$$

The analysis of this system of equations is performed in [4]. As it is quite technical, we only mention here some of the key points that make the equations well posed.

3.1. **Finding the non-hydrostatic correction.** In order to solve the equations, it is necessary to find the non-hydrostatic correction P_{NH} in (38). This is done through an energy conservation argument. For the Boussinesq model (35), there is a local conservation of energy,

$$\partial_t \mathfrak{e} + \partial_x \mathfrak{F} = 0 \tag{40}$$

with

$$\mathfrak{e} = \frac{1}{2}\zeta^2 + \frac{\varepsilon}{6}\zeta^3 + \frac{1}{2}q^2 + \frac{\mu}{6}(\partial_x q)^2 \quad \text{and} \quad \mathfrak{F} = q\big[\zeta + \varepsilon\frac{2}{3}q^2 + \varepsilon\frac{1}{2}\zeta^2 - \frac{\mu}{3}\partial_x\partial_t q\big]$$

In the interior region, the energy density and flux are

$$\mathfrak{e}_{\rm i}=\frac{1}{2}\zeta_{\rm w}^2+\frac{\varepsilon}{6}\zeta_{\rm w}^3+\frac{1}{2}q_{\rm i}^2 \quad \text{and} \quad \mathfrak{F}_{\rm i}=q_{\rm i}\big[\zeta_{\rm w}+\varepsilon\frac{2}{3}q_{\rm i}^2+\varepsilon\frac{1}{2}\zeta_{\rm w}^2+\underline{P}_{\rm i}\big]$$

(recall that $\partial_x q_{\rm i}=0$) and the local conservation of energy reads

$$\partial_t\mathfrak{e}_{\rm i}+\partial_x\mathfrak{F}_{\rm i}=-0. \tag{41}$$

Since the object is fixed, the total energy $E_{\rm tot}$ of the fluid should be constant, where

$$E_{\rm tot}=\int_{|x|<R}\mathfrak{e}_{\rm i}+\int_{|x|>R}\mathfrak{e}.$$

Time differentiating and using (40) and (41), we impose therefore that

$$0=-[\![\mathfrak{F}_{\rm i}]\!]+[\![\mathfrak{F}]\!]$$

With the coupling conditions (37) and (38) this yields

$$[\![\varepsilon\frac{1}{2}\zeta_{\rm w}^2+P_{\rm NH}]\!]=[\![\varepsilon\frac{1}{2}\zeta^2-\frac{\mu}{3}\partial_x\partial_t q]\!].$$

We therefore impose the following value for the non-hydrostatic correction in (38),

$$P_{\rm NH}=\varepsilon\frac{1}{2}(\zeta^2-\zeta_{\rm w}^2)-\frac{\mu}{3}\partial_x\partial_t q \quad \text{at} \quad x=\pm R. \tag{42}$$

This choice corresponds therefore to a conservation of the total energy.

3.2. **Reformulation as a transmission problem.** Integrating the second equation in (36) from $-R$ to R, we get, owing to (42),

$$2R\dot{q}_{\rm i}+[\![\zeta_{\rm w}+\varepsilon\frac{1}{2}\zeta_{\rm w}^2]\!]=-[\![\zeta-\zeta_{\rm w}+\varepsilon\frac{1}{2}(\zeta^2-\zeta_{\rm w}^2)-\frac{\mu}{3}\partial_x\partial_t q]\!]$$

and therefore

$$2R\dot{q}_{\rm i}+[\![\zeta+\varepsilon\frac{1}{2}\zeta^2]\!]=\frac{\mu}{3}[\![\partial_x\partial_t q]\!].$$

We have therefore reduced the problem to the following transmission problem :

$$\begin{cases}\partial_t\zeta+\partial_x q=0,\\ (1-\frac{1}{3}\mu\partial_x^2)\partial_t q+\varepsilon\partial_x(q^2)+h\partial_x\zeta=0\end{cases} \quad \text{on} \quad (-\infty,-R)\cup(R,+\infty); \tag{43}$$

with transmission conditions

$$[\![q]\!]=0, \tag{44}$$

$$-\frac{\mu}{3}\partial_t[\![\partial_x q]\!]+[\![\zeta+\varepsilon\frac{1}{2}\zeta^2]\!]=-2R\dot{q}_{\rm i}, \tag{45}$$

where $q_{\rm i}$ is defined as the common value $q_{\rm i}=q(-R)=q(R)$. The system is completed by the initial condition

$$(\zeta,q)_{|t=0}=(\zeta^{\rm in},q^{\rm in}). \tag{46}$$

Note that the similar problem without the dispersive terms was considered in [14] in the $1D$ case, and in [2] in the $2D$-radial case.

3.3. **Mathematical analysis of the system.** We present here some important steps related to the mathematical analysis of (43)-(46) and more specifically to the proof of a well-posedness result over a time scale of order $O(1/\varepsilon)$ which is the relevant time scale for Boussinesq models without floating objects [13]

3.3.1. *Reducing to a linear transmission condition.* The fact that the transmission condition (45) is nonlinear complicates the analysis; we therefore rewrite the system in (θ, q) variables where

$$\theta = \zeta + \varepsilon\frac{1}{2}\zeta^2 \quad \text{or equivalently } \zeta = \theta + \varepsilon c(\theta) \quad \text{with} \quad c(\theta) = -\frac{2\theta^2}{(1+\sqrt{1+2\varepsilon\theta})^2}.$$

We thus obtain a transmission problem with linear transmission conditions, namely

$$\begin{cases} (1+\varepsilon c'(\theta))\partial_t\theta + \partial_x q = 0, \\ [1 - \frac{\mu}{3}\partial_x^2]\partial_t q + \varepsilon\partial_x(q^2) + \partial_x\theta = 0 \end{cases} \quad \text{on} \quad (-\infty, -R)\cup(R, +\infty) \tag{47}$$

with the linear transmission conditions

$$[\![q]\!] = 0, \tag{48}$$

$$-\frac{\mu}{3}\partial_t[\![\partial_x q]\!] + [\![\theta]\!] = -2R\dot{q}_{\rm i} \tag{49}$$

and the initial condition

$$(\theta, q)|_{t=0} = (\theta^{\rm in}, q^{\rm in}) \tag{50}$$

where $\theta^{\rm in} = \zeta^{\rm in} + \varepsilon\,\frac{1}{2}(\zeta^{\rm in})^2$.

3.3.2. *Reduction to an ODE.* Let R_0 denote the inverse of $(1-\frac{\mu}{3}\partial_x^2)$ on $(-\infty, -R)\cup(R, \infty)$ with Dirichlet condition at $\pm R$. The second equation in (47) can be written

$$\partial_t q = -R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big) + \dot{q}_{\rm i}\exp\big(-\sqrt{\frac{3}{\mu}}|x|_R\big), \tag{51}$$

with $|x|_R = (x-R)$ if $x > R$ and $-R-x$ if $x < -R$. Space differenciating yields

$$\partial_t\partial_x q = -\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big) \mp \sqrt{\frac{3}{\mu}}\dot{q}_{\rm i}\exp\big(-\sqrt{\frac{3}{\mu}}|x|_R\big),$$

so that

$$[\![\partial_t\partial_x q]\!] = -[\![\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big)]\!] - 2\sqrt{\frac{3}{\mu}}\dot{q}_{\rm i}.$$

Using (49) this gives

$$\frac{3}{\mu}[\![\theta]\!] + \frac{6R}{\mu}\dot{q}_{\rm i} = -[\![\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big)]\!] - 2\sqrt{\frac{3}{\mu}}\dot{q}_{\rm i}.$$

We therefore get the following expression for $\dot{q}_{\rm i}$,

$$\dot{q}_{\rm i} = -\frac{1}{6R + 2\sqrt{3\mu}}\big[3[\![\theta]\!] + [\![\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big)]\!]\big] =: q_{{\rm i},1}.$$

Plugging into (51) we have therefore the following formulation of the problem

$$\partial_t\begin{pmatrix}\theta\\ q\end{pmatrix} = \begin{pmatrix} -\partial_x q \\ R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big) - q_{{\rm i},1}\exp\big(-\sqrt{\frac{3}{\mu}}|x|_R\big)\end{pmatrix}. \tag{52}$$

If $n \geq 1$, it is quite easy to check that this is an ODE on $H^n \times H^{n+1}((-\infty, -R)\cup(R, \infty))$ so that one can get a solution in a standard way.

3.3.3. *Uniform estimates.* As seen above, it is possible to construct a solution to the transmission problem (47)-(50); however, the existence time and the bounds on the solution are not uniform when the parameters ε and μ become small. We sketch here the stategy developed in [4] to get uniform estimates:

(1) Uniform L^2-estimates for the linearized equations. This follows from computations similar to these performed above to derive the corrective term $P_{\rm NH}$ for the boundary value of the interior pressure in such a way that the total energy of the system is conserved.
(2) Uniform bounds on the time derivative of the solution of the linearized equations. These estimates can be obtained by time differentiating the equations and using the L^2-estimates. However, the resulting estimates involve the norm of $\partial_t^j \theta$ and $\partial_t^j q$ at $t = 0$. Contrary to what happens for hyperbolic systems, it is not straightforward to control these quantities in terms of Sobolev norms of the initial data. This requires the introduction of compatibility conditions that prevent the creation of *dispersive boundary layers.*
(3) Control of space derivatives. In the hyperbolic case, space derivatives can be controled in terms of time derivatives using the equations. Here, this is not directly possible because of the dispersive terms. The solution is to derive an ODE satisfied by $\partial_x \theta$ and to control it through an analysis of this ODE.
(4) Construction of a solution. Finding a appropriate functional space, one can run an iterative scheme using the previous steps.

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