数理解析研究所講究録 第2109巻 2019年 177-190

Kakinuma model for internal gravity waves in the rigid-lid case

Tatsuo Iguchi
Department of Mathematics, Keio University

1 Introduction

This article is based on an on-going joint research with Vincent Duchêne at Université de Rennes 1 in France. We consider the motion of internal gravity waves at the interface of two immiscible incompressible and inviscid fluids in $(n+1)$-dimensional space. For simplicity, we assume that the water surface of the upper layer is flat, that is, rigid-lid. Let t be the time, $x=(x_1,\ldots,x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume also that the interface, the rigid-lid, and the bottom are represented as $z=\zeta(x,t)$, $z=h_1$, and $z=-h_2+b(x)$, respectively, where $\zeta=\zeta(x,t)$ is the elevation of the internal layer, h_1 and h_2 are mean thicknesses of the upper and lower layers, and $b=b(x)$ represents the bottom topography. Therefore, the upper layer $\Omega_1(t)$ and the lower layer $\Omega_2(t)$ of the water have the form

$$\Omega_1(t)=\{X=(x,z)\in\mathbf{R}^{n+1}\,;\,\zeta(x,t)<z<h_1\},$$
$$\Omega_2(t)=\{X=(x,z)\in\mathbf{R}^{n+1}\,;\,-h_2+b(x)<z<\zeta(x,t)\}.$$

We denote the internal layer, the rigid-lid, and the bottom by $\Gamma(t)$, $\Sigma_{\rm l}$, and $\Sigma_{\rm b}$, respectively. Furthermore, we assume that the waters in the upper and the lower layers have constant densities ρ_1 and ρ_2, respectively, which satisfy Rayleigh's stability condition

$$(\rho_2-\rho_1)g>0,$$

where g is the gravitational constant.

As in the case of water waves, the basic equations for the internal gravity waves have a variational structure and a Lagrangian is given in terms of velocity potentials Φ_1 and Φ_2 in the upper and the lower layers and the interface elevation ζ. T. Kakinuma [7, 8, 9] approximated the velocity potentials Φ_1 and Φ_2 in the Lagrangian by

$$\Phi_1^{\rm app}(x,z,t)=\sum_{i=0}^{N_1}\Psi_{1i}(z;b)\phi_{1i}(x,t),\quad \Phi_2^{\rm app}(x,z,t)=\sum_{i=0}^{N_2}\Psi_{2i}(z;b)\phi_{2i}(x,t),$$

where $\{\Psi_{1i}\}$ and $\{\Psi_{2i}\}$ are appropriate function systems in the vertical coordinate z and may depend on the bottom topography b, whereas $\boldsymbol{\phi}_1 = (\phi_{10}, \phi_{11}, \ldots, \phi_{1N_1})$ and $\boldsymbol{\phi}_2 = (\phi_{20}, \phi_{21}, \ldots, \phi_{2N_2})$ are unknown variables. The Euler–Lagrange equation of the approximated Lagrangian in terms of $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$ is the Kakinuma model for internal gravity waves. Different choice of the function systems $\{\Psi_{1i}\}$ and $\{\Psi_{2i}\}$ yields different Kakinuma models and it is important to choose good function systems. In view of the mathematical analysis to the Isobe–Kakinuma model for water waves given by Y. Murakami and T. Iguchi [12], R. Nemoto and T. Iguchi [13], and T. Iguchi [4, 5], we will choose the approximated velocity potentials as

$$\Phi_1^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N}(z-h_1)^{2i}\phi_{1i}(x,t), \quad \Phi_2^{\mathrm{app}}(x,z,t) = \sum_{i=0}^{N^*}(z+h_2-b(x))^{p_i}\phi_{2i}(x,t), \quad (1)$$

where $p_0, p_1, \ldots, p_{N^*}$ are nonnegative integers satisfying $0 = p_0 < p_1 < \cdots < p_{N^*}$. In this article, according to the presence of the bottom topography we will chose these indices as follows:

(H1) In the case of the flat bottom $b(x) \equiv 0$, $N^* = N$ and $p_i = 2i$ $(i = 0, 1, \ldots, N)$

(H2) In the case of a general bottom topography, $N^* = 2N$ and $p_i = i$ $(i = 0, 1, \ldots, 2N)$

We analyze the linear dispersion relation of the Kakinuma model, which will be compared with that of the basic equations for the internal gravity waves. It is revealed that the Kakinuma model under our choice of the function system would be a higher order shallow water approximation to the internal gravity waves. Then, we will consider the linearized equations to the Kakinuma model around an arbitrary flow. After freezing coefficients, we analyze the linear dispersion relation and derive a stability condition. As was shown by T. Iguchi, N. Tanaka, and A. Tani [6] and D. Lannes [10], the initial value problem to the internal gravity waves is ill-posed and there is no stability regime. However, the initial value problem to the Kakinuma model is well-posed under the stability condition, although the model would be a higher order shallow water approximation. This is one of the advantages of the Kakinuma model.

2 Basic equations for internal gravity waves

The motion of the waters is described by the velocity potentials Φ_1 and Φ_2 and the pressures P_1 and P_2 in the upper and the lower layers satisfying the equations

$$\Delta_X \Phi_1 = 0 \quad \text{in} \quad \Omega_1(t), \quad (2)$$

$$\Delta_X \Phi_2 = 0 \quad \text{in} \quad \Omega_2(t), \quad (3)$$

where Δ_X is the Laplacian with respect to X, that is, $\Delta_X = \Delta + \partial_z^2$ and $\Delta = \partial_1^2 + \cdots + \partial_n^2$. Bernoulli's laws of each layers have the form

$$\rho_1\left(\partial_t\Phi_1 + \frac{1}{2}|\nabla_X\Phi_1|^2 + gz\right) + P_1 = 0 \quad \text{in} \quad \Omega_1(t), \tag{4}$$

$$\rho_2\left(\partial_t\Phi_2 + \frac{1}{2}|\nabla_X\Phi_2|^2 + gz\right) + P_2 = 0 \quad \text{in} \quad \Omega_2(t). \tag{5}$$

The dynamical boundary condition on the interface is given by

$$P_1 = P_2 \quad \text{on} \quad \Gamma(t). \tag{6}$$

The kinematic boundary conditions on the interface, on the rigid-lid, and on the bottom are given by

$$\partial_t\zeta + \nabla\Phi_1 \cdot \nabla\zeta - \partial_z\Phi_1 = 0 \quad \text{on} \quad \Gamma(t), \tag{7}$$

$$\partial_t\zeta + \nabla\Phi_2 \cdot \nabla\zeta - \partial_z\Phi_2 = 0 \quad \text{on} \quad \Gamma(t), \tag{8}$$

$$\partial_z\Phi_1 = 0 \quad \text{on} \quad \Sigma_{\mathrm{l}}, \tag{9}$$

$$\nabla\Phi_2 \cdot \nabla b - \partial_z\Phi_2 = 0 \quad \text{on} \quad \Sigma_{\mathrm{b}}. \tag{10}$$

These are the basic equations for the internal gravity waves. It follows form Bernoulli's laws (4)–(5) and the dynamical boundary condition (6) that

$$\rho_1\left(\partial_t\Phi_1 + \frac{1}{2}|\nabla_X\Phi_1|^2 + g\zeta\right) - \rho_2\left(\partial_t\Phi_2 + \frac{1}{2}|\nabla_X\Phi_2|^2 + g\zeta\right) = 0 \quad \text{on} \quad \Gamma(t). \tag{11}$$

It is easy to see that the basic equations (2)–(10) for unknowns $(\zeta, \Phi_1, \Phi_2, P_1, P_2)$ are equivalent to (2)–(3) and (7)–(11) for unknowns (ζ, Φ_1, Φ_2).

In the case of water waves, J. C. Luke [11] showed that the basic equations have a variational structure and his Lagrangian is given by the vertical integral of the pressure difference $P - P_{\mathrm{atm}}$ in the water region, where P_{atm} is an atmospheric pressure. Therefore, it is natural to expect that even in the case of internal gravity waves the vertical integral of the pressure in the water regions would give a Lagrangian $\mathscr{L}$, so that we first define $\mathscr{L}_{\mathrm{pre}}$ by

$$\mathscr{L}_{\mathrm{pre}} = \int_{-h_2+b(x)}^{\zeta(x,t)} P_2(x,z,t)\mathrm{d}z + \int_{\zeta(x,t)}^{h_1} P_1(x,z,t)\mathrm{d}z.$$

By using Bernoulli's laws (4)–(5) to remove the pressures P_1 and P_2, we see that

$$\begin{aligned}\mathscr{L}_{\mathrm{pre}} = &-\rho_2\int_{-h_2+b}^{\zeta}\left(\partial_t\Phi_2 + \frac{1}{2}|\nabla_X\Phi_2|^2\right)\mathrm{d}z - \rho_1\int_{\zeta}^{h_1}\left(\partial_t\Phi_1 + \frac{1}{2}|\nabla_X\Phi_1|^2\right)\mathrm{d}z \\ &- \frac{1}{2}(\rho_2-\rho_1)g\zeta^2 + \frac{1}{2}\left(\rho_2 g(-h_2+b)^2 - \rho_1 g h_1^2\right).\end{aligned}$$

The last term does not contribute the variation of this Lagrangian, so that we define a Lagrangian $\mathscr{L} = \mathscr{L}(\Phi_1, \Phi_2, \zeta)$ by

$$\begin{aligned}\mathscr{L}(\Phi_1, \Phi_2, \zeta) = -\rho_2 \int_{-h_2+b}^{\zeta} \left(\partial_t \Phi_2 + \frac{1}{2}|\nabla_X \Phi_2|^2\right) \mathrm{d}z - \rho_1 \int_{\zeta}^{h_1} \left(\partial_t \Phi_1 + \frac{1}{2}|\nabla_X \Phi_1|^2\right) \mathrm{d}z \\ - \frac{1}{2}(\rho_2 - \rho_1) g \zeta^2, \qquad (12)\end{aligned}$$

and the action function $\mathscr{J} = \mathscr{J}(\Phi_1, \Phi_2, \zeta)$ by

$$\mathscr{J}(\Phi_1, \Phi_2, \zeta) = \int_{t_0}^{t_1} \int_{\mathbf{R}^n} \mathscr{L}(\Phi_1, \Phi_2, \zeta) \mathrm{d}x \mathrm{d}t.$$

In fact, taking the first variation of this action function we have

$$\begin{aligned}&\delta \mathscr{J}(\Phi_1, \Phi_2, \zeta) \\ &= \rho_1 \int_{t_0}^{t_1} \int_{\Omega_1(t)} (\Delta_X \Phi_1) \delta\Phi_1 \mathrm{d}X \mathrm{d}t + \rho_2 \int_{t_0}^{t_1} \int_{\Omega_2(t)} (\Delta_X \Phi_2) \delta\Phi_2 \mathrm{d}X \mathrm{d}t \\ &\quad + \int_{t_0}^{t_1} \int_{\mathbf{R}^n} \left\{ \rho_1 \left(\partial_t \Phi_1 + \frac{1}{2}|\nabla_X \Phi_1|^2 + g\zeta \right) - \rho_2 \left(\partial_t \Phi_2 + \frac{1}{2}|\nabla_X \Phi_2|^2 + g\zeta \right) \right\} \Big|_{z=\zeta} \delta\zeta \mathrm{d}x \mathrm{d}t \\ &\quad - \rho_1 \int_{t_0}^{t_1} \int_{\mathbf{R}^n} (\partial_t \zeta + \nabla \Phi_1 \cdot \nabla \zeta - \partial_z \Phi_1) \delta\Phi_1 \big|_{z=\zeta} \mathrm{d}x \mathrm{d}t \\ &\quad + \rho_2 \int_{t_0}^{t_1} \int_{\mathbf{R}^n} (\partial_t \zeta + \nabla \Phi_2 \cdot \nabla \zeta - \partial_z \Phi_2) \delta\Phi_2 \big|_{z=\zeta} \mathrm{d}x \mathrm{d}t \\ &\quad - \rho_1 \int_{t_0}^{t_1} \int_{\mathbf{R}^n} (\partial_z \Phi_1) \delta\Phi_1 \big|_{z=h_1} \mathrm{d}x \mathrm{d}t \\ &\quad - \rho_2 \int_{t_0}^{t_1} \int_{\mathbf{R}^n} (\nabla \Phi_2 \cdot \nabla \zeta - \partial_z \Phi_2) \delta\Phi_2 \big|_{z=-h_2+b} \mathrm{d}x \mathrm{d}t,\end{aligned}$$

where we used integration by parts. Therefore, the corresponding Euler–Lagrange equations are exactly the same as the basic equations, that is, (2)–(3) and (7)–(11).

3 Kakinuma model

Plugging (1) into the Lagrangian (12), we obtain an approximate Lagrangian

$$\mathscr{L}^{\mathrm{app}}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta) := \mathscr{L}(\Phi_1^{\mathrm{app}}, \Phi_2^{\mathrm{app}}, \zeta),$$

where $\boldsymbol{\phi}_1 = (\phi_{10}, \phi_{11}, \ldots, \phi_{1N})^{\mathrm{T}}$ and $\boldsymbol{\phi}_2 = (\phi_{20}, \phi_{21}, \ldots, \phi_{2N^*})$. This approximate Lagrangian can be written explicitly as

$$
\begin{aligned}
\mathscr{L}^{\mathrm{app}} = -\rho_1 \Biggl\{ & \sum_{i=0}^{N} \frac{1}{2i+1} H_1^{2i+1} \partial_t \phi_{1i} \\
& + \frac{1}{2} \sum_{i,j=0}^{N} \left(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla\phi_{1i} \cdot \nabla\phi_{1j} + \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} \phi_{1i}\phi_{1j} \right) \Biggr\} \\
- \rho_2 \Biggl\{ & \sum_{i=0}^{N^*} \frac{1}{p_i+1} H_2^{p_i+1} \partial_t \phi_{2i} \\
& + \frac{1}{2} \sum_{i,j=0}^{N^*} \left(\frac{1}{p_i+p_j+1} H_2^{p_i+p_j+1} \nabla\phi_{2i} \cdot \nabla\phi_{2j} - \frac{2p_i}{p_i+p_j} H_2^{p_i+p_j} \phi_{2i} \nabla b \cdot \nabla\phi_{2j} \right. \\
& \qquad \left. + \frac{p_i p_j}{p_i+p_j-1} H_2^{p_i+p_j-1} (1+|\nabla b|^2) \phi_{2i}\phi_{2j} \right) \Biggr\} \\
- \frac{1}{2} & (\rho_2 - \rho_1) g \zeta^2,
\end{aligned}
$$

where H_1 and H_2 are thicknesses of the upper and the lower layers, that is,

$$
H_1(t,x) = h_1 - \zeta(x,t), \qquad H_2(x,t) = h_2 + \zeta(x,t) - b(x). \tag{13}
$$

The corresponding Euler–Lagrange equation is the Kakinuma model, which has the form

$$
\begin{cases}
H_1^{2i} \partial_t \zeta - \displaystyle\sum_{j=0}^{N} \left\{ \nabla \cdot \left(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla\phi_{1j} \right) - \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} \phi_{1j} \right\} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \ldots, N, \\
H_2^{p_i} \partial_t \zeta + \displaystyle\sum_{j=0}^{N^*} \left\{ \nabla \cdot \left(\frac{1}{p_i+p_j+1} H_2^{p_i+p_j+1} \nabla\phi_{2j} - \frac{p_j}{p_i+p_j} H_2^{p_i+p_j} \phi_{2j} \nabla b \right) \right. \\
\qquad\qquad \left. + \dfrac{p_i}{p_i+p_j} H_2^{p_i+p_j} \nabla b \cdot \nabla\phi_{2j} - \dfrac{p_i p_j}{p_i+p_j-1} H_2^{p_i+p_j-1} (1+|\nabla b|^2) \phi_{2j} \right\} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \ldots, N^*, \\
\rho_1 \left\{ \displaystyle\sum_{j=0}^{N} H_1^{2j} \partial_t \phi_{1j} + g\zeta + \frac{1}{2} \left(\left| \sum_{j=0}^{N} H_1^{2j} \nabla\phi_{1j} \right|^2 + \left(\sum_{j=0}^{N} 2j H_1^{2j-1} \phi_{1j} \right)^2 \right) \right\} \\
\quad - \rho_2 \left\{ \displaystyle\sum_{j=0}^{N^*} H_2^{p_j} \partial_t \phi_{2j} + g\zeta \right. \\
\qquad \left. + \dfrac{1}{2} \left(\left| \displaystyle\sum_{j=0}^{N^*} (H_2^{p_j} \nabla\phi_{2j} - p_j H_2^{p_j-1} \phi_{2j} \nabla b) \right|^2 + \left(\sum_{j=0}^{N^*} p_j H_2^{p_j-1} \phi_{2j} \right)^2 \right) \right\} = 0.
\end{cases} \tag{14}
$$

Here and in what follows we use the notational convention $0/0 = 0$.

In the case $N = 0$, that is, if we approximate the velocity potentials in the Lagrangian by functions independent of the vertical spatial variable z as

$$\Phi_1^{\mathrm{app}}(x,z,t) = \phi_1(x,t), \qquad \Phi_2^{\mathrm{app}}(x,z,t) = \phi_2(x,t),$$

then the Kakinuma model is reduced to the nonlinear shallow water equations

$$\begin{cases} \partial_t\zeta - \nabla\cdot((h_1-\zeta)\nabla\phi_1) = 0, \\ \partial_t\zeta + \nabla\cdot((h_2+\zeta-b)\nabla\phi_2) = 0, \\ \rho_1\Bigl(\partial_t\phi_1 + g\zeta + \dfrac{1}{2}|\nabla\phi_1|^2\Bigr) - \rho_2\Bigl(\partial_t\phi_2 + g\zeta + \dfrac{1}{2}|\nabla\phi_2|^2\Bigr) = 0. \end{cases} \tag{15}$$

4 Linear dispersion relation

Assuming that $b(x) \equiv 0$, we linearize the Kakinuma model (14) around the rest state. By putting $\boldsymbol{\varphi}_1 = (\phi_{10}, h_1^2\phi_{11}, \dots, h_1^{2N}\phi_{1N})^{\mathrm{T}}$ and $\boldsymbol{\varphi}_2 = (\phi_{20}, h_2^{p_1}\phi_{21}, \dots, h_2^{p_{N^*}}\phi_{2N^*})^{\mathrm{T}}$, the linearized equations have the form

$$\begin{pmatrix} 0 & -\rho_1\mathbf{1}^{\mathrm{T}} & \rho_2\mathbf{1}^{\mathrm{T}} \\ h_1\mathbf{1} & O & O \\ -h_2\mathbf{1} & O & O \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{pmatrix} + \begin{pmatrix} (\rho_2-\rho_1)g & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & -h_1^2A_1^{(0)}\Delta + A_1^{(1)} & O \\ \mathbf{0} & O & -h_2^2A_2^{(0)}\Delta + A_2^{(1)} \end{pmatrix} \begin{pmatrix} \zeta \\ \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \end{pmatrix} = \mathbf{0}, \tag{16}$$

where $\mathbf{1} = (1,\dots,1)^{\mathrm{T}}$ and matrices $A_k^{(0)}$ and $A_k^{(1)}$ for $k = 1,2$ are given by

$$A_1^{(0)} = \left(\frac{1}{2(i+j)+1}\right)_{0\le i,j\le N}, \qquad A_1^{(1)} = \left(\frac{4ij}{2(i+j)-1}\right)_{0\le i,j\le N},$$
$$A_2^{(0)} = \left(\frac{1}{p_i+p_j+1}\right)_{0\le i,j\le N^*}, \qquad A_2^{(1)} = \left(\frac{p_ip_j}{p_i+p_j-1}\right)_{0\le i,j\le N^*}.$$

Therefore, the linear dispersion relation to the Kakinuma model is given by

$$\det\begin{pmatrix} (\rho_2-\rho_1)g & \mathrm{i}\rho_1\omega\mathbf{1}^{\mathrm{T}} & -\mathrm{i}\rho_2\omega\mathbf{1}^{\mathrm{T}} \\ -\mathrm{i}h_1\omega\mathbf{1} & A_1(h_1\xi) & O \\ \mathrm{i}h_2\omega\mathbf{1} & O & A_2(h_2\xi) \end{pmatrix} = 0,$$

where $\xi \in \mathbf{R}^n$ is the wave vector, $\omega \in \mathbf{C}$ the angular frequency, and $A_k(\xi) = |\xi|^2A_k^{(0)} + A_k^{(1)}$ for $k = 1,2$. We can expand this dispersion relation as

$$\bigl(\rho_1h_1\det\tilde{A}_1(h_1\xi)\det A_2(h_2\xi) + \rho_2h_2\det\tilde{A}_2(h_2\xi)\det A_1(h_1\xi)\bigr)\omega^2 \\ - (\rho_2-\rho_1)g\det A_1(h_1\xi)\det A_2(h_2\xi) = 0,$$

where

$$\tilde{A}_k(\xi) = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A_k(\xi) \end{pmatrix}$$

for $k = 1, 2$. Therefore, the dispersion relation for the Kakinuma model has the form

$$\omega^2 = \frac{(\rho_2 - \rho_1)g \det A_1(h_1\xi) \det A_2(h_2\xi)}{\rho_1 h_1 \det \tilde{A}_1(h_1\xi) \det A_2(h_2\xi) + \rho_2 h_2 \det \tilde{A}_2(h_2\xi) \det A_1(h_1\xi)}. \tag{17}$$

Concerning the determinants appearing in the above dispersion relation, we have the following proposition.

Proposition 1 *1. For any $\xi \in \mathbf{R}^n \setminus \{\mathbf{0}\}$, the symmetric matrices $A_1(\xi)$ and $A_2(\xi)$ are positive.*

2. There exists $c_0 > 0$ such that for any $\xi \in \mathbf{R}^n$ we have $\det \tilde{A}_k(\xi) \geq c_0$ for $k = 1, 2$.

3. $|\xi|^{-2} \det A_1(\xi)$ and $|\xi|^{-2} \det A_2(\xi)$ are polynomials in $|\xi|^2$ of degree N and N^* and the coefficient of $|\xi|^{2N}$ and $|\xi|^{2N^*}$ are $\det A_1^{(0)}$ and $\det A_2^{(0)}$, respectively.

4. $\det \tilde{A}_1(\xi)$ and $\det \tilde{A}_2(\xi)$ are polynomials in $|\xi|^2$ of degree N and N^* and the coefficient of $|\xi|^{2N}$ and $|\xi|^{2N^*}$ are $\det \tilde{A}_1^{(0)}$ and $\det \tilde{A}_2^{(0)}$, respectively.

Thanks of this proposition and the dispersion relation (17), the linearized system (16) is classified into the dispersive system, so that the Kakinuma model is a nonlinear dispersive system of equations. Therefore, we can define the phase speed $c_{\mathrm{K}}(\xi)$ of the plane wave solution to (16) related to the wave vector $\xi \in \mathbf{R}^n$ by

$$c_{\mathrm{K}}(\xi) = \pm\sqrt{\frac{(\rho_2 - \rho_1)g|\xi|^{-2} \det A_1(h_1\xi) \det A_2(h_2\xi)}{\rho_1 h_1 \det \tilde{A}_1(h_1\xi) \det A_2(h_2\xi) + \rho_2 h_2 \det \tilde{A}_2(h_2\xi) \det A_1(h_1\xi)}}.$$

On the other hand, the phase speed $c_{\mathrm{IW}}(\xi)$ to the internal gravity waves is given by

$$c_{\mathrm{IW}}(\xi) = \pm\sqrt{\frac{(\rho_2 - \rho_1)g|\xi|^{-1} \tanh(h_1|\xi|) \tanh(h_2|\xi|)}{\rho_2 \tanh(h_1|\xi|) + \rho_1 \tanh(h_2|\xi|)}}$$

As a shallow water limit $h_1|\xi|, h_2|\xi| \to 0$, we have

$$c_{\mathrm{IW}}(\xi) \simeq c_{\mathrm{LIW}} = \pm\sqrt{\frac{(\rho_2 - \rho_1)gh_1h_2}{\rho_2 h_1 + \rho_1 h_2}}. \tag{18}$$

Here, c_{LIW} is the phase speed of the linear internal gravity waves. The following theorem is one of our main result in this article and shows that the Kakinuma model is a higher order shallow water approximation to the internal gravity waves at least in the linear level.

Theorem 1 *There exists a positive constant C depending only on N such that for any $\xi \in \mathbf{R}^n$ we have*

$$\left| \left(\frac{c_{\rm IW}(\xi)}{c_{\rm LIW}} \right)^2 - \left(\frac{c_{\rm K}(\xi)}{c_{\rm LIW}} \right)^2 \right| \le C(h_1|\xi| + h_2|\xi|)^{4N+2}.$$

Now, let us compare this error estimate with those of well-known models for internal gravity waves. In the case of the shallow water equations (15), the corresponding error estimate is

$$\left| \left(\frac{c_{\rm IW}(\xi)}{c_{\rm LIW}} \right)^2 - 1 \right| \le C(h_1|\xi| + h_2|\xi|)^{2}.$$

As for the Choi–Camassa model given in [2], the corresponding error estimate is

$$\left| \left(\frac{c_{\rm IW}(\xi)}{c_{\rm LIW}} \right)^2 - \left(\frac{c_{\rm CC}(\xi)}{c_{\rm LIW}} \right)^2 \right| \le C(h_1|\xi| + h_2|\xi|)^{4}.$$

Therefore, the Kakinuma model is a much higher shallow water approximation than the well-known models.

5 Stability condition

We linearize the equations in (14) around an arbitrary flow $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$ and denote the variation by $(\dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2, \dot{\zeta})$. By neglecting lower order terms, the linearized equations have the form

$$\begin{cases} \partial_t \dot{\zeta} + \boldsymbol{u}_1 \cdot \nabla \dot{\zeta} - \displaystyle\sum_{j=0}^{N} \frac{1}{2(i+j)+1} H_1^{2j+1} \Delta \dot{\phi}_{1j} = f_{1i} \quad \text{for} \quad i = 0, 1, \dots, N, \\ \partial_t \dot{\zeta} + \boldsymbol{u}_2 \cdot \nabla \dot{\zeta} + \displaystyle\sum_{j=0}^{N^*} \frac{1}{p_i + p_j + 1} H_2^{p_j+1} \Delta \dot{\phi}_{2j} = f_{2i} \quad \text{for} \quad i = 0, 1, \dots, N^*, \\ \rho_1 \displaystyle\sum_{j=0}^{N} H_1^{2j} (\partial_t \dot{\phi}_{1j} + \boldsymbol{u}_1 \cdot \nabla \dot{\phi}_{1j}) - \rho_2 \sum_{j=0}^{N^*} H_2^{p_j} (\partial_t \dot{\phi}_{2j} + \boldsymbol{u}_2 \cdot \nabla \dot{\phi}_{2j}) - a\dot{\zeta} = f_0, \end{cases} \tag{19}$$

where

$$\begin{cases} \boldsymbol{u}_1 = \displaystyle\sum_{i=0}^{N} H_1^{2i} \nabla \phi_{1i} = (\nabla \Phi_1^{\rm app})|_{z=\zeta}, \\ w_1 = -\displaystyle\sum_{i=0}^{N} 2i H_1^{2i-1} \phi_{1i} = (\partial_z \Phi_1^{\rm app})|_{z=\zeta}, \\ \boldsymbol{u}_2 = \displaystyle\sum_{i=0}^{N^*} (H_2^{p_i} \nabla \phi_{2i} - p_i H_2^{p_i-1} \phi_{2i} \nabla b) = (\nabla \Phi_2^{\rm app})|_{z=\zeta}, \\ w_2 = \displaystyle\sum_{i=0}^{N^*} p_i H_2^{p_i-1} \phi_{2i} = (\partial_z \Phi_2^{\rm app})|_{z=\zeta} \end{cases} \tag{20}$$

are approximate horizontal and vertical velocities in the upper and the lower layers on the interface and

$$\begin{aligned} a &= \rho_2\biggl(\sum_{i=0}^{N^*} p_i H_2^{p_i-1}(\partial_t\phi_{2i} + \boldsymbol{u}_2\cdot\nabla\phi_{2i}) + \sum_{i=0}^{N^*} p_i(p_i-1)H_2^{p_i-2}(w_2 - \boldsymbol{u}_2\cdot\nabla b)\phi_{2i} + g\biggr) \\ &\quad + \rho_1\biggl(\sum_{i=0}^{N} 2iH_1^{2i-1}(\partial_t\phi_{1i} + \boldsymbol{u}_1\cdot\nabla\phi_{1i}) + w_1\sum_{i=0}^{N} 2i(2i-1)H^{2(i-1)}\phi_{1i} - g\biggr) \\ &= \bigl(\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}})\bigr)|_{z=\zeta}. \end{aligned} \tag{21}$$

Here, P_1^{app} and P_2^{app} are approximate pressures in the upper and the lower layers calculated from Bernoulli's laws (4)–(5), that is,

$$P_1^{\mathrm{app}} = -\rho_1\biggl(\partial_t\Phi_1^{\mathrm{app}} + \frac{1}{2}|\nabla_X\Phi_1^{\mathrm{app}}|^2 + gz\biggr),$$
$$P_2^{\mathrm{app}} = -\rho_2\biggl(\partial_t\Phi_2^{\mathrm{app}} + \frac{1}{2}|\nabla_X\Phi_2^{\mathrm{app}}|^2 + gz\biggr).$$

Now, we freeze the coefficients in (19). Putting

$$\begin{cases} \dot{\boldsymbol{\varphi}}_1 = (\dot\phi_{10}, H_1^2\dot\phi_{11}, \ldots, H_1^{2N}\dot\phi_{1N})^{\mathrm{T}}, \\ \dot{\boldsymbol{\varphi}}_2 = (\dot\phi_{20}, H_2^{p_1}\dot\phi_{21}, \ldots, H_2^{p_{N^*}}\dot\phi_{2N^*})^{\mathrm{T}}, \end{cases}$$

we can rewrite (19) in a matrix form as

$$\begin{pmatrix} 0 & -\rho_1\mathbf{1}^{\mathrm{T}} & \rho_2\mathbf{1}^{\mathrm{T}} \\ H_1\mathbf{1} & O & O \\ -H_2\mathbf{1} & O & O \end{pmatrix} \partial_t \begin{pmatrix} \dot\zeta \\ \dot{\boldsymbol{\varphi}}_1 \\ \dot{\boldsymbol{\varphi}}_2 \end{pmatrix} + \begin{pmatrix} a & -\rho_1\mathbf{1}^{\mathrm{T}}(\boldsymbol{u}_1\cdot\nabla) & \rho_2\mathbf{1}^{\mathrm{T}}(\boldsymbol{u}_2\cdot\nabla) \\ H_1\mathbf{1}(\boldsymbol{u}_1\cdot\nabla) & -H_1^2A_1^{(0)}\Delta & O \\ -H_2\mathbf{1}(\boldsymbol{u}_2\cdot\nabla) & O & -H_2^2A_2^{(0)}\Delta \end{pmatrix} \begin{pmatrix} \dot\zeta \\ \dot{\boldsymbol{\varphi}}_1 \\ \dot{\boldsymbol{\varphi}}_2 \end{pmatrix} = \begin{pmatrix} -f_0 \\ \boldsymbol{f}_1 \\ -\boldsymbol{f}_2 \end{pmatrix}.$$

Therefore, the corresponding linear dispersion relation is given by

$$\det\begin{pmatrix} a & \mathrm{i}\rho_1(\omega - \boldsymbol{u}_1\cdot\xi)\mathbf{1}^{\mathrm{T}} & -\mathrm{i}\rho_2(\omega - \boldsymbol{u}_2\cdot\xi)\mathbf{1}^{\mathrm{T}} \\ -\mathrm{i}H_1(\omega - \boldsymbol{u}_1\cdot\xi)\mathbf{1} & (H_1|\xi|)^2A_1^{(0)} & O \\ \mathrm{i}H_2(\omega - \boldsymbol{u}_2\cdot\xi)\mathbf{1} & O & (H_2|\xi|)^2A_2^{(0)} \end{pmatrix} = 0,$$

which can be expanded as

$$\frac{\rho_1}{H_1a_1}(\omega - \boldsymbol{u}_1\cdot\xi)^2 + \frac{\rho_2}{H_2a_2}(\omega - \boldsymbol{u}_2\cdot\xi)^2 - a|\xi|^2 = 0, \tag{22}$$

where

$$a_k = \frac{\det A_k^{(0)}}{\det\tilde{A}_k^{(0)}}, \qquad \tilde{A}_k^{(0)} = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A_k^{(0)} \end{pmatrix} \tag{23}$$

for $k = 1, 2$. It is easy to see that the solutions ω to the dispersion relation (22) are real for any $\xi \in \mathbf{R}^n$ if and only if

$$a - \frac{\rho_1 \rho_2}{\rho_1 H_2 a_2 + \rho_2 H_1 a_1} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 \geq 0.$$

This leads us the following stability condition

$$a - \frac{\rho_1 \rho_2}{\rho_1 H_2 a_2 + \rho_2 H_1 a_1} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 \geq c_0 \tag{24}$$

for some positive constant c_0, which is equivalent to

$$\partial_z (P_2^{\mathrm{app}} - P_1^{\mathrm{app}}) - \frac{\rho_1 \rho_2}{\rho_1 H_2 a_2 + \rho_2 H_1 a_1} |\nabla \Phi_2^{\mathrm{app}} - \nabla \Phi_1^{\mathrm{app}}|^2 \geq c_0 \quad \text{on} \quad \Gamma(t).$$

We note that a_1 and a_2 are positive constants depending only on N and converges to 0 as $N \to \infty$. Therefore, as N goes to infinity the regime of the stability diminishes.

6 Well-posedness of the initial value problem

We proceed to consider the initial value problem to the Kakinuma model (14) under the initial condition

$$(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)|_{t=0} = (\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)}, \zeta_{(0)}). \tag{25}$$

Here, we remark that the Kakinuma model has a drawback, that is, the hypersurface $t = 0$ is characteristic for the Kakinuma model, so that the initial value problem to the Kakinuma model (14) and (25) is not solvable in general. In fact, if the problem has a solution $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$, then by eliminating the time derivative $\partial_t \zeta$ from the equations we see that the solution has to satisfy the relations

$$\begin{cases} H_1^{2i} \displaystyle\sum_{j=0}^{N} \nabla \cdot \left(\frac{1}{2j+1} H_1^{2j+1} \nabla \phi_{1j} \right) \\ \quad - \displaystyle\sum_{j=0}^{N} \left\{ \nabla \cdot \left(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla \phi_{1j} \right) - \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} \phi_{1j} \right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 1, 2, \ldots, N, \\ H_2^{p_i} \displaystyle\sum_{j=0}^{N} \nabla \cdot \left(\frac{1}{2j+1} H_1^{2j+1} \nabla \phi_{1j} \right) \\ \quad + \displaystyle\sum_{j=0}^{N^*} \left\{ \nabla \cdot \left(\frac{1}{p_i + p_j + 1} H_2^{p_i + p_j + 1} \nabla \phi_{2j} - \frac{p_j}{p_i + p_j} H_2^{p_i + p_j} \phi_{2j} \nabla b \right) \right. \\ \qquad \left. + \frac{p_i}{p_i + p_j} H_2^{p_i + p_j} \nabla b \cdot \nabla \phi_{2j} - \frac{p_i p_j}{p_i + p_j - 1} H_2^{p_i + p_j - 1} (1 + |\nabla b|^2) \phi_{2j} \right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0, 1, \ldots, N^*. \end{cases} \tag{26}$$

In the following we write $\boldsymbol{\phi}_1 = (\phi_{10}, \boldsymbol{\phi}_1')^{\mathrm{T}}$, $\boldsymbol{\phi}_2 = (\phi_{20}, \boldsymbol{\phi}_2')^{\mathrm{T}}$, $\boldsymbol{\phi}_{1(0)} = (\phi_{10(0)}, \boldsymbol{\phi}_{1(0)}')^{\mathrm{T}}$, and $\boldsymbol{\phi}_{2(0)} = (\phi_{20(0)}, \boldsymbol{\phi}_{2(0)}')^{\mathrm{T}}$. We denote by $H^m = H^m(\mathbf{R}^n)$ and $W^{m,\infty} = W^{m,\infty}(\mathbf{R}^n)$ the L^2 and the L^∞ Sobolev spaces of order m, respectively. The following theorem states that the initial value problem to the Kakinuma model is well-posed locally in time in the Sobolev space H^m under the necessary conditions (26), the stability condition (24), and positivity of the thicknesses of the upper and lower layers.

Theorem 2 *Let $g, \rho_1, \rho_2, h_1, h_2, c_0, M_0$ be positive constants and m an integer such that $m > n/2+1$. There exists a time $T > 0$ such that if the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)}, \zeta_{(0)})$ and the bottom topography b satisfy*

$$\begin{cases} \|(\nabla\phi_{10(0)}, \nabla\phi_{20(0)}, \zeta_{(0)})\|_{H^m} + \|(\boldsymbol{\phi}_{1(0)}', \boldsymbol{\phi}_{2(0)}')\|_{H^{m+1}} + \|b\|_{W^{m+2,\infty}} \le M_0, \\ h_1 - \zeta_{(0)}(x) \ge c_0, \quad h_2 + \zeta_{(0)}(x) - b(x) \ge c_0 \quad \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

the necessary conditions (26)*, and the stability condition* (24)*, then the initial value problem* (14) *and* (25) *to the Kakinuma model has a unique solution $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta)$ satisfying*

$$\nabla\phi_{10}, \nabla\phi_{20}, \zeta \in C([0,T]; H^m), \quad \boldsymbol{\phi}_1', \boldsymbol{\phi}_2' \in C([0,T]; H^{m+1}).$$

We note that the initial value problem to the full equations for internal gravity waves is ill-posed whereas the problem to the Kakinuma model is well-posed, although the Kakinuma model would be a higher order shallow water approximation of the full equations. This interesting inconsistency comes from the fact that as a deep water limit we have

$$\lim_{|\xi|\to\infty} \left(\frac{c_{\mathrm{K}}(\xi)}{c_{\mathrm{LIW}}}\right)^2 = \frac{(\rho_2 h_1 + \rho_1 h_2) \det A_1^{(0)} \det A_2^{(0)}}{\rho_2 h_1 \det A_1^{(0)} \det \tilde{A}_2^{(0)} + \rho_1 h_2 \det A_2^{(0)} \det \tilde{A}_1^{(1)}} > 0,$$

which is not consistent with

$$\lim_{|\xi|\to\infty} \left(\frac{c_{\mathrm{IW}}(\xi)}{c_{\mathrm{LIW}}}\right)^2 = 0.$$

As for the Choi–Camassa model, we have

$$\lim_{|\xi|\to\infty} \left(\frac{c_{\mathrm{CC}}(\xi)}{c_{\mathrm{LIW}}}\right)^2 = 0,$$

which is consistent with the above deep water limit to the full equations. This is one of the reasons why there is no stability regime for the Choi–Camassa model as in the case of the full equations.

If the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)}, \zeta_{(0)})$ and the bottom topography b are sufficiently small, then the stability condition (24) and positivity of the thicknesses of the upper and lower layers are automatically satisfied under Rayleigh's stability condition $(\rho_2 - \rho_1)g > 0$.

However, it is not evident how we prepare the initial data so that they satisfy the necessary conditions (26). On the other hand, as was shown by T. B. Benjamin and T. J. Bridges [1] and W. Craig and M. D. Groves [3] the basic equations (2)–(10) for internal gravity waves have a Hamiltonian structure and the Hamiltonian is given by the total energy

$$\begin{aligned}\mathscr{H} &= \int_{\Omega_1(t)} \frac{1}{2}\rho_1|\nabla_X\Phi_1|^2\mathrm{d}X + \int_{\Omega_2(t)} \frac{1}{2}\rho_2|\nabla_X\Phi_2|^2\mathrm{d}X \\ &\quad + \int_{\mathbf{R}^n}\left(\int_0^\zeta \rho_1(-g)z\mathrm{d}z + \int_0^\zeta \rho_2 gz\mathrm{d}z\right)\mathrm{d}x.\end{aligned}$$

The canonical variables are (ζ, ϕ), where ϕ is defined by

$$\phi = -\rho_1\Phi_1|_{z=\zeta} + \rho_2\Phi_2|_{z=\zeta}.$$

Therefore, it is natural to impose the initial data on these canonical variables. The corresponding quantity to the Kakinuma model is given by

$$\begin{aligned}\phi &= -\rho_1\Phi_1^{\mathrm{app}}|_{z=\zeta} + \rho_2\Phi_2^{\mathrm{app}}|_{z=\zeta} \\ &= -\rho_1\sum_{i=0}^{N} H_1^{2i}\phi_{1i} + \rho_2\sum_{i=0}^{N^*} H_2^{p_i}\phi_{2i},\end{aligned} \tag{27}$$

where H_1 and H_2 are mean thicknesses of the upper and the lower layers given by (13). The following proposition states that once we are given the initial data for the canonical variables (ζ, ϕ) and the bottom topography b satisfying the positivity of the thicknesses of the upper and the lower layers, the necessary conditions (26) and the relation (27) determine uniquely the initial data for the Kakinuma model.

Proposition 2 *Let $\rho_1, \rho_2, h_1, h_2, c_0, M$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant C such that if ζ and b satisfy*

$$\begin{cases}\|\zeta\|_{H^m} + \|b\|_{W^{m,\infty}} \le M, \\ H_1(x) = h_1 - \zeta(x) \ge c_0, \quad H_2(x) = h_2 + \zeta(x) - b(x) \ge c_0 \quad \text{for} \quad x \in \mathbf{R}^n,\end{cases}$$

then for any ϕ satisfying $\nabla\phi \in H^{m-1}$ there exists a solution $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to (26)–(27) satisfying

$$\|(\boldsymbol{\phi}_1', \boldsymbol{\phi}_2')\|_{H^m} + \|(\nabla\phi_{10}, \nabla\phi_{20})\|_{H^{m-1}} \le C\|\nabla\phi\|_{H^{m-1}}.$$

Moreover, the solution is unique up to an additive constant of the form $(C\rho_2, C\rho_1)$ to (ϕ_{10}, ϕ_{20}).

7 Conserved quantities

As in the case of the basic equations for internal gravity waves, the Kakinuma model has conserved quantities: mass and energy, which are given by

$$\begin{aligned}
\text{Mass} &= \int_{\mathbf{R}^n} \zeta \,\mathrm{d}x, \\
\text{Energy} &= \int_{\Omega_1(t)} \frac{1}{2}\rho_1 |\nabla_X \Phi_1^{\mathrm{app}}|^2 \mathrm{d}X + \int_{\Omega_2(t)} \frac{1}{2}\rho_2 |\nabla_X \Phi_2^{\mathrm{app}}|^2 \mathrm{d}X \\
&\quad + \int_{\mathbf{R}^n} \left(\int_0^{\zeta} \rho_1(-g) z \mathrm{d}z + \int_0^{\zeta} \rho_2 g z \mathrm{d}z \right) \mathrm{d}x,
\end{aligned}$$

where the approximate velocity potentials Φ_1^{app} and Φ_2^{app} are given by (1). Moreover, if the bottom is flat, then we have another conserved quantity, that is, the horizontal components of the momentum, which is given by

$$\begin{aligned}
\text{Momentum} &= \int_{\Omega_1(t)} \rho_1 \nabla \Phi_1^{\mathrm{app}} \mathrm{d}X + \int_{\Omega_2(t)} \rho_2 \nabla \Phi_2^{\mathrm{app}} \mathrm{d}X \\
&= \int_{\mathbf{R}^n} \zeta \nabla \phi \,\mathrm{d}x,
\end{aligned}$$

where ϕ is the canonical variable given by (27).

The details in this article will be published elsewhere.

Acknowledgements This work was partially supported by JSPS KAKENHI Grant Number JP17K18742 and JP17H02856.

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Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kohoku-ku
Yokohama 223-8522
JAPAN
E-mail address: `iguchi@math.keio.ac.jp`

慶應義塾大学・理工学部　井口　達雄