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One Parameter Groups Associated with Quantum Girsanov Transformation

Mi Ra Lee
Department of Mathematics Chungbuk National University

1 Preliminary

Let T be a topological space with Borel measure dt and $H = L^2(T, dt)$ be a Hilbert space equipped with a norm $|\cdot|_0$. For a positive self-adjoint operator A on H with Hilbert Schmidt inverse and each $p \geq 0$, define

$$E_p = \{\xi \in H : |\xi|_p = |A^p \xi|_0 < \infty\}$$

and E_{-p} by closure of H with respect to a norm $|\xi|_{-p} := |A^{-p}\xi|_0,\ \xi \in H$. Then we have a chain of Hilbert spaces as

$$\cdots \subset E_p \subset H \subset E_{-p} \subset \cdots .$$

By defining

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p, \quad E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}$$

the Gelfand triple is obtained :

$$E \subset H \subset E^*.$$

The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$. For each $p \geq 0$, the Boson Fock spaces of E_p given by

$$\Gamma(E_p) = \{\phi = (f_n)_{n=0}^{\infty} : \|\phi\|_p = \sum_{n=0}^{\infty} n! |f_n|_p < \infty,\ f_n \in E_p^{\hat{\otimes} n}\}$$

derive a chain of Fock spaces

$$\cdots \subset \Gamma(E_p) \subset \Gamma(H) \subset \Gamma(E_{-p}) \subset \cdots .$$

Define

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \quad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p})$$

then we again have the Gelfand triple

$$(E) \subset \Gamma(H) \subset (E)^*.$$

In particular, it is known as *Hida–Kubo–Takenaka space* with the Wiener-Itô-Segal isomorphism $L^2(E^*, \mu) \cong \Gamma(H)$ where μ is a standard Gaussian probability measure on E^*. Note that the set of exponential vectors

$$\phi_\xi = (1, \xi, \cdots, \frac{1}{n!}\xi^{\otimes n}, \cdots), \quad \xi \in E$$

spans a dense subspace of (E). The topology of (E) is given by the norm

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|f_n|^2, \qquad \phi = (f_n)_{n=0}^{\infty},\ p \in \mathbb{R}.$$

Moreover, for $\Phi \in (E)^*$, there exists $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$, that is

$$\|\Phi\|_{-p}^2 = \sum_{n=0}^{\infty} n!|F_n|_{-p}^2 < \infty, \quad \Phi = (F_n)_{n=0}^{\infty}.$$

The canonical bilinear form on $(E)^* \times (E)$ is given by

$$\langle\langle \Phi,\, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \left\langle F_n,\, f_n \right\rangle, \qquad \Phi = (F_n)_{n=0}^{\infty},\, \phi = (f_n)_{n=0}^{\infty}.$$

2 White Noise Operators

An operator $\Xi \in \mathcal{L}((E),(E)^*)$ is called a *white noise operator* where $\mathcal{L}((E),(E)^*)$ is the space of continuous linear operator from (E) to $(E)^*$. For $x \in (E)^*$, the *annihilation operator* $a(x) \in \mathcal{L}((E),(E))$ is given by

$$a(x) : \phi = (f_n)_{n=0}^{\infty} \mapsto ((n+1)x \otimes_1 f_{n+1})_{n=0}^{\infty}$$

where $x \otimes_1 f_n$ is a contraction of tensor product. The adjoint of annihilation operator $a^*(x) \in \mathcal{L}((E)^*,(E)^*)$ is called a *creation operator*, and its action is given by

$$a^*(x) : \phi = (f_n)_{n=0}^{\infty} \mapsto (x \hat{\otimes} f_{n-1})_{n=0}^{\infty}.$$

For $\kappa \in (E^{\otimes(l+m)})^*$ and $\phi = (f_n)_{n=0}^{\infty} \in (E)$, we define a sequence $(g_n)_{n=0}^{\infty}$ by

$$g_n = 0,\, 0 \leq n < l, \quad g_{l+n} = \frac{(n+m)!}{n!} \kappa \otimes_m f_{n+m},\, n \geq 0.$$

The operator $\Xi_{l,m}$ defined by $\Xi_{l,m}\phi = (g_n)_{n=0}^{\infty}$ is called the *integral kernel operator* with kernel distribution κ. The following theorem shows that each white noise operator has unique infinite series expansion called a *Fock expansion*.

Theorem 2.1 (Obata [12]) *For each $\Xi \in \mathcal{L}((E),(E)^*)$, there exists a unique family of $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$ such that*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$$

whenever the sum converges in $\mathcal{L}((E),(E)^*)$.

For $S \in \mathcal{L}(E, E^*)$ associated distribution $\tau_S \in (E \otimes E)^*$ is given by

$$\langle \tau_S,\, \xi \otimes \eta \rangle = \langle S\eta,\, \xi \rangle\,, \quad \xi, \eta \in E.$$

If $\langle S\eta, \xi\rangle = \langle S\xi, \eta\rangle$, i.e. $S = S^*$, then S is called symmetric. If $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of H, then τ_S has infinite series expansion

$$\tau_S = \sum_{n=0}^{\infty} Se_n \otimes e_n.$$

The *quadratic annihilation operator* associated with S is defined by $\Delta_G(S) = \Xi_{0,2}(\tau_S)$. Applying to the exponential vector ϕ_ξ of $\xi \in E$, its action is understood by

$$\Delta_G(S)\phi_\xi = \langle S\xi, \xi\rangle \phi_\xi.$$

Moreover, $\Delta_G(S)$ has infinite series expansion

$$\Delta_G(S) = \sum_{n=0}^{\infty} a(Se_n)a(e_n).$$

The dual of $\Delta_G(S)$ with respect to the canonical bilinear form is called the *quadratic creation operator*

$$\Delta_G^*(S) = \Xi_{2,0}(\tau_S) = \sum_{n=0}^{\infty} a^*(Se_n)a^*(e_n).$$

The *conservation operator* $\Lambda(S) \in \mathcal{L}((E), (E)^*)$ is defined by $\Lambda(S) = \Xi_{1,1}(\tau_S)$, and its infinite series expansion is obtained as

$$\Lambda(S) = \sum_{n=0}^{\infty} a^*(Se_n)a(e_n).$$

The *second quantization* of S is defined by

$$\Gamma(S)\phi = (S^{\otimes n} f_n)_{n=0}^{\infty}, \quad \phi = (f_n)$$

and is related to the conservation operator as

$$\frac{d\Gamma(e^{tS})}{dt}\Big|_{t=0} = \Lambda(S).$$

$\Lambda(S)$ is also called the differential second quantization operator.

3 Quantum White Noise Differential Equations

For $\Xi \in \mathcal{L}((E), (E)^*)$ and $\zeta \in E$, a commutator

$$[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi a(\zeta), \quad [a^*\zeta, \Xi] = a^*(\zeta)\Xi - \Xi a^*(\zeta)$$

are well-defined by composition since $a(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$ and $a^*(\zeta) \in \mathcal{L}((E)^*, (E)^*) \cap \mathcal{L}((E), (E))$. Define

$$D_\zeta^+\Xi = [a(\zeta), \Xi], \quad D_\zeta^-\Xi = -[a^*\zeta, \Xi],$$

which are called the *creation* and *annihilation derivatives* respectively. Both together are called quantum white noise derivatives.

Theorem 3.1 (Ji–Obata [6]) *$(\zeta,\Xi)\mapsto D_\zeta^{\pm}\Xi$ is a continuous bilinear map from $E\times\mathcal{L}((E),(E)^*)$ to $\mathcal{L}((E),(E)^*)$.*

Example 3.2 (Ji–Obata [8]) For $S\in\mathcal{L}(E,E^*)$ and $\zeta\in E$ we have

$$\begin{array}{ll} D_\zeta^+\Delta_G(S)=0, & D_\zeta^-\Delta_G(S)=a(S\zeta)+a(S^*\zeta),\\ D_\zeta^+\Delta_G^*(S)=a^*(S\zeta)+a^*(S^*\zeta), & D_\zeta^-\Delta_G^*(S)=0,\\ D_\zeta^+\Lambda(S)=a(S^*\zeta), & D_\zeta^-\Lambda(S)=a^*(S\zeta). \end{array}$$

For $\Xi\in\mathcal{L}((E),(E)^*)$, a function $\hat{\Xi}$ on $E\times E$ defined by

$$\hat{\Xi}(\xi,\eta)=\langle\langle\Xi\phi_\xi,\,\phi_\eta\rangle\rangle$$

is called the operator symbol of Ξ. Note that the mapping $\Xi\mapsto\hat{\Xi}$ is injective.

Proposition 3.3 (Obata [11]) *Let Θ be a function on $E\times E$ with values on $\mathbb{C}$. Then there exists a continuous operator $\Xi\in\mathcal{L}((E),(E)^*)$ such that $\Theta=\hat{\Xi}$ if and only if*

(1) Θ is Gâteaux entire function;

(2) for any $p\ge 0$ and $\epsilon>0$, there exist $C\ge 0$ and $q\ge 0$ such that

$$|\Theta(\xi,\eta)|\le C\exp\{\epsilon(|\xi|_{p+q}^2+|\eta|_{-p}^2)\},\quad \xi,\eta\in E.$$

Let $\Xi_1,\Xi_2\in\mathcal{L}((E),(E)^*)$. Then there exists a unique $\Xi\in\mathcal{L}((E),(E)^*)$ satisfying

$$\hat{\Xi}(\xi,\eta)=e^{-\langle\xi,\,\eta\rangle}\hat{\Xi}_1(\xi,\eta)\hat{\Xi}_2(\xi,\eta),\quad \xi,\eta\in E$$

where $\hat{\Xi}$ is an operator symbol of Ξ. Then Ξ is called the *Wick product* of Ξ_1 and Ξ_2 and denoted by $\Xi=\Xi_1\diamond\Xi_2$. For examples,

$$\begin{array}{ll} a(x)\diamond a(y)=a(x)a(y), & a^*(x)\diamond a^*(y)=a^*(x)a^*(y),\\ a(x)\diamond a^*(y)=a^*(y)a(x), & a^*(x)\diamond a(y)=a^*(x)a(y). \end{array}$$

The right hand sides of above examples are called the *wick ordered form* of given operators. More generally, for $\Xi\in\mathcal{L}((E),(E)^*)$ one has

$$a^*(x_1)\cdots a^*(x_2)\Xi a(y_1)\cdots a(y_m)=(a^*(x_1)\cdots a^*(x_2)a(y_1)\cdots a(y_m))\diamond\Xi.$$

Equipped with Wick product, $\mathcal{L}((E),(E)^*)$ becomes a commutative $*$-algebra. For $Y\in\mathcal{L}((E),(E)^*)$ the *wick exponential* of Y is defined by

$$\operatorname{wexp}Y=\sum_{n=0}^\infty\frac{1}{n!}Y^{\diamond n}$$

whenever the series converges in $\mathcal{L}((E),(E)^*)$.

A continuous map $\mathcal{D}:\mathcal{L}((E),(E)^*)\to\mathcal{L}((E),(E)^*)$ satisfying

$$\mathcal{D}(\Xi_1\diamond\Xi_2)=(\mathcal{D}\Xi_1)\diamond\Xi_2+\Xi_2\diamond(\mathcal{D}\Xi_2)$$

is called a *wick derivation.*

Theorem 3.4 (Ji–Obata [8]) *The creation and annihilation derivatives are wick derivations.*

According to wick derivation, Ji–Obata [8] introduced first order homogeneous linear differential equation of wick type.

Theorem 3.5 (Ji–Obata [8]) *Let $G \in \mathcal{L}((E),(E)^*)$. If there is an operator $U \in \mathcal{L}((E),(E)^*)$ such that $\mathcal{D}U = G$ and $\mathrm{wexp}\, U \in \mathcal{L}((E),(E)^*)$, then the solution of linear differential equation:*

$$\mathcal{D}\Xi = G \diamond \Xi$$

is given by

$$\Xi = F \diamond (\mathrm{wexp}\, U)$$

where the operator $F \in \mathcal{L}((E),(E)^*)$ satisfies $\mathcal{D}F = 0$.

For non-homogeneous type, we refer [9].

Proposition 3.6 (Ji–Obata [8]) *Let $\Xi \in \mathcal{L}((E),(E)^*)$.*

(1) $D_\zeta^+\Xi = 0$ for all $\zeta \in E$ if and only if Ξ is of the form:

$$\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}).$$

(2) $D_\zeta^-\Xi = 0$ for all $\zeta \in E$ if and only if Ξ is of the form:

$$\Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}).$$

(3) If Ξ satisfies $D_\zeta^+\Xi = D_\zeta^-\Xi = 0$ for all $\zeta \in E$,then Ξ is a scalar operator.

Example 3.7 Let $\eta, \zeta \in E$ and $S \in \mathcal{L}(E,E)$. By solving wick type differential equations, we can get the wick ordered form of Ξ as following:

$$e^{a(\zeta)}e^{a^*(\eta)} = e^{\langle \zeta,\eta\rangle}e^{a^*(\eta)}e^{a(\zeta)}, \tag{3.1}$$

$$\Gamma(S)e^{a^*(\eta)} = e^{a^*(S\eta)}\Gamma(S), \tag{3.2}$$

$$e^{a(\zeta)}\Gamma(S) = \Gamma(S)e^{a(S^*\zeta)}. \tag{3.3}$$

Indeed, for (3.1), take creation and annihilation derivatives for $\xi \in E$ to Ξ, then we have

$$\begin{aligned} D_\xi^+\Xi &= D_\xi^+ e^{a(\eta)}e^{a^*(\zeta)} = e^{a(\eta)}(D_\xi^+ e^{a^*(\zeta)}) \\ &= e^{a(\eta)}(D_\xi^+ a^*(\zeta))e^{a^*(\zeta)} = \langle \xi, \zeta\rangle\, e^{a(\eta)}e^{a^*(\zeta)} = \langle \xi, \zeta\rangle\, I \diamond \Xi, \\ D_\xi^-\Xi &= D_\xi^- e^{a(\eta)}e^{a^*(\zeta)} = (D_\xi^- a(\eta))e^{a(\eta)}e^{a^*(\zeta)} \\ &= \langle \eta, \xi\rangle\, e^{a(\eta)}e^{a^*(\zeta)} = \langle \eta, \xi\rangle\, I \diamond \Xi. \end{aligned}$$

Let Y satisfy $D_\xi^+ Y = \langle \xi, \zeta\rangle\, I$ and $D_\xi^- Y = \langle \eta, \xi\rangle\, I$. Then from creation derivative one has

$$Y = a^*(\zeta) + Y_1, \quad D_\xi^+ Y_1 = 0$$

and from annihilation derivative we have

$$D_\xi^- Y = D_\xi^- Y_1 = \langle \eta, \xi \rangle$$

which implies that $Y_1 = a(\eta) + Y_2$ where $D_\xi^- Y_2 = 0$. So $Y = a^*(\zeta) + a(\eta) + C$ for some scalar operator C. Then

$$\Xi = C \cdot \mathrm{wexp}\, Y = C \cdot e^{a^*(\zeta)} e^{a(\eta)}$$

and C is obtained by

$$C = \langle\!\langle \Xi \phi_0, \phi_0 \rangle\!\rangle = \left\langle\!\!\left\langle e^{a^*(\zeta)} \phi_0, e^{a^*(\eta)} \phi_0 \right\rangle\!\!\right\rangle = e^{\langle \zeta, \eta \rangle}.$$

Similarly we can get (3.2) and (3.3). Furthermore, the wick ordered form of white noise operators including up to quadratic annihilation and creation operators are well known. For more details see Ji–Obata [9, 10].

4 One Parameter Group

Motivated from [10], we construct one parameter group involving annihilation, creation and conservation operators. For a locally convex space X, let $GL(X)$ be a group of linear homomorphisms in X. A one-parameter family $\{T_\theta\}_{\theta \in \mathbb{R}}$ is called a group if

(1) $T_0 = I$;

(2) $T_{\theta_1 + \theta_2} = T_{\theta_1} T_{\theta_2}$, for $\theta_1, \theta_2 \in \mathbb{R}$.

Let η, ζ be differentiable functions from $\mathbb{R}$ with values on E and A be a differentiable function from $\mathbb{R}$ to $\mathcal{L}(E, E)$. Let C be a differentiable function on $\mathbb{R}$. For each $\theta \in \mathbb{R}$ we put

$$T_\theta = C(\theta) e^{a^*(\eta(\theta))} \Gamma(A(\theta)) e^{a(\zeta(\theta))}.$$

To satisfy the group conditions, we consider the compositions of T_{θ_1} and T_{θ_2}

$$\begin{aligned} T_{\theta_1} T_{\theta_2} &= C(\theta_1) C(\theta_2) e^{a^*(\eta(\theta_1))} \Gamma(A(\theta_1)) e^{a(\zeta(\theta_1))} e^{a^*(\eta(\theta_2))} \Gamma(A(\theta_2)) e^{a(\zeta(\theta_2))} \\ &= C(\theta_1) C(\theta_2) e^{\langle \zeta(\theta_1), \eta(\theta_2) \rangle} e^{a^*(\eta(\theta_1) + A(\theta_1)\eta(\theta_2))} \Gamma(A(\theta_1) A(\theta_2)) e^{a(\zeta(\theta_2) + A(\theta_2)^* \zeta(\theta_1))} \end{aligned}$$

by applying (3.1), (3.2) and (3.3). Then the group property $T_{\theta_1 + \theta_2} = T_{\theta_1} T_{\theta_2}$ induces following equations:

$$\eta(\theta_1 + \theta_2) = \eta(\theta_1) + A(\theta_1)\eta(\theta_2), \tag{4.1}$$
$$A(\theta_1 + \theta_2) = A(\theta_1) A(\theta_2), \tag{4.2}$$
$$\zeta(\theta_1 + \theta_2) = \zeta(\theta_2) + A(\theta_2)^* \zeta(\theta_1), \tag{4.3}$$
$$C(\theta_1 + \theta_2) = C(\theta_1) C(\theta_2) e^{\langle \zeta(\theta_1), \eta(\theta_2) \rangle} \tag{4.4}$$

and initial values are obtained as $A(0) = I$, $\eta(0) = \zeta(0) = 0$.

Proposition 4.1 *The one parameter family* $\{T_\theta\}_{\theta\in\mathbb{R}}$ *is a group if* E*-valued functions* η, ζ, $\mathcal{L}(E,E)$*-valued function* A *and real valued function* C *satisfy following differential equations:*

$$\eta'(\theta) = A(\theta)\eta'(0), \tag{4.5}$$
$$A'(\theta) = A(\theta)A'(0), \tag{4.6}$$
$$\zeta'(\theta) = A(\theta)^*\zeta'(0), \tag{4.7}$$
$$C'(\theta) = C'(0)C(\theta). \tag{4.8}$$

Proof. Take $\theta_1 = \theta$ and $\theta_2 = h$. Then from (4.2) we see

$$A(\theta+h) - A(\theta) = A(\theta)[A(h) - I]$$

which shows (4.6). In the same way, by taking $\theta_1 = \theta$ and $\theta = h$, we get

$$\eta(\theta+h) - \eta(\theta) = A(\theta)\eta(h).$$

Thus (4.5) is obtained. Similarly we have (4.7). For (4.8), consider a function $f(x) = C(x)e^{\langle\zeta(\theta),\,\eta(x)\rangle}$. Then f is differentiable and

$$f'(x) = C'(x)e^{\langle\zeta(\theta),\,\eta(x)\rangle} + C(x)\,\langle\zeta(\theta),\,\eta(x)\rangle\,e^{\langle\zeta(\theta),\,\eta(x)\rangle}$$

shows that $f'(0) = C'(0)$. So

$$C'(\theta) = C(\theta)\lim_{h\to 0}\frac{C(h)e^{\langle\zeta(\theta),\,\eta(h)\rangle}}{h} = C(\theta)C'(0)$$

is obtained. ∎

Theorem 4.2 *Let* $\eta = \eta'(0)$, $\zeta = \zeta'(0) \in E$, $A = A'(0) \in \mathcal{L}(E,E)$ *and* $c = C'(0) \in \mathbb{R}$ *be given. The solutions of* (4.5)–(4.8) *are obtained as followings:*

$$A(\theta) = e^{\theta A}, \qquad C(\theta) = e^{c\theta},$$
$$\eta(\theta) = \int_0^\theta e^{tA}\eta dt, \quad \zeta(\theta) = \int_0^\theta e^{tA^*}\zeta dt.$$

The proof is straightforward.

Theorem 4.3 $\{T_\theta\}_{\theta\in\mathbb{R}}$ *is a one-parameter group with the infinitesimal generator*

$$\left.\frac{dT_\theta}{d\theta}\right|_{\theta=0} = cI + a^*(\eta) + \Lambda(A) + a(\zeta)$$

where $\eta = \eta'(0)$, $\zeta = \zeta'(0)$, $A = A'(0)$, $c = C'(0)$.

Proof. We can see by applying characterization theorem of operator symbol of T_θ. ∎

Corollary 4.4 *Let* $A(\theta) = I$ *for* $\theta \in \mathbb{R}$. *Then one parameter family of*

$$T_\theta = C(\theta)e^{a^*(\eta(\theta))}e^N e^{a(\zeta(\theta))}$$

is a group with the infinitesimal generator $I + a^*(\eta) + N + a(\zeta)$ *where* $\eta = \eta'(0)$, $\zeta = \zeta'(0)$ *and* $N = \Lambda(I)$ *is a number operator. For more details, see [2].*

As a general case, one parameter group with the infinitesimal generator which is a linear combination of $a^*(\eta), a(\zeta), \Lambda(B), \Delta_G(A)$ and $\Delta_G^*(C)$ is studied in [4], where A, B, C, η, ζ satisfy certain conditions.

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Department of Mathematics
Chungbuk National University
Cheongju, 28644 Korea
E-Mail: mrlee1012@chungbuk.ac.kr