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Integral kernels of the renormalized Nelson Hamiltonian

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Abstract

In this article we consider the ground state of the renormalized Nelson Hamiltonian in quantum field theory by using the integral kernel of the semigroup generated by the Hamiltonian. By introducing an infrared cutoff, the existence of the ground state is shown and the expectation values of observables with respect to the ground state are given in terms of a probability measure.

1 Introduction

This is a joint work with Oliver Matte [19]. Since the end of the last century several interaction models between quantum mechanical matters and quantum fields have been investigated; the Pauli-Fierz model [25] in non-relativistic quantum electrodynamics and spin-boson model have been typical examples. There are a lot of contributions to studying ground states of models. Here the ground state describes an eigenvector associated with the bottom of the spectrum of a self-adjoint operator. The Hamiltonian of the interaction system can be realised as a self-adjoint operator and we are interested in investigating the ground state of the Hamiltonian, e.g., the existence of the ground state and its properties.

In this article we discuss the ground state of a renormalized Hamiltonian introduced by Edward Nelson in 1964 [23, 24] to consider the removal of ultraviolet cutoffs. This

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model is nowadays the so-called renormalized Nelson Hamiltonian. The renormalized Nelson Hamiltonian describes a linear interaction between non-relativistic spinless nucleons and spinless scalar mesons, where the non-relativistic nucleons are governed by a Schrödinger operator acting in $L^2(\mathbb{R}^{dN})$, where N denotes the number of nucleons and d the spatial dimension. The physical reasonable choice is $d = 3$. In this article we assume that $N = 1$ and $d = 3$ for simplicity.

In mathematics field operator

$$\phi(f) = \frac{1}{\sqrt{2}}(a^\dagger(\hat{f}) + a(\bar{\hat{f}})) \tag{1.1}$$

can be defined for $f \in L^2(\mathbb{R}^3)$, but in physics $\phi(x)$ is defined by $\phi(f)$ with $f = \delta(\cdot - x)$. It is not straightforward however to define $\phi(x)$. It is common to define $\phi(x)$ as the limit of $\phi(f_n)$ as $n \to \infty$, where $f_n \to \delta(\cdot - x)$ as $n \to \infty$ in some sense. f_n is called cutoff function or ultraviolet cutoff function in this article. The Nelson Hamiltonian is defined by introducing cutoff functions and it can be realised as a self-adjoint operator acting in the Hilbert space given by

$$\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{F},$$

where $\mathscr{F}$ denotes the boson Fock space over $L^2(\mathbb{R}^3)$ defined by

$$\mathscr{F} = \oplus_{n=0}^{\infty}[L^2_{\mathrm{sym}}(\mathbb{R}^{3n})],$$

where $L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$ denotes the set of symmetric L^2-functions with $L^2_{\mathrm{sym}}(\mathbb{R}^0) = \mathbb{C}$. We set $\mathscr{F}^{(n)} = L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$ and

$$\mathscr{F}_0 = \{\Phi \in \mathscr{F} \mid \exists m \text{ such that } \Phi^{(n)} = 0 \text{ for } \forall n \geq m\}.$$

$\mathscr{F}_0$ is called the finite particle subspace of $\mathscr{F}$. Subtracting a renormalization term from the Nelson Hamiltonian, we can define the renormalized Nelson Hamiltonian H_∞. A crucial point is that H_∞ is defined by the limit of the Nelson Hamiltonian and consequently it is given as a semi-bounded quadratic form. Then it is impossible to see an explicit form of H_∞ as an operator in $\mathscr{H}$. Recently J. Møller and O. Matte [22] however succeeded in constructing a Feynman-Kac type formula of e^{-TH_∞} explicitly. More precisely it is shown that

$$\left(F, e^{-TH_\infty} G\right)_{\mathscr{H}} = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[(F(B_0), K_T G(B_T))_{\mathscr{F}} \right].$$

Here $(\cdot,\cdot)_{\mathscr{F}}$ is the inner product on $\mathscr{F}$, $(B_t)_{t\geq 0}$ denotes 3-dimensional Brownian motion and K_T is an integral kernel which is of the form

$$K_T = e^{-\int_0^T V(B_s)ds} e^{a^\dagger(U)} e^{-TH_{\mathrm{f}}} e^{a(\tilde{U})}.$$

In this article we shall show (1) and (2) below:

(1) If an infrared cut off is introduced, then H_∞ has the ground state φ_{g} and it is unique up to multiple constants.

(2) The ground state $\varphi_{\mathrm{g}} = \varphi_{\mathrm{g}}(x,\phi)$ is localized in the sense of Gaussian domination with respect to field operators ϕ, and super-exponential decay of the number of bosons:
$$\sum_{n=0}^\infty e^{2\beta n} \|\varphi_{\mathrm{g}}^{(n)}\|^2_{L^2(\mathbb{R}^3_x \times \mathbb{R}^{3n}_k)} < \infty, \quad \forall \beta > 0.$$

Above (1) and (2) can be proven by using Feynman-Kac type formula mentioned above.

2 Renormalized Nelson Hamiltonian

2.1 Definition of the Nelson Hamiltonian with cutoffs

In this section we define the renormalized Nelson Hamiltonian as a self-adjoint operator acting in $\mathcal{H}$. The Nelson Hamiltonian with ultraviolet cutoff Λ is defined by

$$H_\Lambda = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{I}}.$$

We explain H_{p}, H_{f} and H_{I} below. First

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V$$

denotes a Schrödinger operator acting in $L^2(\mathbb{R}^3)$. Here we assume that the mass of the particle is one and we shall give an assumption on V below. Operator $H_{\mathrm{f}} = d\Gamma(\omega)$ denotes the free field Hamiltonian acting in $\mathscr{F}$ defined by the second quantization of ω:

$$d\Gamma(\omega) = 0 \oplus \left\{ \oplus_{n=1}^{\infty} \left[\sum_{j=1}^{n} \omega(k_j) \right] \right\},$$

where $\omega(k)$ describes the dispersion relation (the multiplication operator by $\omega(k)$). Note that H_{f} acts as

$$H_{\mathrm{f}}\Phi = \oplus_{n=1}^{\infty} \left(\sum_{j=1}^{n} \omega(k_j) \right) \Phi^{(n)}(k_1, \cdots, k_n)$$

for $\Phi = \oplus_{n=0}^{\infty}\Phi^{(n)}(k_1, \cdots, k_n)$. Finally H_{I} is the interaction term defined by

$$H_{\mathrm{I}} = \frac{1}{\sqrt{2}} \int \left(a^{\dagger}(k) e^{-ikx} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{\omega(k)}} + a(k) e^{ikx} \frac{\hat{\varphi}_{\kappa,\Lambda}(k)}{\sqrt{\omega(k)}} \right) dk.$$

Here $a(k)$ and $a^{\dagger}(k)$ are the formal kernel of the annihilation and creation operators, respectively, satisfying canonical commutation relations:

$$[a(k), a^{\dagger}(k')] = \delta(k - k').$$

Using this notation we have $a(f) = \int f(k)a(k)dk$ and $a^{\dagger}(f) = \int f(k)a^{\dagger}(k)dk$. Both $a(f)$ and $a^{\dagger}(f)$ are closed operators and they satisfy canonical commutation realtions:

$$[a(f), a^{\dagger}(g)] = (\bar{f}, g).$$

$\hat{\varphi}_{K,\Lambda}$ is the cutoff function given by

$$\hat{\varphi}_{\kappa,\Lambda}(k) = \begin{cases} 0, & |k| < \kappa, \\ 1, & \kappa \leq |k| \leq \Lambda, \\ 0, & |k| > \Lambda \end{cases}$$

with infrared cutoff parameter κ and ultraviolet cutoff parameter Λ such that $0 \leq \kappa < \Lambda$.

In order to give an assumption on V we define the Kato-class of potentials [4, 21]. Potential $V : \mathbb{R}^d \to \mathbb{R}$ is called d-dimensional Kato-class potential whenever

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le r}|g(x-y)V(y)|dy = 0$$

holds, where function g depends on the dimension and is given by

$$g(x) = \begin{cases} |x|, & d = 1, \\ -\log|x|, & d = 2, \\ |x|^{2-d}, & d \ge 3. \end{cases}$$

We introduce assumptions used through this article unless otherwise stated.

Assumption 2.1 (Dispersion relation and potential) We assume (1) and (2):

(1) $\omega(k) = |k|$.

(2) V is 3-dimensional Kato-class potential.

It can be seen that H_{I} is infinitesimally small with respect to $H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}$. By the Kato-Rellich theorem [20] H_Λ is self-adjoint on $D(H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}})$ and bounded from below. We mention it as proposition below.

Proposition 2.2 Let $\kappa \ge 0$ and $\Lambda < \infty$. Then H_Λ is self-adjoint and bounded from below on $D(H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}})$.

2.2 Definition of the renormalized Nelson Hamiltonian

According to [23], we introduce the renormalization term defined by

$$E_\Lambda = -\frac{1}{2}\int_{\mathbb{R}^3}\frac{|\hat{\varphi}_{\kappa,\Lambda}(k)|^2}{\omega(k)}\beta(k)dk,$$

where $\beta(k)$ describes a propagator given by

$$\beta(k) = \left(\omega(k) + \frac{1}{2}|k|^2\right)^{-1}.$$

We notice that $\lim_{\Lambda\to\infty} E_\Lambda = -\infty$.

Proposition 2.3 (Nelson [23]) Let $\kappa \geq 0$. Then there exists a self-adjoint operator H_∞ bounded below such that for any $T \geq 0$

$$\lim_{\kappa\to\infty} e^{-T(H_\Lambda - E_\Lambda)} = e^{-TH_\infty}.$$

Nelson [23] proved the convergence in Proposition 2.3 in the strong sense. It is however shown that this convergence is in the uniform sense in e.g. [22].

We shall see that e^{-TH_∞} can be represented in terms of path measures. Let $(B_t)_{t\geq 0}$ be 3-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathcal{W}^x)$, where $\mathcal{W}^x$ denotes a probability measure on Ω such that $\mathcal{W}^x(B_0 = x) = 1$. Let

$$U = \int_0^T \frac{e^{-s\omega(k)}}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\kappa} e^{-ikB_s} ds, \quad \tilde{U} = \int_0^T \frac{e^{-|T-s|\omega(k)}}{\sqrt{\omega(k)}} \mathbb{1}_{|k|\geq\kappa} e^{ikB_s} ds.$$

Both integrals are finite for arbitrary $\kappa \geq 0$ and $\mathbb{R}^3 \ni k \neq 0$. Furthermore since $\mathbb{E}[\int_{\mathbb{R}^3} |U|^2 dk] < \infty$ and $\mathbb{E}[\int_{\mathbb{R}^3} |\tilde{U}|^2 dk] < \infty$, we can see that $U, \tilde{U} \in L^2(\mathbb{R}^3)$ almost surely. Hence both $a^\dagger(U)$ and $a(U)$ are well-defined closed operators almost surely.

Here we review the exponentials of annihilation operators and creation operators. See [17, Appendix B] for the detail. Let $f \in L^2(\mathbb{R}^3)$ and we define the exponential of creation operators F_f by

$$F_f = \sum_{n=0}^\infty \frac{1}{n!} a^\dagger(f)^n$$

and the domain is given by

$$D(F_f) = \left\{ \Phi \in \cap_{n=1}^\infty D(a^\dagger(f)^n) \mid \sum_{n=0}^\infty \frac{1}{n!} \|a^\dagger(f)^n \Phi\| < \infty \right\}.$$

Let $\Phi \in \mathscr{F}^{(m)}$. Thus we have

$$\|F_f\Phi\| \leq \|\Phi\| + \sum_{n=1}^\infty \frac{\sqrt{m+n-1}\cdots\sqrt{m}}{n!} \|f\|^n \|\Phi\| < \infty.$$

$\mathscr{F}_0 \subset D(F_f)$ follows. We also define the exponential of annihilation operators by

$$G_f = \sum_{n=0}^\infty \frac{1}{n!} a(f)^n$$

with the domain

$$D(G_f) = \left\{ \Phi \in \cap_{n=1}^{\infty} D(a(f)^n) \mid \sum_{n=0}^{\infty} \frac{1}{n!} \|a(f)^n \Phi\| < \infty \right\}.$$

We simply write $F_f = e^{a^\dagger(f)}$ and $G_f = e^{a(\bar{f})}$ whenever confusion may arise. Then we can see that $(e^{a^\dagger(f)})^* \supset e^{a(\bar{f})}$ and this implies that $e^{a^\dagger(f)}$ is closable. The closure of $e^{a^\dagger(f)}$ is denoted by the same symbol. Similarly the closure of $e^{a(f)}$ is denoted by the same symbol.

Proposition 2.4 (Algebraic relations) Let $f, g \in L^2(\mathbb{R}^3)$ and P be a polynomial. Suppose that $\Omega \in \mathscr{F}$ be the Fock vacuum. Then (1) $e^{a^\dagger(g)} e^{a^\dagger(f)} \Omega = e^{a^\dagger(f+g)} \Omega$, (2) $P(a(g)) e^{a^\dagger(f)} \Omega = P((\bar{g}, f)) e^{a^\dagger(f)} \Omega$ and (3) $e^{a(g)} e^{a^\dagger(f)} \Omega = e^{(\bar{g}, f)} e^{a^\dagger(f)} \Omega$.

Proof See [17, Appendix B]. □

We note that $e^{a^\dagger(f)}\Omega$ is an eigenvector for $a(f)$ such that $a(f)e^{a^\dagger(f)}\Omega = (\bar{g}, f)e^{a^\dagger(f)}\Omega$. We conclude that the spectrum of $a(f)$ is $\mathbb{C}$. $e^{a^\dagger(f)}\Omega$ is called a coherent vector.

Proposition 2.5 (Boundedness) Let $t > 0$ and $f \in D(1/\sqrt{\omega})$. Then both $e^{a^\dagger(f)} e^{-tH_{\rm f}}$ and $\overline{e^{-tH_{\rm f}} e^{a(f)}}$ are bounded operators.

Proof See [17, Appendix B]. □

Let $A = e^{a^\dagger(U)} e^{-\frac{T}{2} H_{\rm f}}$ and $\tilde{A} = e^{-\frac{T}{2} H_{\rm f}} e^{a(\tilde{U})}$.

Lemma 2.6 Let $\kappa \geq 0$. Then A and $\tilde{A}$ are bounded.

Proof Since $\mathbb{E}[\int_{\mathbb{R}^3} |U|^2/\omega dk] < \infty$ and $\mathbb{E}[\int_{\mathbb{R}^3} |\tilde{U}|^2/\omega dk] < \infty$, the lemma follows from Proposition 2.5. □

Theorem 2.7 (Matte and Møller[22]) Let $\kappa \geq 0$. Let $F, G \in \mathcal{H}$. Then it follows that

$$\left(F, e^{-TH_\infty} G\right)_{\mathscr{H}} = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_0^T V(B_s) ds} e^{\frac{1}{2} S_{\rm ren}} \left(F(B_0), A\tilde{A} G(B_T) \right)_{\mathscr{F}} \right], \tag{2.1}$$

where phase factor $S_{\rm ren}$ is given by

$$S_{\rm ren} = 2\int_0^T \left(\int_0^t \nabla\varphi_0(B_s - B_t, s-t)ds\right) dB_t - 2\int_0^T \varphi_0(B_s - B_T, s-T)ds$$

and

$$\varphi_0(X,t) = \int_{\mathbb{R}^3} \frac{e^{-ikX}e^{-|t|\omega(k)}}{2\omega(k)}\beta(k)\mathbb{1}_{|k|\geq\kappa}dk.$$

In particular e^{-TH_∞} for $T>0$ is positivity improving, and if H_∞ has a ground state, then it is unique up to multiple constants.

We give several remarks on Theorem 2.7.

(1) In (2.1) we identify $\mathcal{H}$ with $L^2(\mathbb{R}^3;\mathscr{F})$: the set of $\mathscr{F}$-valued L^2-functions on $\mathbb{R}^3$. I.e., $F\in\mathcal{H}$ implies that $F(x)\in\mathscr{F}$ for each x.

(2) We revive a coupling constant g in H_Λ by replacing $H_{\rm I}$ with $gH_{\rm I}$. Then renormalization term E_Λ is identical with the coefficient of g^2 in the expansion of the ground state energy of H_Λ with $V=0$ on g^2, i.e.,

$$E(g) = E_\Lambda g^2 + \sum_{2\leq n} a_n g^{2n},$$

and we can see that $\lim\limits_{\Lambda\to\infty}\sum\limits_{2\leq n} a_n g^{2n} < \infty$. See [16].

(3) Gubinelli,Hiroshima and Lörinczi [13] derived (2.1) for $F,G\in D$ with some dense domain D.

3 Ground states

3.1 Existence of the ground state

The difficulty in establishing the existence of the ground state in quantum field theory comes from the fact that the bottom of the spectrum lies in the essential spectrum, not below it, as is the case for usual Schrödinger operator. Let us consider Schrödinger operator $-\Delta/2+V$ with external potential V which satisfies $V \in L^\infty(\mathbb{R}^3)$ and $|V(x)| \to 0$ as $|x| \to \infty$. This assumption yields that V is relatively compact with respect to $-\Delta/2$; $V(-\Delta/2+m)^{-1}$ is compact for any $m > 0$ and the essential spectrum of $-\Delta/2+V$ is $[0,\infty)$. Let e be the bottom of the spectrum of $-\Delta/2+V$ and e_0 that of $-\Delta/2$. e_0 is equal to the bottom of the essential spectrum of $-\Delta/2+V$. Then $e_0 = 0$. If $e < e_0$, then e is discrete and we can conclude that $-\Delta/2+V$ has the ground state. Consider H_Λ. Similar to $-\Delta/2+V$, we denote the bottom of the spectrum of H_Λ and H_Λ with no external potential by E and E_0, respectively. In the case of the Nelson Hamiltonian, despite inequality $E < E_0$, E lies in the bottom of the essential spectrum, and it is unclear that H_Λ admits a ground state.

In the case of $\Lambda < \infty$ it is shown that the ground state of H_Λ exists and it is unique up to multiple constants. This is due to e.g., [2, 5, 6, 27]. In [14] the existence of a ground state of the renormalized Nelson Hamiltonian without ultraviolet cutoff is shown but only for sufficiently small coupling constants.

In this section using Feynman- Kac type formula mentioned above we can also show that the ground state of H_∞ exists for arbitrary values of coupling constants for $\kappa > 0$. Note that in our setting the coupling constant is absorbed in coupling function $\hat{\varphi}_{\kappa,\Lambda}$.

Theorem 3.1 (Hiroshima and Matte [19]) Suppose that $\kappa > 0$. Then the ground state of H_∞ exists and it is unique.

Proof The uniqueness is due to Theorem 2.7. We shall show the existence. Outline of a proof is as follows. See [19] for the detail.

Step 1: We assume that $\omega(k) = \sqrt{|k|^2+\nu^2}$ with some artificial constant $\nu > 0$. Let $G \subset \mathbb{R}^3$ be a bounded and open subset. Let

$$\tau_G(x) = \inf\{t > 0 | B_t + x \notin G\}$$

be the exit time from G. In particular when $x \notin G, \tau_G(x) = 0$. Define the quadratic form $Q_t : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ by

$$Q_t : \Psi \times \Phi \mapsto \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[\mathbb{1}_{\tau_G(x) \geq t} e^{\frac{1}{2} S_{\mathrm{ren}}} e^{-\int_0^t V(B_s) ds} \Big(\Psi(B_0), A\tilde{A}\Phi(B_t) \Big) \right].$$

Thus it can be seen that there exists a self-adjoint operator H_G bounded from below such that $(\Psi, e^{-tH_G}\Phi) = Q_t(\Psi, \Phi)$. The self-adjoint operator H_G can be regarded as a self-adjoint operator on $L^2(G) \otimes \mathscr{F}$. Under the identifications $\mathscr{F} \cong L^2(Q)$ with some probability space $(Q, \mathcal{B}, \mu)$, and

$$L^2(G) \otimes \mathscr{F} \cong L^2(G \times Q),$$

it can be seen that e^{-tH_G} is hypercontractive for $t > 0$. Hence H_G must have the ground state in $L^2(G \times Q)$ by [12, 26], since $\lambda \times \mu$ is a finite measure on $G \times Q$ and e^{-tH_G}, $t > 0$, is hypercontractive. Here λ denotes the Lebesgue measure on G.

Step 2: Let φ_G be the unique ground state of H_G with $\mu > 0$ and we extend φ_G to the vector on $L^2(\mathbb{R}^3 \times Q)$ by zero-extension, i.e.,

$$\tilde{\varphi}_G(x, \phi) = \begin{cases} \varphi_G(x, \phi) & (x, \phi) \in G \times Q \\ 0 & (x, \phi) \notin G \times Q. \end{cases}$$

Let $\varphi_n = \varphi_{G_n}$ and $G_n \uparrow \mathbb{R}^3$. It can be seen that $\{\varphi_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^3 \times Q)$ and $\lim_{n \to \Lambda} \varphi_n$ exists for each $\Lambda < \infty$. The limit is denoted by φ_Λ and it is the ground state of H_Λ with $\mu > 0$.

Step 3: It is established that if H_Λ with $\mu > 0$ has the ground state, then H_Λ with $\mu = 0$ also has the ground state, since $\kappa > 0$. This trick is used in e.g. in [5]. We denote the ground state of H_Λ with $\mu = 0$ by the same symbol φ_Λ.

Step 4: Suppose that $\Lambda \to \infty$. Hence it can be also shown that $\{\varphi_\Lambda\}$ is a compact set in $L^2(\mathbb{R}^3 \times Q)$ by using the uniform convergence of e^{-tH_Λ} to e^{-tH_∞} as $\Lambda \to \infty$, the pull-through formula, spatial exponential decay [2, 11] and the Kolmogorov-Riesz-Fréchet type theorem [18]. This implies that $\{\varphi_\Lambda\}$ includes a strongly convergent subsequence $\varphi_{\Lambda'}$ and we can conclude that $\lim_{\Lambda' \to \infty} \varphi_{\Lambda'} = \varphi_{\mathrm{g}}$ is the ground state of H_∞. □

3.2 Ground state expectations

Problems we are interested in are the expectation values of observables with respect to the ground state of H_∞. Let O be an observable realised as a self-adjoint operator in $\mathscr{H}$. We want to estimate $(\varphi_{\mathrm{g}}, O\varphi_{\mathrm{g}})$. Typical examples of O are $e^{+\beta N}$ and $e^{+\beta\phi(f)^2}$.

3.2.1 Super-exponential decay of the number of bosons

We consider the Nelson Hamiltonian without the interaction: $H_{\rm p}\otimes \mathbb{1}+\mathbb{1}\otimes H_{\rm f}$. The ground state of it is $f\otimes\Omega$, where f is the ground state of $H_{\rm p}$. Then the number of bosons of $f\otimes\Omega$ is zero. I.e., $(\mathbb{1}\otimes N)f\otimes\Omega=0$, where N denotes the number operator defined by $N=d\Gamma(\mathbb{1})$. We want to estimate the number of bosons of the ground state $\varphi_{\rm g}$ of H_∞. We define the number operator with momenta grater than one by[1] $N_+=\int_{|k|\geq 1}a^\dagger(k)a(k)dk$ and $N_-=\int_{|k|<1}a^\dagger(k)a(k)dk$. Then $N=N_++N_-$. We set $\mathbb{1}\otimes N$ by N.

Lemma 3.2 $\varphi_{\rm g}\in D(e^{\beta N_+})$ for any $\beta\geq 0$.

Proof Let E be the bottom of the spectrum of H_∞. We have $\varphi_{\rm g}=e^{tE}e^{-tH_\infty}\varphi_{\rm g}$ for any $t\geq 0$. Then $\|e^{\beta N}\varphi_{\rm g}\|=e^{tE}\|e^{\beta N}e^{-tH_\infty}\varphi_{\rm g}\|$ and by Feynman-Kac formula we can see that

$$\left(F,e^{\beta N}e^{-tH_\infty}G\right)=\int_{\mathbb{R}^3}dx\mathbb{E}^x\left[e^{\frac12 S_{\rm ren}}e^{-\int_0^t V(B_s)ds}\Big(F(B_0),e^{\beta N}A\tilde{A}G(B_t)\Big)\right].$$

It can be also seen that

$$e^{\beta N_+}A\tilde{A}=e^{a^*(e^{\beta\mathbb{1}_{|k|\geq 1}}U)}e^{\beta N_+}e^{-tH}e^{a(\tilde{U})},$$

and $e^{\beta N_+}e^{-tH}$ is bounded for $t>\beta$, since

$$e^{\beta N_+}e^{-tH}=\Gamma(e^{-t|k|+\beta\mathbb{1}_{|k|\geq 1}})=\Gamma(e^{\mathbb{1}_{|k|\geq 1}(-t|k|+\beta)})\Gamma(e^{-t|k|\mathbb{1}_{|k|<1}})$$

and $\mathbb{1}_{|k|\geq 1}(-t|k|+\beta)<0$ for any $|k|\geq 1$. Then $|(F,e^{\beta N_+}e^{-tH_\infty}G)|\leq C\|F\|\|G\|$ with some constant C depending only t and β. Thus $e^{\beta N_+}e^{-tH_\infty}$ is bounded and the lemma follows. $\square$

Lemma 3.3 $\varphi_{\rm g}\in D(e^{\beta N_-})$ for any $\beta\geq 0$

[1] $N_+=d\Gamma(\mathbb{1}_{|k|\geq 1})$ and $N_-=d\Gamma(\mathbb{1}_{|k|<1})$.

Proof To show the lemma we use the Gibbs measure associated with the ground state $\varphi_{\rm g}$. Let $(B_t)_{t\in\mathbb{R}}$ be 3-dimensional Brownian motion on the whole real line $\mathbb{R}$. Let $\mathcal{F}_t=\sigma(B_r;-t\le r\le t)$ be the sigma-field generated by $\{B_r;-t\le r\le t\}$. We set $\mathcal{G}=\sigma(\cup_{t\ge 0}\mathcal{F}_t)$. Define the probability measure $\mu_t(\cdot)$ by

$$\mu_t(A)=\frac{1}{Z_t}\int_{\mathbb{R}^3}dx\mathbb{E}^x\left[\mathbb{1}_A L_t\right],\quad A\in\mathcal{G},$$

where Z_t denotes the normalising constant and

$$L_t=f(B_{-t})f(B_t)e^{\frac{1}{2}\bar{S}_{\rm ren}}e^{-\int_{-t}^t V(B_s)ds}$$

with

$$\bar{S}_{\rm ren}=2\int_{-T}^{T}\left(\int_{-T}^{t}\nabla\varphi_0(B_s-B_t,s-t)ds\right)dB_t-2\int_{-T}^{T}\varphi_0(B_s-B_T,s-T)ds.$$

By a direct computation we then have

$$(\varphi_{\rm g},e^{-BN_-}\varphi_{\rm g})=\lim_{t\to\infty}\frac{(e^{-tH_\infty}f\otimes\mathbb{1},e^{-\beta N_-}e^{-tH_\infty}f\otimes\mathbb{1})}{\|e^{-tH_\infty}f\otimes\mathbb{1}\|^2}=\lim_{t\to\infty}\mathbb{E}_{\mu_t}\left[e^{-(1-e^{-\beta})\int_{-t}^0 ds\int_0^t drW}\right],$$

where

$$W=\int_{\kappa\le|k|\le 1}\frac{1}{\omega(k)}e^{-|r-s|\omega(k)}e^{-ik(B_r-B_s)}dk.$$

We see that

$$|W|\le\int_{\kappa\le|k|\le 1}\omega(k)^{-3}dk<\infty$$

which implies that W is unifprmly bounded with respect to Brownian motion and $t\ge 0$. By the existence of the positive ground state, and

$$\lim_{t\to\infty}\frac{e^{-tH_\infty}f\otimes\mathbb{1}}{\|e^{-tH_\infty}f\otimes\mathbb{1}\|}=\varphi_{\rm g},$$

there exists a probability measure μ_∞ on $(\Omega,\mathcal{G})$ such that

$$(\varphi_{\rm g},e^{-\beta N_-}\varphi_{\rm g})=\mathbb{E}_{\mu_\infty}\left[e^{-(1-e^{-\beta})\int_{-\infty}^0 ds\int_0^\infty W dr}\right].$$

The proof of the existence of the measure μ_∞ is due to [15]. By the analytic continuation in β we can extend above identity to whole $\beta \in \mathbb{C}$. Thus it follows that

$$(\varphi_{\mathrm{g}}, e^{\beta N_-}\varphi_{\mathrm{g}}) = \mathbb{E}_{\mu_\infty}\left[e^{-(1-e^\beta)\int_{-\infty}^0 ds \int_0^\infty W dr}\right] \quad \text{for} \quad \beta \in \mathbb{C}.$$

In particular $\|e^{+\beta N_-}\varphi_{\mathrm{g}}\| < \infty$ for any $\beta > 0$, and the lemma is proven. □

Theorem 3.4 (Hiroshima and Matte [19]) Let $\kappa > 0$. Then

$$\varphi_{\mathrm{g}} \in D(e^{\beta N}), \quad \forall \beta > 0.$$

Proof By Lemmas 3.2 and 3.3 we have

$$(\varphi_{\mathrm{g}} e^{\beta N}\varphi_{\mathrm{g}}) = (\varphi_{\mathrm{g}}, e^{\beta N_+}e^{\beta N_-}\varphi_{\mathrm{g}}) = (e^{\beta N_+}\varphi_{\mathrm{g}}, e^{\beta N_-}\varphi_{\mathrm{g}}) = \|e^{\beta N_+}\varphi_{\mathrm{g}}\|\|e^{\beta N_-}\varphi_{\mathrm{g}}\| < \infty.$$

Hence the theorem is proven. □

From Theorem 3.4 we can say that the number of bosons of the ground state of H_∞ is a few.

3.2.2 Gaussian dominations

In a similar manner to the proof of the super-exponential decay of φ_{g} we can also show a Gaussian domination of the ground state φ_{g} by the path measure μ_∞.

Let $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$ and $a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$. Set $\varphi(x) = e^{-|x|^2/2}/\pi^{1/4}$. The harmonic oscillator in $L^2(\mathbb{R})$ is defined by $h = a^\dagger a$. The spectrum of h is given by $\{n\}_{n=0}^\infty$ and $h\varphi_n = n\varphi_n$ for $\varphi_n = (n!)^{-1/2}(\prod^n a^\dagger)\varphi$. Precisely $\varphi_n(x) = h_n(x)e^{-|x|^2/2}$ with some n-degree polynomial $h_n(x)$. In particular we have

$$\lim_{\beta\uparrow 1}\|e^{(\beta/2)|x|^2}\varphi_n\|_{L^2(\mathbb{R})} \to \infty. \tag{3.1}$$

Now we consider the Nelson Hamiltonian without the interaction: $H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}$. The ground state of it is $\varphi_0 = f \otimes \Omega$, where f is the normalised ground state of H_{p}. The free field Hamiltonian H_{f} can be regarded as an infinite freedom version of harmonic oscillator h. We have a counterpart of (3.1). Let $\phi(g)$ be given by (1.1). We have $(\Omega, e^{(\beta/2)\phi(g)}\Omega) = (1-\beta\|\hat{g}/\sqrt{\omega}\|^2)^{-1/2}$. In particular

$$\lim_{\beta\uparrow\|\hat{g}/\sqrt{\omega}\|^{-2}} \|e^{(\beta/2)\phi(g)^2}\varphi_0\| = \infty.$$

The renormalized Nelson Hamiltonian H_∞ has a similar properties. We only mention the statement.

Theorem 3.5 (Hiroshima and Matte [19]) Let $\kappa > 0$. Suppose that $\hat{g}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, $\mathbb{1}_{|k|\geq\kappa}\hat{g}/\omega^2 \in L^1(\mathbb{R}^3)$ and $\beta < 1/\|\hat{g}/\sqrt{\omega}\|^2$. Let $\phi(g)$ be given by (1.1). Then

$$\varphi_{\rm g} \in D(e^{(\beta/2)\phi(g)^2})$$

and

$$\|e^{(\beta/2)\phi(g)^2}\varphi_{\rm g}\|^2 = \frac{1}{\sqrt{1-\beta\|\hat{g}/\sqrt{\omega}\|^2}}\mathbb{E}_{\mu_\infty}\left[e^{\frac{\beta K(g)^2}{1-\beta\|\hat{g}/\sqrt{\omega}\|^2}}\right], \tag{3.2}$$

where $K(g)$ denotes the random variable defined by

$$K(g) = \frac{1}{2}\int_{-\infty}^{\infty} dr \int_{\kappa\leq|k|} dk \frac{e^{-|r|\omega(k)}\hat{g}(k)e^{-ikB_r}}{\omega(k)}.$$

In particular

$$\lim_{\beta\uparrow\|\hat{g}/\sqrt{\omega}\|^{-2}} \|e^{(\beta/2)\phi(g)^2}\varphi_{\rm g}\| = \infty.$$

Proof In a similar manner to the proof of Theorem 3.4 we have

$$(\varphi_{\rm g}, e^{i\beta\phi(g)}\varphi_{\rm g}) = \mathbb{E}_{\mu_\infty}[e^{-k^2\beta^2 I_1/2}e^{-i\beta I_2}],$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} dk \frac{|\hat{g}(k)|^2}{2\omega(k)}, \\ I_2 &= \int_{\mathbb{R}^3} dk \mathbb{1}_{|k|\geq\kappa} \frac{\overline{\hat{g}(k)}}{2\omega(k)} \int_{-\infty}^{\infty} ds e^{-\omega(k)|s|}e^{-ikB_s}. \end{aligned}$$

Using the identity

$$e^{-\phi(g)^2/2} = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ik\phi(g)}e^{-k^2/2}dk,$$

and taking analytic continuation of β to some region in the complex plan, we have the theorem. $\square$

4 Concluding remarks

4.1 Comparison with H_Λ

In this article we show that the renormalized Nelson Hamiltonian has the unique ground state and the number of bosons in the ground state is super-exponential decay which can be proven by using the Gibbs measure derived from Feynman-Kac type formula. In [19] it is also shown that for $\kappa = 0$, H_∞ has no ground state, but Gross-transformed renormalized Nelson Hamiltonian H_∞^G has the ground state for all $\kappa \geq 0$. We note that H_∞ and H_∞^G are unitary equivalent if and only if $\kappa > 0$. We can also see localization such that $\|e^{\gamma|x|}\varphi_{\mathrm{g}}\| < \infty$ for some $\gamma > 0$. These results are counterparts of the results for H_Λ established in [1, 2, 3, 5, 11, 14, 15].

4.2 The Nelson model on a Lorenzian manifold

In [10] the Nelson model is defined on a static Lorenzian manifold instead of $\mathbb{R}^3$ and ultraviolet cutoff is removed. It is also interesting to studying ground states of the renormalized Nelson Hamiltonian defined on a static Lorenzian manifold. The Nelson Hamiltonian with ultraviolet cutoff defined on a static Lorenzian manifold has the ground state according to local properties (curvature) of the manifold. See [7, 8, 9]. We conjecture that the renormalized Nelson Hamiltonian defined on a static Lorenzian manifold also has the ground state in the same condition on local properties of manifold as those of the Nelson Hamiltonian with ultraviolet cutoff.

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