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Degeneracy structure of the spectrum of the asymmetric quantum Rabi model

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1 Introduction

The semi-classical Rabi model was originally defined by Isidor Isaac Rabi in 1936 [13] to describe the effect of a rapidly varying weak magnetic field on an oriented atom possessing nuclear spin. The fully quantized version, known now as the the *quantum Rabi model* (QRM), was introduced by Jaynes and Cummings in 1963 [6]. The QRM describes the simplest interaction between a two-level atom and a light field, making it one of the basic models of quantum optics.

Let $\mathcal{H}$ be a Hilbert space satisfying the hypothesis of the Stone-von Neumann theorem, with raising and lowering operators $a^\dagger$ and a, respectively. Then, the QRM is the model with Hamiltonian acting on $\mathcal{H} \otimes \mathbb{C}^2$ given by

$$H_{\mathrm{Rabi}} = \omega a^\dagger a + g(a + a^\dagger)\sigma_x + \Delta\sigma_z,$$

where σ_x, σ_z are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\omega > 0$ is the classical frequency of light field (modeled by a quantum harmonic oscillator), $2\Delta > 0$ is the energy difference of the two-level system and $g > 0$ is the interaction strength between the two systems.

It is not difficult to verify that H_{Rabi} is self-adjoint and that its spectrum consist only of the (discrete) set of eigenvalues (see e.g. Prop. 2.1-2.3 of [15]). In Figure 1, for $\Delta = 1$, we show the plot of the spectral curves of the QRM, obtained by considering the eigenvalues as functions of the parameter $g > 0$. The apparent crossings on the plots actually correspond to multiplicity two degeneracies in the spectrum of QRM.

An important feature of the QRM is the presence of a $\mathbb{Z}/2\mathbb{Z}$-symmetry. This symmetry amounts to the existence of the parity operator $\Pi = -\sigma_z(-1)^{a^\dagger a}$ satisfying $[\Pi, H_{\mathrm{Rabi}}] = 0$ and having eigenvalues $p = \pm 1$ (c.f. [12]). The presence of this symmetry allows one to write

$$H_{\mathrm{Rabi}} = H_{+\Delta} \oplus H_{-\Delta},$$

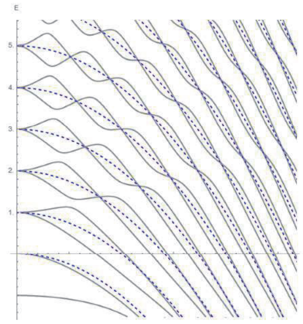


Figure 1: Spectral graph of QRM for $\Delta = 1$

for Hamiltonians $H_{\pm\Delta}$ acting on appropriate subspaces of $\mathcal{H} \otimes \mathbb{C}^2$. Degeneracies of multiplicity two consisting of an eigenvalue of $H_{+\Delta}$ and one eigenvalue of $H_{-\Delta}$ appear naturally, manifesting as crossings in the spectral curves (cf. in Figure 1).

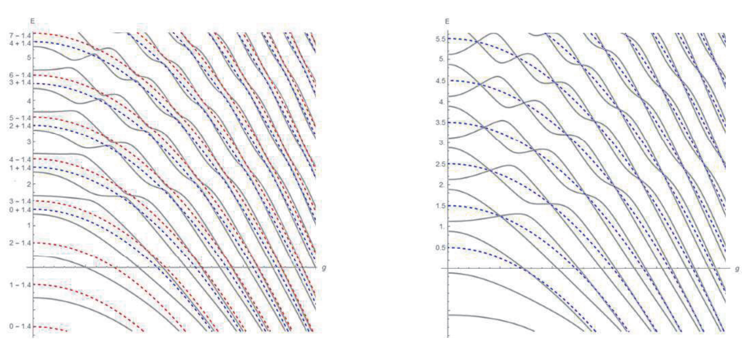
The symmetry in the QRM is broken by the introduction of a non-trivial interaction term, resulting in a model called *asymmetric quantum Rabi model* (AQRM). Concretely, the AQRM is the model described by the Hamiltonian

$$H_{\text{Rabi}}^{\varepsilon} = \omega a^{\dagger} a + \Delta \sigma_z + g \sigma_x (a^{\dagger} + a) + \varepsilon \sigma_x,$$

acting on $\mathcal{H} \otimes \mathbb{C}^2$, with $\varepsilon \in \mathbb{R}$. Clearly, the Hamiltonian of the QRM is recovered by setting $\varepsilon = 0$, that is, $H_{\text{Rabi}}^{0} = H_{\text{Rabi}}$.

In the same way as the QRM, it is verified that the spectrum of the AQRM consist only on the discrete set of eigenvalues of $H_{\text{Rabi}}^{\varepsilon}$. In general, due to the absence of a symmetry operator acting on the Hamiltonian $H_{\text{Rabi}}^{\varepsilon}$ for nonzero parameter $\varepsilon \in \mathbb{R}$, the presence is degenerate eigenvalues is a prior not to be expected.

In Figure 2, we show plots of spectral curves of the AQRM for fixed $\Delta = 1$ and different values of $\varepsilon \in \mathbb{R}$. Notice that for $\varepsilon = 1.4$, the spectral graph does not have crossings, that is, there is not degenerate eigenvalues of AQRM. However, in the case $\varepsilon = \frac{1}{2}$, (apparent) crossings in the spectral graphs were first observed by Li and Batchelor in [10].



(a) $\varepsilon = 1.4$ and $\Delta = 1$ (b) $\varepsilon = \frac{1}{2}$ and $\Delta = 1$

Figure 2: Spectral graphs AQRM

The presence of degenerate eigenvalues was proved for the case $\varepsilon = \frac{1}{2}$ by Masato Wakayama in [16], where he also conjecture that degeneracies are present for general $\varepsilon \in \frac{1}{2}\mathbb{Z}$. The conjecture was settled and the degeneracy structure of the spectrum of the AQRM was completed in [8]. In this document we present an overview and introduction to these results.

The document is organized as follows. First, in Section 2 we introduce the classification of the spectrum of the AQRM and its degeneracy structure. In Section 3 we explain the relation between constraint polynomials and Juddian solutions, leading to the proof of existence of degeneracies in the AQRM for half-integer ε. In Section 4 we give a brief description of the non-degenerate states of the AQRM via the study of the G-function and T-function.

2 The spectrum of the AQRM

In this section, we introduce the classification of the spectrum of the AQRM. As we have explained in the introduction, the spectrum of the AQRM consists only on the discrete set of (real) eigenvalues of $H^{\varepsilon}_{\mathrm{Rabi}}$, in other words, the continuous and residual spectrum of $H^{\varepsilon}_{\mathrm{Rabi}}$ are empty.

To discuss the classification of eigenvalues we introduce first the Segal-Bargmann Hilbert space (cf. [1, 5]). Let $\mathcal{V}(\mathbb{C})$ be the space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ with the inner-product $(\cdot,\cdot)_{\mathcal{H_B}}$ defined for $f, g \in \mathcal{V}(\mathbb{C})$ by

$$(f, g)_{\mathcal{B}} = \int_{\mathbb{C}} \overline{f(z)} g(z) d\mu(z) \tag{1}$$

where the measure $d\mu(z)$ is given by $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dxdy$ for $z = x + iy$, and $dxdy$ is the Lebesgue measure in $\mathbb{C} \simeq \mathbb{R}^2$.

The Segal-Bargmann space $\mathcal{H_B}$ is the space of (entire) functions $f \in \mathcal{V}(\mathbb{C})$ satisfying

$$\|f\|_{\mathcal{B}} = (f, f)_{\mathcal{B}}^{1/2} = \left(\int_{\mathbb{C}} |f(z)|^2 d\mu(z) \right)^{1/2} < \infty.$$

The Segal-Bargmann space $\mathcal{H_B}$ is a complete Hilbert space (cf. Proposition 14.15 of [5]). Moreover, in $\mathcal{H_B}$ the multiplication operator $Z = z$ and and differentiation operator $Y = \partial_z = \frac{d}{dz}$ acting on $\mathcal{H_B}$ satisfy the commutation relation

$$[Y, Z] = 1,$$

and in particular, are verified to be realizations of the raising and lowering operators $a^\dagger$ and a.

Next, we consider the representation of the eigenvalue problem of the AQRM in the Segal-Bargmann space $\mathcal{H_B}$. The Hamiltonian $H^{\varepsilon}_{\mathrm{Rabi}}$, realized as an operator acting on $\mathcal{H_B} \otimes \mathbb{C}^2$, corresponds to the operator

$$\tilde{H}^{\varepsilon}_{\mathrm{Rabi}} := \begin{bmatrix} z\partial_z + \Delta & g(z + \partial_z) + \varepsilon \\ g(z + \partial_z) + \varepsilon & z\partial_z - \Delta \end{bmatrix}.$$

Then, the time-independent Schrödinger equation $H^{\varepsilon}_{\text{Rabi}}\varphi = \lambda\varphi\,(\lambda \in \mathbb{R})$ is equivalent to the system of first order differential equations

$$\tilde{H}^{\varepsilon}_{\text{Rabi}}\psi = \lambda\psi, \quad \psi = \begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix},$$

where eigenfunctions of $H^{\varepsilon}_{\text{Rabi}}$ associated to a given eigenvalue $\lambda \in \mathbb{R}$ correspond to solutions $\psi_i \in \mathcal{H}_{\mathcal{B}}$ $i = 1, 2$.

Therefore, the eigenvalue problem of the AQRM amounts to finding entire functions $\psi_1, \psi_2 \in \mathcal{H}_{\mathcal{B}}$ and real number λ satisfying

$$\begin{cases} (z\partial_z + \Delta)\psi_1 + (g(z + \partial_z) + \varepsilon)\psi_2 = \lambda\psi_1, \\ (g(z + \partial_z) + \varepsilon)\psi_1 + (z\partial_z - \Delta)\psi_2 = \lambda\psi_2. \end{cases}$$

Now, by setting $f_{\pm} = \psi_1 \pm \psi_2$, we get

$$\begin{cases} (z + g)\dfrac{d}{dz}f_+ + (gz + \varepsilon - \lambda)f_+ + \Delta f_- = 0, \\ (z - g)\dfrac{d}{dz}f_- - (gz + \varepsilon + \lambda)f_- + \Delta f_+ = 0. \end{cases} \tag{2}$$

Notice that the system (2) has an (unramified) irregular singular point at $z = \infty$ in addition to regular singular points at $z = \pm g$. It is known (see e.g. [4]) that actually, any entire solution ψ of (2) is actually $\psi \in \mathcal{H}_{\mathcal{B}} \otimes \mathbb{C}^2$.

By using the substitution $\phi_{1,\pm}(z) := e^{gz}f_{\pm}(z)$ and the change of variable $y = \frac{g+z}{2g}$, we obtain

$$\begin{cases} y\dfrac{d}{dy}\phi_{1,+}(y) = (\lambda + g^2 - \varepsilon)\phi_{1,+}(y) - \Delta\phi_{1,-}(y), \\ (y - 1)\dfrac{d}{dy}\phi_{1,-}(y) = (\lambda + g^2 - \varepsilon - 4g^2 + 4g^2y + 2\varepsilon)\phi_{1,-}(y) - \Delta\phi_{1,+}(y). \end{cases} \tag{3}$$

Defining $a := -(\lambda + g^2 - \varepsilon)$, we get

$$\begin{cases} y\dfrac{d}{dy}\phi_{1,+}(y) = -a\phi_{1,+}(y) - \Delta\phi_{1,-}(y), \\ (y - 1)\dfrac{d}{dy}\phi_{1,-}(y) = -(4g^2 - 4g^2y + a - 2\varepsilon)\phi_{1,-}(y) - \Delta\phi_{1,+}(y). \end{cases} \tag{4}$$

We remark here that we can define a system of linear differential equations (similar to (4)) by applying the substitutions $\phi_{2,\pm}(z) := e^{-gz}f_{\pm}(z)$ and $\bar{y} = \frac{g-z}{2g}$. In order to make the full analysis of the holomorphicity of solutions, it is necessary to consider both systems. For simplicity, in this document we consider only system (4) and leave the detailed discussion to [8].

The exponents of the equation system can be obtained by standard computation, and are shown in Table 1 for reference.

Due to the presence of finite singularities, solutions of (4) are not to be automatically assumed to correspond to solutions of the eigenvalue problem of the AQRM. The verification of the holomorphicity (on the complex plane) of the Frobenius solution

Table 1: Exponents of system (4).

	$\phi_{1,-}(y)$	$\phi_{1,+}(y)$
$y=0$	$0, -a+1$	$0, -a$
$y=1$	$0, -a+2\varepsilon$	$0, -a+2\varepsilon+1$

of the system (4) depends on the value of the parameter a, in other words, of the eigenvalue λ. For instance, if $\lambda = N \pm \varepsilon - g^2$ (i.e. $-a = N$), then the difference between the two exponents (at $y = 0$) is an integer and the system may develop a logarithmic branch-cut at $y = 0$.

The ongoing considerations motivate the classification of the eigenvalues of AQRM. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $H^{\varepsilon}_{\text{Rabi}}$, then

1. if there is an integer $\mathbb{N} \in \mathbb{Z}$ such that $\lambda = N \pm \varepsilon - g^2$, λ is called *exceptional eigenvalue*,
2. if λ is not an exceptional eigenvalue, we say that λ is a *regular eigenvalue*.

In the case that λ is an exceptional eigenvalues, it may be the case that the solution $\phi_{1,-}(y)$ of (4) is polynomial, in which case it is automatically entire and thus, it corresponds to a solution of the eigenvalues problem. Such a solution (and the corresponding eigenvalue) is called *Juddian* (also known as quasi-exact). Otherwise, we say that the solution (resp. the eigenvalue) is *non-Juddian exceptional*. In this context, a non-Juddian eigenvalue is either a regular eigenvalue or a non-Juddian exceptional eigenvalue.

Historically, the first eigenvalues of QRM to be described were the Juddian eigenvalues, studied by Judd in [7] and Kuś in [9]. Concretely, Kuś showed the presence of degenerate eigenvalues of the form $\lambda = N - g^2$ in the spectrum of the QRM, subject to a polynomial equation. These eigenvalues constitute the crossings of the spectral graph in Figure 1. In fact, it was shown in [8] (see also [10, 16] for the case $\varepsilon = \frac{1}{2}$) that for $\varepsilon \in \frac{1}{2}\mathbb{Z}$ the degenerate solutions are exactly the Juddian ones and that any other solution is non-degenerate. The complete degeneracy picture for the AQRM is shown in Table 2.

Type			
Eigenvalue		Solution	Degenerate
Regular			$\times$
Exceptional	$\epsilon \neq \ell/2$	Juddian	$\times$
		Non-Juddian	$\times$
		Juddian /Non-Juddian	$\times$
Exceptional	$\epsilon = \ell/2$	Juddian	✓
		Non-Juddian	$\times$
		Juddian/ Non-Juddian	$\times$

Table 2: Eigenvalue structure of AQRM

The non-degeneracy of regular solution was proved in [2], along with the non-Juddian solutions for the case $\varepsilon \notin \frac{1}{2}\mathbb{Z}$.

In Section 3 we given an overview of the proof of the existence of degeneracy of Juddian solutions for the case $\varepsilon \in \frac{1}{2}\mathbb{Z}$ and in Section 4 we describe the conditions for the existence of non-Juddian solutions.

3 Juddian solutions: constraint polynomials

The presence of a Juddian eigenvalue $\lambda = N \pm \varepsilon - g^2\,(N \in \mathbb{N})$ in the spectrum of $H^{\varepsilon}_{\mathrm{Rabi}}$ for parameters $\Delta, g > 0$ is equivalent (cf. [10, 16]) to the existence of solution of the polynomial equation

$$P_N^{(N,\pm\varepsilon)}((2g)^2, \Delta^2) = 0. \tag{5}$$

The polynomial $P_N^{(N,\varepsilon)}(x,y)$ is known as *constraint polynomial* and equation (5) is the *constraint relation* for the Juddian eigenvalue $\lambda = N \pm \varepsilon - g^2$. The constraint polynomial $P_N^{(N,\varepsilon)}(x,y)$ is the N-th member of a family of polynomials defined by a recurrence relation.

Definition 3.1. *Let $N \in \mathbb{Z}_{\geq 0}$. The polynomials $P_k^{(N,\varepsilon)}(x,y)$ of degree k are defined recursively by*

$$\begin{aligned}
&P_0^{(N,\varepsilon)}(x,y) = 1,\\
&P_1^{(N,\varepsilon)}(x,y) = x + y - 1 - 2\varepsilon,\\
&P_k^{(N,\varepsilon)}(x,y) = (kx + y - k(k+2\varepsilon))P_{k-1}^{(N,\varepsilon)}(x,y) - k(k-1)(N-k+1)xP_{k-2}^{(N,\varepsilon)}(x,y).
\end{aligned}$$

For brevity, we set $c_k^{(\varepsilon)} = k(k+2\varepsilon)$ and $\lambda_k = k(k-1)(N-k+1)$.

A necessary condition for two exceptional eigenvalues $\lambda_1 = N + \varepsilon - g^2$ and $\lambda_2 = M - \varepsilon - g^2$ with $N, M \in \mathbb{Z}_{\geq 0}$ and $N \neq M$, to be equal is that

$$\varepsilon = \frac{M-N}{2} = \frac{\ell}{2} \in \frac{1}{2}\mathbb{Z},$$

that is, ε must be half-integer. In terms of constraint polynomials, this is equivalent to the simultaneous satisfaction of the two constraint relations

$$P_N^{(N,\ell/2)}((2g)^2, \Delta^2) = 0 = P_{N+\ell}^{(N+\ell,-\ell/2)}((2g)^2, \Delta^2), \tag{6}$$

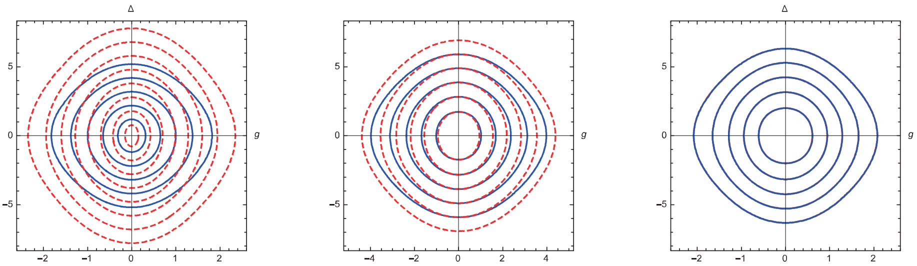
where $N \in \mathbb{Z}_{\geq 0}$ and $\ell \geq 0$.

Following this argumentation, Masato Wakayama conjectured in [16] that the relation

$$P_{N+\ell}^{(N+\ell,-\ell/2)}(x,y) = A_N^{\ell}(x,y)P_N^{(N,\ell/2)}(x,y), \tag{7}$$

holds for $N, \ell \in \mathbb{Z}_{\geq 0}$ and that the polynomials $A_N^{\ell}(x,y)$ have no positive roots for $x, y > 0$.

The divisibility condition (7) and the positivity of the factor $A_N^{\ell}(x,y)$ are illustrated in Figure 3 where the curves described by the zeros of constraint polynomials



(a) $\epsilon = 0.3$ (b) $\epsilon = 1$ (c) $\epsilon = 3/2$

Figure 3: Curves defined by constraint polynomials for $N = 5, \ell = 3$

$P_{N+\ell}^{(N+\ell,-\varepsilon)}(x,y)$ and $P_N^{(N,\varepsilon)}(x,y)$, with $N = 5, \ell = 8$, are plotted in the (g,Δ)-plane for different values of ε. Notice that in the case $\varepsilon = \frac{1}{2}\mathbb{Z}$, the zeros of both constraint polynomials exactly coincide.

As mentioned in the introduction, the conjecture above was settled in [8]. Actually, we have the following generalization (also conjectured in [16]), these results are the topic of the paper by the author [14].

Theorem 3.2. *Let $\ell, k \in \mathbb{Z}_{\geq 0}$, then*

$$P_{k+\ell}^{(N+\ell,-\frac{\ell}{2})}(x,y) = A_k^{(\ell)}(x,y)P_k^{(N,\frac{\ell}{2})}(x,y) + B_k^{(N,\ell)}(x,y)$$

with $B_N^{(N,\ell)}(x,y) = 0$.

Notice that the conjecture (7) is recovered from Theorem 3.2 by setting $k = N$. The positivity part of the conjecture also holds in the general case.

Theorem 3.3. *With the notation of Theorem 3.2, $A_k^{(\ell)}(x,y) > 0$ for $x, y > 0$.*

Next, we sketch the proof of the Theorems 3.2 and 3.3. For a tridiagonal matrix we write

$$\operatorname{tridiag}\begin{bmatrix} a_i & b_i \\ c_i & \end{bmatrix}_{1\leq i\leq n} := \begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & 0 & c_{n-1} & a_n \end{bmatrix}.$$

Since the polynomials $P_k^{(N,\varepsilon)}(x,y)$ are defined by a recurrence relation, the naturally have a representation as the determinant of a $k \times k$ tridiagonal matrix,

$$P_k^{(N,\varepsilon)}(x,y) = \det(\mathbf{I}_k y + \mathbf{A}_k^{(N)} x + \mathbf{U}_k^{(\varepsilon)}) \tag{8}$$

where $\mathbf{I}_k$ is the identity matrix of size k and

$$\mathbf{A}_k^{(N)} = \operatorname{tridiag}\begin{bmatrix} i & 0 \\ \lambda_{i+1} & \end{bmatrix}_{1\leq i\leq k}, \quad \mathbf{U}_k^{(\varepsilon)} = \operatorname{tridiag}\begin{bmatrix} -c_i^{(\varepsilon)} & 1 \\ 0 & \end{bmatrix}_{1\leq i\leq k}.$$

The key to the proof of Theorem 3.2 is the fact that the polynomials $P_k^{(N,\varepsilon)}(x,y)$ can be expressed as the determinant of a tridiagonal matrix plus a rank-one perturbation.

Proposition 3.4. *Let $k \in \mathbb{Z}_{\geq 0}$, then*

$$P_k^{(N,\varepsilon)} = \det\left(\mathbf{I}_k y + \mathbf{D}_k x + \mathbf{C}_k^{(N,\epsilon)} + \mathbf{e}_k{}^T \mathbf{u}\right),$$

where $\mathbf{I}_k$ is the identity matrix, $\mathbf{D}_k = \operatorname{diag}(1, 2, \ldots, k)$ and $\mathbf{C}_k^{(N,\epsilon)}$ is the tridiagonal matrix given by

$$\mathbf{C}_k^{(N,\epsilon)} = \operatorname{tridiag}\begin{bmatrix} -i(2(N-i)+1+2\varepsilon) & 1 \\ i(i+1)c_{N-i}^{(\varepsilon)} & \end{bmatrix}_{1\leq i\leq k},$$

$\mathbf{e}_k \in \mathbb{R}^k$ is the k-th standard basis vector and $\mathbf{u} \in \mathbb{R}^k$ is given entrywise by

$$\mathbf{u}_j = (-1)^{k-j+2}\binom{k+1}{j}\frac{k!(N-j)!}{(j-1)!(N-k-1)!}$$

Sketch of the proof of Theorem 3.2. By elementary linear algebra, from Proposition 3.4 we obtain the expression

$$P_k^{(N+\ell,-\frac{\ell+N-k}{2})}(x,y) = \det\left(\mathbf{I}_k y + \mathbf{D}_k x + \mathbf{C}_k^{(N+\ell,-\frac{\ell+N-k}{2})}\right) + q_k(x,y),$$

for some polynomial $q_k(x,y)$ divisible by $N-k$. Next, observe that the matrix $\mathbf{I}_k y + \mathbf{D}_k x + \mathbf{C}_k^{(N+\ell,-\frac{\ell+N-k}{2})}$ is block diagonal and therefore, the determinant is given by the product of

$$\bar{A}_k^{(N,\ell)}(x,y) = \frac{(k+\ell)!}{k!}\det\operatorname{tridiag}\begin{bmatrix} x+\frac{y}{k+i}+2i-1+k-N-\ell & 1 \\ c_{-i}^{\frac{N+\ell-k}{2}} & \end{bmatrix}_{1\leq i\leq \ell}.$$

and

$$P_k^{(N,\frac{\ell+N-k}{2})}(x,y) + q'(x,y;N,\ell,k)$$

for some polynomial $q'(x,y;N,\ell,k)$ divisible by $N-k$.

Note that the matrices in the determinant expressions of $\bar{A}_k^{(N,\ell)}(x,y)$ and

$$A_k^{(\ell)}(x,y) := \frac{(k+\ell)!}{k!}\det\operatorname{tridiag}\begin{bmatrix} x+\frac{y}{k+i}+2i-1-\ell & 1 \\ c_{-i}^{(\frac{\ell}{2})} & \end{bmatrix}_{1\leq i\leq \ell}$$

differ entrywise only by multiples of $N-k$. The result then follows from the multi-linearity of the determinant. □

Next, we give a sketch of the proof of positivity. First, note that we can find a matrix $\mathbf{M}_\ell^{(N)}(x)$ such that

$$\operatorname{Spec}(-\mathbf{M}_\ell^{(N)}(x)) = \{y \in \mathbb{R} : A_N^{(\ell)}(x,y) = 0\}$$

for any fixed $x > 0$.

Then, we establish the properties of $\mathbf{M}_\ell^{(N)}(x)$:

- $\det(\mathbf{M}_\ell^{(N)}(x)) = \frac{(N+\ell)!}{N!}x^\ell$.

- For $x \geq 0$, the eigenvalues $\lambda \in \mathrm{Spec}(\mathbf{M}_\ell^{(N)}(x))$ are real.
- We have $\mathrm{Spec}(\mathbf{M}_\ell^{(N)}(0)) = \{i(\ell - i) : i = 1, 2, \cdots, \ell\}$.
- If $x' > \ell - 1$, all eigenvalues $\lambda \in \mathrm{Spec}(\mathbf{M}_\ell^{(N)}(x'))$ satisfy $\lambda > 0$.

To prove the positivity it is enough to show that all the eigenvalues of $\mathbf{M}_\ell^{(N)}(x)$ are positive for $x > 0$.

Sketch of the proof of Theorem 3.3. Suppose there is a positive x' such that $\mathbf{M}_\ell^{(N)}(x')$ has an eigenvalue $\lambda(x') < 0$.

Since $\lambda(x) \in \mathrm{Spec}(\mathbf{M}_\ell^{(N)}(x))$ is a continuous real-valued function and $\lambda(\ell) > 0$, there is x'' with $x' < x'' < \ell$ such that

$$\lambda(x'') = 0 \in \mathrm{Spec}(\mathbf{M}_\ell^{(N)}(x'')).$$

Thus, $0 = \det(\mathbf{M}_\ell^{(N)}(x'')) = \frac{(N+\ell)!}{N!}(x'')^\ell > 0$. □

We summarize the discussion of constraint polynomials in terms of the spectrum of the AQRM in the following theorem.

Theorem 3.5. *If $x = (2g)^2$ is a root of the equation $P_N^{(N,\ell/2)}(x, \Delta^2) = 0$, then the Juddian eigenvalue $\lambda = N + \ell/2 - g^2$ is a degenerate exceptional eigenvalue of multiplicity 2. Moreover, the two linearly independent solutions are Juddian.* □

For a proof of the linear independence of the solution using techniques from representation theory of $\mathfrak{sl}_2$, we refer the reader to [16].

4 Non-juddian eigenvalues: constraint functions

The G-function was introduced in 2011 by Daniel Braak [2] to describe analytically the regular solutions of the QRM. It was defined by considering the conditions for the solutions of the system (4) to be entire, and thus constitute solutions of the eigenvalue problem of AQRM.

Definition 4.1. *The G-function for the Hamiltonian $H^\varepsilon_{\mathrm{Rabi}}$ is defined as*

$$G_\varepsilon(x; g, \Delta) := \Delta^2 \bar{R}^+(x; g, \Delta, \varepsilon)\bar{R}^-(x; g, \Delta, \varepsilon) - R^+(x; g, \Delta, \varepsilon)R^-(x; g, \Delta, \varepsilon)$$

where

$$R^\pm(x; g, \Delta, \varepsilon) = \sum_{n=0}^{\infty} K_n^\pm(x) g^n \quad \text{and} \quad \bar{R}^\pm(x; g, \Delta, \varepsilon) = \sum_{n=0}^{\infty} \frac{K_n^\pm(x)}{x - n \pm \varepsilon} g^n, \tag{9}$$

whenever $x \mp \varepsilon \notin \mathbb{Z}_{\geq 0}$, respectively. For $n \in \mathbb{Z}_{\geq 0}$, define the functions $f_n^\pm = f_n^\pm(x, g, \Delta, \varepsilon)$ by

$$f_n^\pm(x, g, \Delta, \varepsilon) = 2g + \frac{1}{2g}\Big(n - x \pm \varepsilon + \frac{\Delta^2}{x - n \pm \varepsilon}\Big), \tag{10}$$

then, the coefficients $K_n^\pm(x) = K_n^\pm(x, g, \Delta, \varepsilon)$ are given by the recurrence relation

$$nK_n^\pm(x) = f_{n-1}^\pm(x, g, \Delta, \varepsilon)K_{n-1}^\pm(x) - K_{n-2}^\pm(x) \quad (n \geq 1) \tag{11}$$

with initial condition $K_{-1}^\pm = 0$ and $K_0^\pm = 1$.

For fixed parameters $\{g, \Delta, \varepsilon\}$ the zeros x_n of $G_\varepsilon(x; g, \Delta)$ correspond to regular eigenvalues $\lambda_n = x_n - g^2$ of $H^\varepsilon_{\text{Rabi}}$ (cf. [2, 3, 12]).

On the other hand, if $x \in \mathbb{R}$ is fixed, the equation

$$G_\varepsilon(x; g, \Delta) = 0,$$

is the constraint condition for the regular eigenvalue $\lambda = x - g^2$.

In a similar way, it is possible to define a T-function $T^{(N)}_\varepsilon(g, \Delta)$ such that the solutions $g, \Delta > 0$ of the equations

$$T^{(N)}_\varepsilon(g, \Delta) = 0,$$

correspond to the values of the parameters such that the the spectrum of $H^\varepsilon_{\text{Rabi}}$ contains the non-Juddian exceptional eigenvalue $\lambda = N + \varepsilon - g^2$.

The constraint T-function $T^{(N)}_\varepsilon(g, \Delta)$ of the AQRM is given by

$$T^{(N)}_\varepsilon(g, \Delta) = \bar{R}^{(N,+)}(g, \Delta; \varepsilon)\bar{R}^{(N,-)}(g, \Delta; \varepsilon) - R^{(N,+)}(g, \Delta; \varepsilon)R^{(N,-)}(g, \Delta; \varepsilon), \tag{12}$$

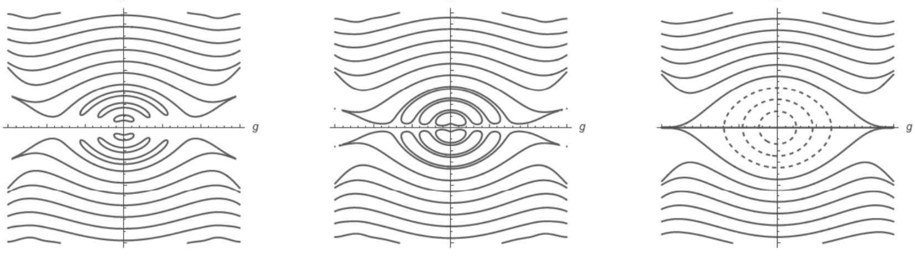
with

$$\bar{R}^{(N,-)}(g, \Delta; \varepsilon) = \phi_{1,+}\Big(\frac{1}{2}; \varepsilon\Big), \qquad \bar{R}^{(N,+)}(g, \Delta; \varepsilon) = \phi_{2,+}\Big(\frac{1}{2}; -\varepsilon\Big), \tag{13}$$

$$R^{(N,-)}(g, \Delta; \varepsilon) = \phi_{1,-}\Big(\frac{1}{2}; \varepsilon\Big), \qquad R^{(N,+)}(g, \Delta; \varepsilon) = \phi_{2,-}\Big(\frac{1}{2}; -\varepsilon\Big), \tag{14}$$

where $\phi_{1,\pm}$ and $\phi_{2,\pm}$ are solutions of (4) (see [8] for the precise definition).

The curves of the constraint relations for non-Juddian eigenvalues (either G-function or T-function) are shown in Figure 4. In the case of exceptional eigenvalues (i.e. the case $N = 3, x = 3.5$) the constraint relations for Juddian eigenvalues are shown in dashed lines.



(a) $x = 3.2$ (b) $x = 3.4$ (c) $N = 3, x = 3.5$)

Figure 4: Curves defined by constraint relations (G-function for (a) and (b), T-function and constraint polynomials for (c))

In fact, the G-function contains almost all the information regarding the spectrum of AQRM (not just the regular spectrum). First, from the definition we see that at the point $x = N \pm \varepsilon\,(N \in \mathbb{Z}_{\geq 0})$ the G-function has a singularity which is immediately seen to be either a simple, pole or a removable singularity. These poles are also seen to be the only singularities of the G-function.

For the case $\varepsilon \notin \frac{1}{2}\mathbb{Z}$, the residues at the simple poles of the G-function are given in terms of the constraint functions.

Proposition 4.2. *Let $\varepsilon \notin \frac{1}{2}\mathbb{Z}$. Then any pole of the G-function $G_\varepsilon(x;g,\Delta)$ is simple. If $N \in \mathbb{Z}_{\geq 0}$, the residue of $G_\varepsilon(x;g,\Delta)$ at the points $x = N \pm \varepsilon$ is given by*

$$\operatorname*{Res}_{x=N\pm\varepsilon} G_\varepsilon(x;g,\Delta) = C(N)\Delta^2 P_N^{(N,\pm\varepsilon)}((2g)^2,\Delta^2)T_{\pm\varepsilon}^{(N)}(g,\Delta),$$

where $C(N) = \frac{1}{N!(N+1)!}$.

In particular, we see that residues at the poles vanish when there is an exceptional eigenvalue $\lambda = N \pm \varepsilon - g^2$ corresponding to the parameters $g, \Delta > 0$.

The computation of the residues for the case of half-integer ε is more complicated and we refer the reader to [8] for the details. However, the situation is summarized in the following result.

Proposition 4.3. *Suppose $\ell \in \mathbb{Z}_{\geq 0}$ and let $\Delta > 0$ be fixed. The G-function $G_{\ell/2}(x;g,\Delta)$ has ℓ poles of order ≤ 1 at $x = N - \ell/2$ for $0 \leq N < \ell$ and poles of order ≤ 2 at $x = N + \ell/2$ for $N \in \mathbb{Z}_{\geq 0}$. Moreover, for $N \in \mathbb{Z}_{\geq 0}$, we have:*

- *If $\lambda = N \pm \ell/2 - g^2$ is a Juddian eigenvalue of $H_{\text{Rabi}}^{\ell/2}$, then $x = N \pm \ell/2$ is not a pole of $G_{\ell/2}(x;g,\Delta)$.*
- *For $0 \leq N < \ell$, the function $G_{\ell/2}(x;g,\Delta)$ does not have a pole at $x = N - \ell/2$ if and only if $\lambda = N - \ell/2 - g^2$ is a non-Juddian exceptional eigenvalue of $H_{\text{Rabi}}^{\ell/2}$.*
- *If $G_{\ell/2}(x;g,\Delta)$ has a simple pole at $x = N + \ell/2$, then $\lambda = N + \ell/2 - g^2$ is a non-Juddian exceptional eigenvalue of $H_{\text{Rabi}}^{\ell/2}$.*
- *If $G_{\ell/2}(x;g,\Delta)$ has a double pole at $x = N \pm \ell/2$, then there is no exceptional eigenvalue $\lambda = N \pm \ell/2 - g^2$ of $H_{\text{Rabi}}^{\ell/2}$.*

In this way, for a general $\varepsilon \in \mathbb{R}$, the residues at the poles $x = N \pm \varepsilon$ (or the coefficient of the -2 power term in the Laurent series expansion for the case of double poles) are determined by the constraint functions for exceptional eigenvalues $\lambda = N \pm \varepsilon - g^2$. It is then possible to define a generalized (or extended) G-function for the AQRM that is holomorphic in the complex plane.

Definition 4.4. *The generalized G-function of the AQRM is*

$$\mathcal{G}_\varepsilon(x;g,\Delta) := G_\varepsilon(x;g,\Delta)\Gamma(\varepsilon - x)^{-1}\Gamma(-\varepsilon - x)^{-1}. \tag{15}$$

As a consequence of the discussion above on the poles of the G-function, we can establish the following result.

Theorem 4.5. *For fixed $g, \Delta > 0$, x is a zero of $\mathcal{G}_\varepsilon(x;g,\Delta)$ if and only if $\lambda = x - g^2$ is an eigenvalue of $H_{\text{Rabi}}^{\varepsilon}$.*

A related definition for the generalized G-function $\mathcal{G}_\varepsilon(x;g,\Delta)$ was used in [11] to compute all the eigenvalues of the AQRM in a unified way. This numerical computation is justified by Theorem 4.5 above.

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