数理解析研究所講究録 第2128巻 2019年 90-102

内部重力波の斜め反射により生じる平均流について

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要旨

Weakly nonlinear analysis is utilized to discuss the mean flow induced by the reflection of an internal gravity wave beam at a uniform rigid slope, in the case where the beam planes of constant phase meet the slope at an arbitrary direction, not necessarily parallel to the isobaths, and the flow cannot be taken as two-dimensional. Along the vertical, the Eulerian mean flow due to such an oblique reflection is equal and opposite to the Stokes drift so the Lagrangian mean flow vanishes, similar to a two-dimensional reflection. The horizontal Eulerian mean flow, however, is controlled by the mean potential vorticity (PV) and the corresponding Lagrangian mean flow is generally nonzero, in contrast to two-dimensional flow where PV identically vanishes. For an oblique reflection, furthermore, viscous dissipation can trigger generation of horizontal mean flow via irreversible production of mean PV, a phenomenon akin to streaming. This report is based on Kataoka & Akylas (2019).

1. 緒言

Internal waves obey unusual rules of reflection at a sloping boundary: the incident and reflected wave rays make the same angle to the direction of gravity, rather than the normal to the boundary as would be the case for isotropic wave propagation (Phillips 1966). Also, because the internal wave frequency is independent of the wave vector magnitude, the same reflection rules also apply to wave beams. These are time-harmonic plane waves that propagate at a given angle as determined by the dispersion relation, but whose spatial profile involves a superposition of sinusoidal plane waves with wave vectors of different magnitude but fixed direction (Tabaei & Akylas 2003). Internal wave beams arise in various geophysical contexts, particularly in connection with the ocean internal tide (Lamb 2004; Cole et al. 2009) and the generation of atmospheric gravity waves due to thunderstorms (Fovell et al. 1992).

An interesting feature of internal wave beam reflection at a rigid boundary is the possibility of radiating beams at higher harmonics of the incident wave frequency. Such secondary reflected beams were noted in numerical simulations of the internal tide (Lamb 2004) and later explained theoretically (Tabaei et al. 2005) as the result of nonlinear interactions between the incident and the primary reflected beam. These interactions occur in the region where the primary-harmonic beams overlap close to the boundary, and give rise to a second-harmonic and a mean disturbance (to leading, quadratic order). If the second harmonic is below the buoyancy frequency, the former of these disturbances then radiates away as a secondary beam at an angle determined by the dispersion relation. The induced mean flow, by contrast, is confined in the interaction region and results in no net transport as the associated Lagrangian mean flow turns out to be zero. The radiation of secondary reflected beams has also been confirmed experimentally (Peacock & Tabaei 2005; Rodenborn et al. 2011).

In the present report, we revisit the reflection of an internal wave beam at a sloping boundary using the weakly nonlinear approach of Tabaei et al. (2005). Here the focus is on the case where the beam planes of constant phase meet the slope at an arbitrary direction, not necessarily parallel to the isobaths, so unlike Tabaei et al. (2005) the flow cannot be taken as two-dimensional. Such an oblique reflection can radiate secondary reflected beams as before, but there are serious ramifications in regard to the induced mean flow. Along the vertical, the Lagrangian mean flow vanishes as in a two-dimensional reflection, but the horizontal mean flow is now tied to the evolution of mean potential vorticity (PV). The induced horizontal mean flow cannot be determined by steady-state analysis alone, and the corresponding Lagrangian mean flow is generally non-zero. A similar ambiguity regarding the induced along-slope mean flow at steady state was found in the special case of obliquely reflecting sinusoidal plane waves (Thorpe 1997). Furthermore, the evolution equation for the mean PV due to an oblique reflection reveals that viscous dissipation can trigger the generation of horizontal mean flow via irreversible production of mean PV (Mcintyre & Norton 1997). This mechanism is akin to that responsible for ‘streaming’ due to attenuating waves in acoustics (Lighthill 1978) and propagating internal wave beams in the presence of three-dimensional variations (Bordes et al. 2012; Kataoka & Akylas 2015; Fan et al. 2018).

2. 定式化

The present analysis assumes the same general setting as Tabaei et al. (2005), namely a propagating time-harmonic internal wave beam that reflects at a rigid slope of constant angle

α to the horizontal, in a uniformly stratified Boussinesq fluid with (constant) buoyancy frequency N_0. However, here the incident and reflected beams are taken to meet the slope at an arbitrary direction, not necessarily parallel to the isobaths, so the flow is no longer two-dimensional (see Figure 1). To discuss such an oblique reflection, we shall use the rotated coordinate system (ξ,η,ζ) relative to (x,y,z), as shown in Figure 2. Here, x and y are horizontal coordinates, y being along the slope isobaths, and z is a vertical coordinate pointing upwards, so the plane $z = x\tan\alpha$, $-\infty < y < \infty$, coincides with the slope boundary. Now, ζ is perpendicular to this plane and ξ, η are in-plane coordinates such that η is along the direction where the beam planes of constant phase intersect the slope; thus, the angle β of η relative to the isobaths (y-) direction controls the reflection obliqueness, with $\beta = 0$ corresponding to the normal (two-dimensional) reflection discussed in Tabaei et al. (2005).

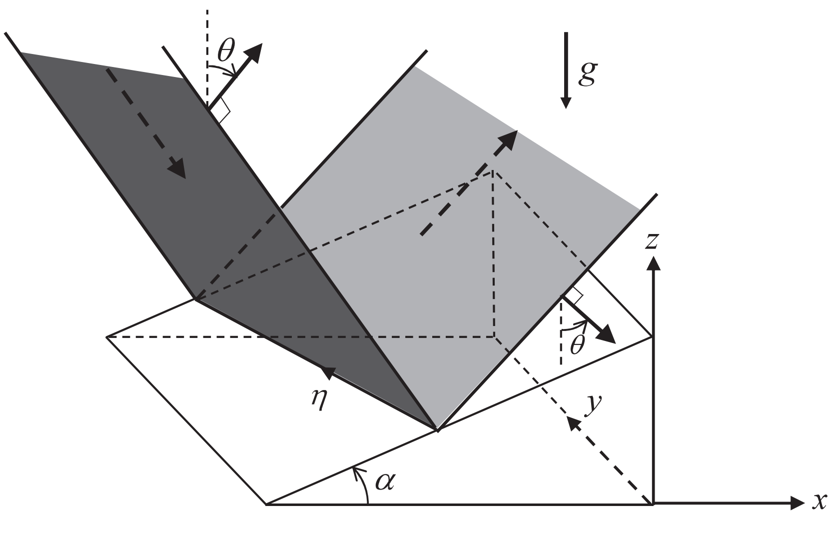

Figure 1. Geometry of obliquely incident and (primary) reflected beam at a slope α.

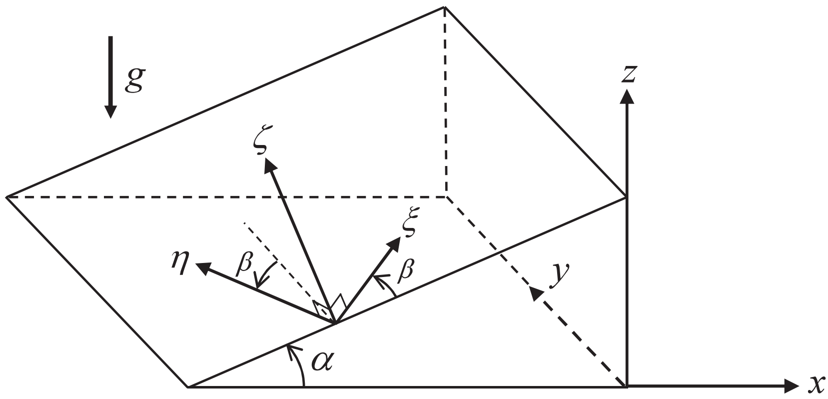

Figure 2. The rotated coordinate system (ξ, η, ζ).

An advantage of the rotated coordinates (ξ,η,ζ) is that the flow is independent of η, so incompressibility can be automatically satisfied by introducing a streamfunction $\psi(\xi,\zeta,t)$ for the velocity field (u,w) in the (ξ,ζ) plane:

$$u=\psi_{\zeta}, \quad w=-\psi_{\xi}. \tag{1}$$

Then, from mass and momentum conservation, the equations governing ψ, the transverse (along η) velocity $v(\xi,\zeta,t)$ and the reduced density $\rho(\xi,\zeta,t)$ are

$$\rho_t+J(\rho,\psi)+\psi_{\chi}+v\sin\alpha\sin\beta=0, \tag{2a}$$

$$(\nabla^2\psi)_t+J(\nabla^2\psi,\psi)-\rho_{\chi}=0, \tag{2b}$$

$$v_t+J(v,\psi)-\rho\sin\alpha\sin\beta=0, \tag{2c}$$

where

$$J(a,b)=a_{\xi}b_{\zeta}-a_{\zeta}b_{\xi}, \quad \frac{\partial}{\partial\chi}=\cos\alpha\frac{\partial}{\partial\xi}-\sin\alpha\cos\beta\frac{\partial}{\partial\zeta}, \quad \nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\zeta^2}.$$

Furthermore, the inviscid boundary condition at the slope is

$$w=0 \quad (\zeta=0). \tag{3}$$

(In the equations above and throughout this paper, all variables have been made dimensionless as in Tabaei et al. (2005), employing a characteristic length associated with the incident wave beam as lengthscale and $1/N_0$ as timescale.)

It will also prove convenient for the ensuing analysis to introduce the potential vorticity (PV), which in the present setting can be expressed as

$$q = \Omega - J(v, \rho)\,, \tag{4a}$$

where

$$\Omega = -\sin\alpha \sin\beta \nabla^2 \psi + v_\chi \tag{4b}$$

is the vertical vorticity. It can be deduced from equations (2) that in an inviscid fluid q is a materially conserved quantity

$$\frac{Dq}{Dt} = 0\,, \tag{5}$$

where $D/Dt \equiv \partial/\partial t + J(\bullet, \psi)$ is the material derivative. Unlike the case of normal reflection ($\beta = 0$) where $v \equiv 0$ and hence $q \equiv 0$, here PV turns out to play an important part in the generation of mean flow owing to nonlinear interactions of obliquely incident and reflected beams.

Assuming weakly nonlinear incident and reflected beams, we solve equations (2), (3) using perturbation expansions in an amplitude parameter $\varepsilon << 1$:

$$\left(\psi, \rho, v\right) = \varepsilon\left(\psi^{(1)}, \rho^{(1)}, v^{(1)}\right) + \varepsilon^2\left(\psi^{(2)}, \rho^{(2)}, v^{(2)}\right) + \cdots. \tag{6}$$

The leading-order problem is

$$\rho_t^{(1)} + \psi_\chi^{(1)} + v^{(1)} \sin\alpha \sin\beta = 0\,, \tag{7a}$$

$$\left(\nabla^2 \psi^{(1)}\right)_t - \rho_\chi^{(1)} = 0\,, \tag{7b}$$

$$v_t^{(1)} - \rho^{(1)} \sin\alpha \sin\beta = 0\,, \tag{7c}$$

$$\psi^{(1)} = 0 \quad (\zeta = 0)\,. \tag{8}$$

The appropriate solution representing an incident and reflected beam of frequency $\omega = \sin\theta$, where θ is the beam inclination to the horizontal, takes the form

$$\left(\psi^{(1)}, \rho^{(1)}, v^{(1)}\right) = \left(\Psi(\xi,\zeta), R(\xi,\zeta), V(\xi,\zeta)\right) \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}\,, \tag{9a}$$

where

$$\Psi = \int_0^\infty A(l) \left\{ \exp\left[\mathrm{i}l(\xi + m^{\mathrm{inc}}\zeta)\right] - \exp\left[\mathrm{i}l(\xi + m^{\mathrm{refl}}\zeta)\right] \right\} \mathrm{d}l\,, \tag{9b}$$

$$R = \frac{-\mathrm{i}\omega \Psi_\chi}{\omega^2 - \sin^2\alpha \sin^2\beta}\,, \quad V = \frac{\sin\alpha \sin\beta\, \Psi_\chi}{\omega^2 - \sin^2\alpha \sin^2\beta}\,, \tag{9c}$$

and

$$m^{\text{inc}} = \frac{\cos\beta(\omega r\cos\theta - \sin\alpha\cos\alpha)}{\omega^2 - \sin^2\alpha}, \quad m^{\text{refl}} = -\frac{\cos\beta(\omega r\cos\theta + \sin\alpha\cos\alpha)}{\omega^2 - \sin^2\alpha}, \qquad (9d)$$

with

$$r = \frac{\sqrt{\omega^2 - \sin^2\alpha\sin^2\beta}}{\omega\cos\beta}. \qquad (9e)$$

In equation (9b), $A(l)$ is a given function that specifies the incident (and reflected) beam profile; for $A(l) = \delta(l - l_0)$, in particular, one recovers the solution for the oblique reflection of a sinusoidal plane wave. It should be noted that $\omega^2 = \sin^2\theta > \sin^2\alpha\sin^2\beta$; this condition, which ensures that r in equation (9c) is real, is trivially met when $\theta > \alpha$ but also can be shown to hold for general θ, assuming that the incident beam is propagating (i.e., it is a superposition of plane waves with real wavevectors pointing along a fixed direction). Furthermore, in the limit $\theta \to \alpha$ where the reflected beam lies on the slope, it follows from equations (9d,e) that $m^{\text{refl}} \to \infty$; thus, the reflected beam thickness tends to zero while the beam velocity diverges. The possibility of healing this singular behavior by re-scaling near the critical angle $\theta = \alpha$ including nonlinear and viscous effects, was examined in Dauxois & Young (1999) for the case of normal reflection. The analysis here assumes that θ is not close to α.

3. 非線形相互作用

We now proceed to compute nonlinear corrections to the linear solution (9). These arise from quadratic interactions in the overlap region of the incident and reflected beams near the slope, as each of these beams separately is an exact nonlinear state (Tabaei & Akylas 2003). Specifically, the problem governing the $O(\varepsilon^2)$ term in expansion (6) reads

$$\rho_t^{(2)} + \psi_\chi^{(2)} + v^{(2)}\sin\alpha\sin\beta = F^{(2)}, \qquad (10a)$$

$$(\nabla^2\psi^{(2)})_t - \rho_\chi^{(2)} = G^{(2)}, \qquad (10b)$$

$$v_t^{(2)} - \rho^{(2)}\sin\alpha\sin\beta = H^{(2)}, \qquad (10c)$$

$$\psi^{(2)} = 0 \quad (\zeta = 0), \qquad (11)$$

$$\begin{Bmatrix} F^{(2)} \\ G^{(2)} \\ H^{(2)} \end{Bmatrix} = -\begin{Bmatrix} J(R,\Psi^*) \\ J(\nabla^2\Psi,\Psi^*) \\ J(V,\Psi^*) \end{Bmatrix} - \begin{Bmatrix} J(R,\Psi) \\ J(\nabla^2\Psi,\Psi) \\ J(V,\Psi) \end{Bmatrix} \mathrm{e}^{-2\mathrm{i}\omega t} + \text{c.c.}. \tag{12}$$

These forcing terms, as expected, comprise a mean and second harmonic that derive from quadratic interactions of the primary harmonic. Thus, similar to the normal beam reflection, the $O(\varepsilon^2)$ correction here as well involves an induced mean flow and a second-harmonic component:

$$\begin{aligned}\left(\psi^{(2)},\rho^{(2)},v^{(2)}\right) &= \left(\Psi_0(\xi,\zeta),R_0(\xi,\zeta),V_0(\xi,\zeta)\right) \\ &+ \left\{\left(\Psi_2(\xi,\zeta),R_2(\xi,\zeta),V_2(\xi,\zeta)\right)\mathrm{e}^{-2\mathrm{i}\omega t} + \text{c.c.}\right\}.\end{aligned} \tag{13}$$

Upon substituting expressions (13) into equations (10), the second-harmonic response, which includes a radiating beam if $2\omega < 1$, is readily computed by an analogous procedure to that followed in the case of normal reflection. This is in sharp contrast to the induced mean flow, however, which cannot be fully determined at this stage. Specifically, equation (10a) specifies the vertical (along z) mean flow velocity,

$$-\Psi_{0,\chi} - V_0 \sin\alpha\sin\beta = \frac{-\mathrm{i}\omega}{\omega^2 - \sin^2\alpha\sin^2\beta}\frac{\partial J(\Psi,\Psi^*)}{\partial\chi}, \tag{14a}$$

while equations (10b,c) yield identical information for the density perturbation,

$$R_0 = \frac{J(\Psi_\chi,\Psi^*)}{\omega^2 - \sin^2\alpha\sin^2\beta} + \text{c.c.}, \tag{14b}$$

thus leaving the horizontal mean flow undetermined. (Alternatively, equation (14a) specifies V_0 in terms of Ψ_0, which remains undetermined.) It should be noted that this difficulty does not arise in the case of normal reflection ($\beta = 0$) where $V_0 \equiv 0$ and Ψ_0 (also consistent with the slope boundary condition (11)) is specified via equation (14a): $\Psi_0 = (\mathrm{i}/\omega)J(\Psi,\Psi^*)$.

To compute the $O(\varepsilon^2)$ induced horizontal mean flow, it turns out that one has to carry the perturbation expansions (6) to $O(\varepsilon^4)$. Instead, as the horizontal mean flow is closely connected to the mean vertical vorticity, it is more instructive to appeal to equation (5) for the PV. The evolution of mean PV is key to determine Ψ_0, as discussed below.

4. 平均ポテンシャル渦度

Attention is now focused on equations (4) and (5) for the PV, q; the goal is to obtain an equation governing the mean PV and thereby determine the induced horizontal mean flow. To this end, we first compute q by making use of the perturbation expansions (6). Specifically, from equations (4), (9) and (10), it is deduced that the primary and second harmonic carry no PV to leading order, so the dominant contribution to q comes from the mean PV, $\varepsilon^2 Q_0$,

$$q = \varepsilon^2 Q_0 + \varepsilon^3 \left\{ Q_1 e^{-\mathrm{i}\omega t} + \text{c.c.} \right\} + \cdots, \tag{15}$$

where

$$Q_0 = \Omega_0 - \left(J(V, R^*) + \text{c.c.} \right), \tag{16a}$$

with

$$\Omega_0 = -\sin\alpha \sin\beta\, \nabla^2 \Psi_0 + V_{0\chi}. \tag{16b}$$

Furthermore, the $O(\varepsilon^2)$ primary-harmonic PV in equation (15) is given by

$$Q_1 = \Omega_1 - \left(J(V_0, R) + J(V, R_0) + J(V_2, R^*) + J(V^*, R_2) \right), \tag{17a}$$

with

$$\Omega_1 = -\sin\alpha \sin\beta\, \nabla^2 \Psi_1 + V_{1,\chi}. \tag{17b}$$

Here the vertical vorticity Ω_1 derives from the $O(\varepsilon^2)$ correction to the $O(\varepsilon)$ primary incident and reflected beams in the perturbation expansions (6). In analogy with equation (9a), this correction takes the form

$$\varepsilon^3 \left\{ \left(\Psi_1(\xi,\zeta), R_1(\xi,\zeta), V_1(\xi,\zeta) \right) \mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \right\},$$

where Ψ_1, R_1, V_1 satisfy a forced problem obtained by substituting expansions (6) into equations (2) and collecting primary-harmonic terms correct to $O(\varepsilon^3)$. Equations (2b,c), in particular, yield

$$\mathrm{i}\omega \nabla^2 \Psi_1 + R_{1\chi} = G_1, \tag{18a}$$

$$\mathrm{i}\omega V_1 + R_1 \sin\alpha \sin\beta = H_1, \tag{18b}$$

where

$$G_1 = J(\nabla^2 \Psi_0, \Psi) + J(\nabla^2 \Psi, \Psi_0) + J(\nabla^2 \Psi_2, \Psi^*) + J(\nabla^2 \Psi^*, \Psi_2), \tag{19a}$$

$$H_1 = J(V_0, \Psi) + J(V, \Psi_0) + J(V_2, \Psi^*) + J(V^*, \Psi_2). \tag{19b}$$

Thus, by combining equation (17b) with equations (18) and (19), we find

$$\Omega_1 = -\frac{\mathrm{i}}{\omega}\left(J(\Omega_0, \Psi) + J(V_0, \Psi_\chi) + J(V, \Psi_{0\chi}) + J(V_2, \Psi^*_\chi) + J(V^*, \Psi_{2\chi}) \right). \quad (20)$$

Finally, inserting expression (20) into equation (17a) yields

$$\begin{aligned} Q_1 = &-\frac{\mathrm{i}}{\omega} J(\Omega_0, \Psi) - J\left(V_0, R + \frac{\mathrm{i}}{\omega}\Psi_\chi \right) - J\left(V, R_0 + \frac{\mathrm{i}}{\omega}\Psi_{0\chi} \right) \\ &- J\left(V_2, R^* + \frac{\mathrm{i}}{\omega}\Psi^*_\chi \right) - J\left(V^*, R_2 + \frac{\mathrm{i}}{\omega}\Psi_{2\chi} \right), \end{aligned} \quad (21)$$

which, in view of equations (14), can be simplified to

$$Q_1 = -\frac{\mathrm{i}}{\omega} J(\Omega_0, \Psi) + \frac{2\sin\alpha\sin\beta}{(\omega^2 - \sin^2\alpha\sin^2\beta)^2}\left\{ -J\left[J(\Psi_\chi, \Psi), \Psi^*_\chi \right] + J\left[J(\Psi^*_\chi, \Psi), \Psi_\chi \right] \right\}. \quad (22)$$

The desired equation for Q_0 is now derived by collecting mean terms in the PV equation (5). Since the primary harmonic PV is $O(\varepsilon^3)$ according to equation (5), the mean of the convective derivative $J(q,\psi)$ is $O(\varepsilon^4)$ and takes the form

$$\varepsilon^4 \left\{ J(Q_0, \Psi_0) + \left(J(Q_1, \Psi^*) + \text{c.c.} \right) \right\}, \quad (23)$$

where Q_1 is given by equation (22). Hence, the mean PV dynamics occurs on the 'slow' time $T = \varepsilon^2 t$ and is governed by the following evolution equation for $Q_0(\xi, \zeta, \tau)$

$$\frac{\partial Q_0}{\partial T} + J(Q_0, \Psi_0) + \left(J(Q_1, \Psi^*) + \text{c.c.} \right) = 0. \quad (24)$$

(The analysis remains virtually unchanged if the incident and reflected beams also depend on T.)

Equation (24) can be re-cast in a form that is more revealing physically, by first substituting expression (22) for Q_1 and then using the identity $J[J(a,b),c] = J[J(a,c),b] - J[J(b,c),a]$ to simplify the resulting triple Jacobians. Finally, it turns out that

$$\frac{\partial Q_0}{\partial T} + J\left[Q_0, \Psi_0 - \frac{\mathrm{i}}{\omega} J(\Psi, \Psi^*) \right] = 0. \quad (25)$$

The above form of the mean PV equation brings out the fact that Q_0 is convected by a mean flow with streamfunction

$$\bar{\Psi} = \Psi_0 - \frac{\mathrm{i}}{\omega} J(\Psi, \Psi^*), \quad (26)$$

which turns out to be the Lagrangian mean flow associated with the (Eulerian) induced mean flow Ψ_0. This can be readily verified by computing the flow particle paths correct to $O(\varepsilon^2)$, from which it follows that the Stokes mean drift streamfunction $\varepsilon^2\Psi^{\mathrm{D}}$ and transverse velocity $\varepsilon^2 V^{\mathrm{D}}$ are

$$\Psi^{\mathrm{D}} = -\frac{\mathrm{i}}{\omega} J(\Psi, \Psi^*), \tag{27a}$$

$$V^{\mathrm{D}} = -\frac{\mathrm{i}}{\omega} \frac{\sin\alpha \sin\beta}{\omega^2 - \sin^2\alpha \sin^2\beta} \frac{\partial J\left(\Psi, \Psi^*\right)}{\partial \chi}; \tag{27b}$$

thus, the Lagrangian mean flow (Eulerian mean flow + mean drift) has streamfunction $\overline{\Psi} = \Psi_0 + \Psi^{\mathrm{D}}$ and transverse velocity $\overline{V} = V_0 + V^{\mathrm{D}}$.

According to equations (27), the vertical (along z) drift is given by

$$-\Psi_\chi^{\mathrm{D}} - V^{\mathrm{D}} \sin\alpha \sin\beta = \frac{\mathrm{i}\omega}{\omega^2 - \sin^2\alpha \sin^2\beta} \frac{\partial J\left(\Psi, \Psi^*\right)}{\partial \chi}, \tag{28}$$

and cancels the (Eulerian) vertical induced mean flow found earlier (cf. equation (14a)); hence, the Lagrangian vertical mean flow vanishes, as was also found in Thorpe (1997) for obliquely reflecting sinusoidal plane waves. The along-slope (along y) Lagrangian mean flow, $\overline{\Psi}_\zeta \sin\beta + \overline{V}\cos\beta$, is generally non-zero, however, as it depends on Ψ_0, which solves the evolution equation (25) subject to the boundary condition $\Psi_0 = 0$ on the slope ($\zeta = 0$) and an appropriate initial condition at $T = 0$. A particular steady-state solution of this problem is

$$\Psi_0 = \frac{\mathrm{i}}{\omega} J(\Psi, \Psi^*) = -\Psi^{\mathrm{D}}, \tag{29}$$

in which case not only $\overline{\Psi} = 0$, but also $\overline{V} = 0$ since $V_0 = -V^{\mathrm{D}}$ by virtue of equation (14a); hence, the corresponding Lagrangian mean flow identically vanishes as in the case of normal reflection ($\beta = 0$). Finally, it is worth noting that for obliquely reflecting sinusoidal waves where the Eulerian induced mean flow as well as the Stokes drift are independent of ξ, the Jacobian in the evolution equation (25) identically vanishes. Thus, in this instant any function of ζ is a solution and by suitably choosing Ψ_0, Eulerian or Lagrangian mean flow can be taken to be zero.

Another steady-state solution of equation (25) is found by setting $Q_0 = 0$. This condition combined with equations (14a) and (16a) then yields

$$\left(a\frac{\partial^2}{\partial\zeta^2}+2b\frac{\partial^2}{\partial\zeta\partial\xi}+c\frac{\partial^2}{\partial\xi^2}\right)\Psi_0=f_0, \tag{30a}$$

where

$$f_0=\frac{\mathrm{i}\omega}{\omega^2-\sin^2\alpha\sin^2\beta}\frac{\partial^2 J(\Psi,\Psi^*)}{\partial\chi^2}-\frac{2\mathrm{i}\omega\sin^2\alpha\sin^2\beta}{(\omega^2-\sin^2\alpha\sin^2\beta)^2}J(\Psi_\chi,\Psi^*_\chi) \tag{30b}$$

and

$$a=\sin^2\alpha,\quad b=-\sin\alpha\cos\alpha\cos\beta,\quad c=\cos^2\alpha+\sin^2\alpha\sin^2\beta.$$

Since $b^2-ac=-\sin^2\alpha\sin^2\beta<0$, equation (30a) is an elliptic equation for Ψ_0, whose solution subject to conditions $\Psi_0=0$ on $\zeta=0$ and $\Psi_0\to 0$ as $\zeta\to\infty$, is readily found by taking Fourier transform in ξ. The Lagrangian horizontal mean flow corresponding to this steady-state solution Ψ_0 is non-zero.

5. 平均流

Specifically, allowing for viscous dissipation, the inviscid PV equation (5) is replaced by

$$\frac{Dq}{Dt}=\nu\left(\nabla^2\Omega-J(\nabla^2 v,\rho)\right), \tag{31}$$

where ν is an inverse Reynolds number. Now, upon collecting mean terms, as the mean vertical vorticity is $O(\varepsilon^2)$ in view of equations (15) and (16), we take $\nu=\mu\varepsilon^2$, with $\mu=O(1)$, in order for the viscous term in equation (31) to be comparable to the inviscid left-hand side, which is $O(\varepsilon^4)$ as was argued earlier. Thus, the inviscid mean-PV transport equation (25) in the presence of viscous dissipation becomes

$$\begin{aligned}\frac{\partial Q_0}{\partial T}+J(Q_0,\bar{\Psi})&=\mu\nabla^2 Q_0+\mu\left\{\frac{1}{2}\nabla^2 J(V,R^*)+J(\nabla V,\nabla R^*)+\text{c.c.}\right\}\\&=\mu\nabla^2 Q_0+\frac{\mathrm{i}\mu\omega\sin\alpha\sin\beta}{(\omega^2-\sin^2\alpha\sin^2\beta)^2}\left\{\nabla^2 J(\Psi_\chi,\Psi^*_\chi)+2J(\nabla\Psi_\chi,\nabla\Psi^*_\chi)\right\}.\end{aligned} \tag{32}$$

The second term on the right-hand side of equation (32) is clearly of viscous origin and accounts for the production of mean PV owing to nonlinear interactions in the overlap region of the incident and reflected beams. This process is similar to the mean-PV production mechanism that gives rise to streaming in the propagation of a beam with transverse variations, although for an oblique reflection where $V\neq 0$ such variations are not necessary. However, here, mean-PV production is at the same level as mean-PV dissipation, described by the first term on the right-hand side of equation (32); as a result, resonant (proportional to T) growth of Q_0 is not possible and the streaming effect may be less pronounced than in the case of a propagating beam with transverse variations.

6. 結言

The present analysis of oblique reflection of internal wave beams at a sloping boundary has brought out the key role of mean PV in determining the induced horizontal mean flow. Unlike two-dimensional reflections where PV vanishes identically and the induced mean flow is readily found and results in no net transport, the horizontal induced mean flow due to an oblique reflection evolves according to its own dynamics, governed by the mean PV equation. We have identified two possible steady-state solutions of this evolution equation: one for which the associated Lagrangian mean flow vanishes, implying no net transport, and another that corresponds to zero mean PV but for which the associated Lagrangian mean flow is non-zero. In order to decide whether any of these steady states is reached, it would be necessary to compute the evolution of mean PV by solving an appropriate initial-value problem. Since PV is conserved in an inviscid fluid, assuming that PV vanishes initially, the latter steady-state solution is the most physically relevant. Thus, in a high-Reynolds-number environment, oblique reflection of internal waves is expected to induce horizontal mass transport near a sloping boundary.

We have also explored the effect of viscous dissipation on the mean PV dynamics. Apart from the expected damping of mean PV, viscous dissipation in conjunction with nonlinear interactions in the region where the incident and reflected beams overlap, can act as a driving mechanism of horizontal mean flow via the irreversible production of mean PV. This process, which applies to oblique reflections only, is akin to the generation of streaming by attenuating acoustic waves and propagating internal wave beams in the presence of transverse variations. A complete study of viscous effects on the induced mean flow would require solving the mean PV evolution equation subject to viscous (nonslip) conditions on the slope, a task that is not attempted here.

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