数理解析研究所講究録 第2128巻 2019年 156-163

非零境界条件下での多成分 Fokas-Lenells 方程式の多重ソリトン公式

Multisoliton formulas for the multi-component Fokas-Lenells equation with nonzero boundary conditions

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Abstract. The multi-component Fokas-Lenells equation is considered. In particular, we present the multisoliton formulas for the system with plane-wave boundary conditions, as well as with mixed zero and plane-wave boundary conditions. A direct approach is employed to construct solutions, showing that for both boundary conditions, the multisoliton solutions have compact determinantal expressions.

1. Introduction

The Lax pair of the integrable multi-component Fokas-Lenells system is given by

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \tag{1.1a}$$

$$U = \begin{pmatrix} \frac{\mathrm{i}}{2}\zeta^2 & -\mathrm{i}\zeta\mathbf{u}_x \\ \mathrm{i}\zeta\mathbf{v}_x^T & -\frac{\mathrm{i}}{2}\zeta^2 I \end{pmatrix} = (u_{jk}), \quad V = \begin{pmatrix} -\frac{\mathrm{i}}{2\zeta^2} - \mathrm{i}\mathbf{u}\mathbf{v}^T & \frac{1}{\zeta}\mathbf{u} \\ \frac{1}{\zeta}\mathbf{v}^T & \frac{\mathrm{i}}{2\zeta^2}\, I + \mathrm{i}\mathbf{v}^T\mathbf{u} \end{pmatrix} = (v_{jk}), \tag{1.1b}$$

where ζ is the spectral parameter, and $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$, $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are n-component row vectors.

It follows from the compatibility condition of the Lax pair that $U_t - V_x + UV - VU = O$. This yields the system of nonlinear PDEs for $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}_{xt} - \mathbf{u} + \mathrm{i}(\mathbf{u}_x\mathbf{v}^T\mathbf{u} + \mathbf{u}\mathbf{v}^T\mathbf{u}_x) = \mathbf{0}, \tag{1.2a}$$

$$\mathbf{v}_{xt} - \mathbf{v} - \mathrm{i}(\mathbf{v}_x\mathbf{u}^T\mathbf{v} + \mathbf{v}\mathbf{u}^T\mathbf{v}_x) = \mathbf{0}. \tag{1.2b}$$

The system (1.2) can be reduced from the first negative flow of the matrix derivative NLS hierarchy [1-4]. There arise several integrable PDEs from the reductions of the system (1.2). Specifically, if we put $v_j = \sigma_j u_j^*$, $\sigma_j = \pm 1$ $(j = 1, 2, ..., n)$, then the above system reduces to

$$u_{j,xt} = u_j - \mathrm{i}\left\{\left(\sum_{k=1}^n \sigma_k u_{k,x} u_k^*\right) u_j + \left(\sum_{k=1}^n \sigma_k u_k u_k^*\right) u_{j,x}\right\}, \ (j = 1, 2, ..., n). \tag{1.3}$$

The system of PDEs (1.3) is the basic equation that we consider here. The two special cases reducing from the system (1.3) are particularly important:

1) $n = 1$: FL equation [5, 6]

$$u_{xt} = u - 2\mathrm{i}\sigma|u|^2 u_x, \ \ (u \equiv u_1, \sigma_1 = \sigma). \tag{1.4}$$

2) $n = 2$: two-component FL system [4, 7]

$$u_{1,xt} = u_1 - \mathrm{i}\left\{(2|u_1|^2 + \sigma|u_2|^2)u_{1,x} + \mathrm{i}\sigma u_1 u_2^* u_{2,x}\right\}, \tag{1.5a}$$

$$u_{2,xt} = u_2 - \mathrm{i}\left\{(|u_1|^2 + 2\sigma|u_2|^2)u_{2,x} + \mathrm{i}\sigma u_2 u_1^* u_{1,x}\right\}, \tag{1.5b}$$

$$(\sigma_1 = 1, \sigma_2 = \sigma).$$

The N-soliton solutions of the FL equation have been constructed for both zero and plane-wave boundary conditions [8, 9] while for the general n-component system, we have obtained the bright N-soliton solution with zero boundary conditions [10]. The purpose of the current work is to present the N-soliton formulas of the system (1.3) with the following two types of the boundary conditions:

1) Plane-wave boundary conditions

$$u_j \sim \rho_j \exp \mathrm{i}\left(k_j x - \omega_j t + \phi_j^{(\pm)}\right), \quad x \to \pm\infty, \quad (j = 1, 2, ..., n), \tag{1.6a}$$

with the linear dispersion relation

$$k_j \omega_j = 1 + \sum_{s=1}^{n} \sigma_s k_s \rho_s^2 + \sum_{s=1}^{n} \sigma_s \rho_s^2 k_j, \quad (j = 1, 2, ..., n). \tag{1.6b}$$

2) Mixed type boundary conditions

$$u_j \sim 0, \quad x \to \pm\infty, \quad (j = 1, 2, ..., m), \tag{1.7a}$$

$$u_{m+j} \sim \rho_j \exp \mathrm{i}\left(k_j x - \omega_j t + \phi_j^{(\pm)}\right), \quad x \to \pm\infty, \quad (j = 1, 2, ..., n - m), \tag{1.7b}$$

with the linear dispersion relation

$$k_j \omega_j = 1 + \sum_{s=1}^{n-m} \sigma_s k_s \rho_s^2 + \sum_{s=1}^{n-m} \sigma_s \rho_s^2 k_j, \quad (j = 1, 2, ..., n - m). \tag{1.7c}$$

In this short note, we provide the main results only, and the details will be reported elsewhere.

2. The N-soliton formula with plane-wave boundary conditions

2.1. Bilinearization

Here, we present the multisoliton solutions of the system (1.3) with plane-wave boundary conditions (1.6). The direct approach is used to obtain solutions. To this end, we start from the following proposition:

Proposition 1. *Under the dependent variable transformations*

$$u_j = \rho_j \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)} \frac{g_j}{f}, \ (j = 1, 2, ..., n), \tag{2.1}$$

the multi-component FL system (1.3) can be decoupled into the system of equations

$$D_t f \cdot f^* = \mathrm{i} \sum_{s=1}^{n} \sigma_s \rho_s^2 (g_s g_s^* - f f^*), \tag{2.2a}$$

$$D_x D_t f \cdot f^* - \mathrm{i} \sum_{s=1}^{n} \sigma_s \rho_s^2 D_x g_s \cdot g_s^* + \mathrm{i} \sum_{s=1}^{n} \sigma_s \rho_s^2 D_x f \cdot f^* + 2 \sum_{s=1}^{n} \sigma_s k_s \rho_s^2 (g_s g_s^* - f f^*) = 0, \tag{2.2b}$$

$$\begin{aligned} & f^* \left[g_{j,xt} f - (f_x - \mathrm{i} k_j f) g_{j,t} - \mathrm{i} \frac{1}{k_j} \left(1 + \sum_{s=1}^{n} \sigma_s k_s \rho_s^2 \right) D_x g_j \cdot f \right] \\ & = f_t^* (g_{j,x} f - g_j f_x + \mathrm{i} k_j g_j f), \ (j = 1, 2, ..., n), \end{aligned} \tag{2.2c}$$

where $f = f(x,t)$ and $g_j = g_j(x,t)$ are the complexed-valued functions of x and t, and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x,t) g(x',t') \Big|_{x'=x,\, t'=t}$$

with m and n being nonnegative integers.

Remark 1.

1) We can decouple the last equation into a system of bilinear equations

$$g_{j,xt} f - (f_x - \mathrm{i} k_j f) g_{j,t} - \mathrm{i} \frac{1}{k_j} \left(1 + \sum_{s=1}^{n} \sigma_s k_s \rho_s^2 \right) D_x g_j \cdot f = h_j f_t^*, \tag{2.3a}$$

$$g_{j,x} f - g_j f_x + \mathrm{i} k_j f g_j = h_j f^*, \tag{2.3b}$$

where $h_j = h_j(x,t)$ are the complexed-valued functions of x and t.

2) If we introduce the variables $q_j = u_{j,x}$, then

$$q_j = \left(\rho_j e^{i(k_j x - \hat{\omega}_j t)} \frac{g_j}{f}\right)_x = \rho_j e^{i(k_j x - \hat{\omega}_j t)} \frac{h_j f^*}{f^2}, \quad \hat{\omega}_j = k_j^2 + 2\sum_{s=1}^{n} \sigma_s \rho_s^2 k_j, \quad (j = 1, 2, ..., n), \tag{2.4}$$

solve the n-component derivative NLS system

$$iq_{j,t} + q_{j,xx} + 2i\left[\left(\sum_{s=1}^{n} \sigma_s |q_s|^2\right) q_j\right]_x = 0, \quad (j = 1, 2, ..., n). \tag{2.5}$$

2.2. N-soliton solution

Theorem 1. *The N-soliton solution of the system of bilinear equations (2.2) is given in terms of the following determinants.*

$$f = |D|, \quad g_s = |G_s|, \quad (s = 1, 2, ..., n), \tag{2.6a}$$

$$D = (d_{jk})_{1\leq j,k\leq N}, \quad d_{jk} = \delta_{jk} - \frac{ip_j}{p_j + p_k^*} z_j z_k^*, \tag{2.6b}$$

$$G_s = (g_{jk}^{(s)})_{1\leq j,k\leq N}, \quad g_{jk}^{(s)} = \delta_{jk} - \frac{ip_k^*}{p_j + p_k^*} \frac{p_j - ik_s}{p_k^* + ik_s} z_j z_k^*, \tag{2.6c}$$

$$z_j = \exp\left[p_j x + \frac{1}{p_j}\left(1 + \sum_{s=1}^{n} \sigma_s k_s \rho_s^2\right) t + \zeta_{j0}\right], \quad (j = 1, 2, ..., N). \tag{2.6d}$$

Here, p_j and ζ_{j0} $(j = 1, 2, ..., N)$ are arbitrary complex parameters. The former parameters are impozed on N constraints

$$\sum_{s=1}^{n} \sigma_s (k_s \rho_s)^2 \frac{i(p_j - p_j^*) + k_s}{(p_j - ik_s)(p_j^* + ik_s)} = -1, \quad (j = 1, 2, ..., N). \tag{2.7}$$

The expressions (2.1) with the tau-functions (2.6) give the dark soliton solutions with plane-wave boundary conditions. The analysis of the one-component system (i.e., FL equation) has been performed in [9] where the detailed description of the dark soliton solutions has been given.

Remark 2.

1) The proof of the N-soliton solution can be done by means of an elementary calculation using the basic formulas of determinants, i.e.,

$$\frac{\partial}{\partial x}|D| = \sum_{j,k=1}^{N} \frac{\partial d_{jk}}{\partial x} D_{jk}, \quad (D_{jk} : \text{cofactor of } d_{jk}),$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{j,k=1}^{N} D_{jk} a_j b_k,$$

$$|D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d})||D| = |D(\mathbf{a};\mathbf{c})||D(\mathbf{b};\mathbf{d})| - |D(\mathbf{a};\mathbf{d})||D(\mathbf{b};\mathbf{c})|, \ \text{(Jacobi's identity)},$$

with the notation

$$\begin{vmatrix} D & \mathbf{b}^T \\ \mathbf{a} & 0 \end{vmatrix} = |D(\mathbf{a};\mathbf{b})|, \qquad \begin{vmatrix} D & \mathbf{c}^T & \mathbf{d}^T \\ \mathbf{a} & 0 & 0 \\ \mathbf{b} & 0 & 0 \end{vmatrix} = |D(\mathbf{a},\mathbf{b};\mathbf{c},\mathbf{d})|.$$

2) The tau-functions h_s are given by

$$h_s = \mathrm{i}k_s|H_s|, \quad H_s = (h_{jk}^{(s)})_{1\le j,k\le N}, \quad h_{jk}^{(s)} = \delta_{jk} + \frac{\mathrm{i}p_j}{p_j + p_k^*}\frac{p_j - \mathrm{i}k_s}{p_k^* + \mathrm{i}k_s} z_j z_k^*.$$

2.3. Derivation of constraints (2.7)

In the case of plane-wave boundary conditions, the n constraints must be imposed among the complex parameters p_j ($j = 1, 2, ..., N$). We derive these constraints from the Lax pair (1.1) of the system. The spatial part of the Lax pair with seed solutions

$$u_j = \rho_j \mathrm{e}^{\mathrm{i}\theta_j}, \quad \theta_j = k_j x - \omega_j t, \quad (j = 1, 2, ..., n),$$

are given by

$$\Psi_x = U\Psi, \quad U = \begin{pmatrix} \frac{\mathrm{i}}{2}\zeta^2 & k_1\rho_1\zeta \mathrm{e}^{\mathrm{i}\theta_1} & \cdots & k_n\rho_n\zeta\mathrm{e}^{\mathrm{i}\theta_n} \\ \sigma_1 k_1\rho_1\zeta\mathrm{e}^{-\mathrm{i}\theta_1} & -\frac{\mathrm{i}}{2}\zeta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n k_n \rho_n \zeta \mathrm{e}^{-\mathrm{i}\theta_n} & 0 & \cdots & -\frac{\mathrm{i}}{2}\zeta^2 \end{pmatrix}. \tag{2.8}$$

Introduce a new wavefunction Ψ_0 by $\Psi = P\Psi_0$, where P is a diagonal matrix $P = \mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\theta_1}, ..., , \mathrm{e}^{\mathrm{i}\theta_n})$. Then, Ψ_0 satisfies the matrix equation

$$\Psi_{0,x} = (P_x P^{-1} + PUP^{-1})\Psi_0 \equiv U_0\Psi_0, \quad U_0 = \begin{pmatrix} \frac{\mathrm{i}}{2}\zeta^2 & k_1\rho_1\zeta & \cdots & k_n\rho_n\zeta \\ \sigma_1 k_1\rho_1\zeta & \mathrm{i}k_1 - \frac{\mathrm{i}}{2}\zeta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n k_n\rho_n\zeta & 0 & \cdots & \mathrm{i}k_n - \frac{\mathrm{i}}{2}\zeta^2 \end{pmatrix}. \tag{2.9}$$

The characteristic equation of U_0 reads $|U_0 - I_{n+1}\mu| = 0$, i.e.,

$$\begin{vmatrix} \frac{\mathrm{i}}{2}\zeta^2 - \mu & k_1\rho_1\zeta & \cdots & k_n\rho_n\zeta \\ \sigma_1 k_1\rho_1\zeta & \mathrm{i}k_1 - \frac{\mathrm{i}}{2}\zeta^2 - \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n k_n\rho_n\zeta & 0 & \cdots & \mathrm{i}k_n - \frac{\mathrm{i}}{2}\zeta^2 - \mu \end{vmatrix} = 0. \tag{2.10}$$

Expanding the above determinant in μ yields

$$\frac{\mathrm{i}}{2}\zeta^2 - \mu = -\zeta^2 \sum_{s=1}^{n} \frac{\sigma_s (k_s\rho_s)^2}{\mu + \frac{\mathrm{i}}{2}\zeta^2 - \mathrm{i}k_s}. \tag{2.11}$$

Let $\mu + \frac{\mathrm{i}}{2}\zeta^2 = p$ and assume ζ^2 be real and p be complex. Then

$$\mathrm{i}\zeta^2 - p = -\zeta^2 \sum_{s=1}^{n} \frac{\sigma_s (k_s\rho_s)^2}{p - \mathrm{i}k_s}, \quad -\mathrm{i}\zeta^2 - p^* = -\zeta^2 \sum_{s=1}^{n} \frac{\sigma_s (k_s\rho_s)^2}{p^* + \mathrm{i}k_s}. \tag{2.12}$$

It follows from the above two relations that

$$\sum_{s=1}^{n} \sigma_s (k_s\rho_s)^2 \frac{\mathrm{i}(p-p^*) + k_s}{(p - \mathrm{i}k_s)(p^* + \mathrm{i}k_s)} = -1, \tag{2.13}$$

which yields (2.7) upon putting $p = p_j$.

3. The N-soliton formula with mixed type boundary conditions

3.1. Bilinearization

The bilinearization of the system (1.3) with mixed type boundary conditions (1.7) can be performed by the following proposition.

Proposition 2. *Under the dependent variable transformations*

$$u_j = \mathrm{e}^{-\mathrm{i}\hat{\lambda}t}\frac{h_j}{f}, \quad \left(j = 1, 2, ..., m,\ \hat{\lambda} = \sum_{s=1}^{n-m} \sigma_{m+s}\rho_s^2\right), \tag{3.1a}$$

$$u_{m+j} = \rho_j\, \mathrm{e}^{\mathrm{i}(k_j x - \omega_j t)}\frac{g_j}{f}, \quad (j = 1, 2, ..., n-m), \tag{3.1b}$$

the multi-component FL system (1.3) can be decoupled into the system of equations

$$D_t f \cdot f^* = \mathrm{i}\sum_{s=1}^{m} \sigma_s h_s h_s^* + \mathrm{i}\sum_{s=1}^{n-m} \sigma_{m+s}\rho_s^2 (g_s g_s^* - f f^*), \tag{3.2a}$$

$$D_x D_t f \cdot f^* - \mathrm{i}\sum_{s=1}^{m} \sigma_s D_x h_s \cdot h_s^* - \mathrm{i}\sum_{s=1}^{n-m} \sigma_s \rho_s^2 D_x g_s \cdot g_s^* + \mathrm{i}\sum_{s=1}^{n-m} \sigma_{m+s}\rho_s^2 D_x f \cdot f^*$$
$$+2\sum_{s=1}^{n-m} \sigma_s k_s \rho_s^2 (g_s g_s^* - f f^*) = 0, \tag{3.2b}$$

$$f^*(h_{j,xx} f - h_{j,t} f_x - \lambda h_j f) = f_t^*(h_{j,x} f - h_j f_x), (j = 1, 2, ..., m), \tag{3.2c}$$

$$f^*\left\{g_{j,xt}f-(f_x-\mathrm{i}k_jf)g_{j,t}-\frac{\mathrm{i}\lambda}{k_j}D_xg_j\cdot f\right\}=f_t^*(g_{j,x}f-g_jf_x+\mathrm{i}k_jg_jf),\ (j=1,2,...,n-m), \tag{3.2d}$$

where $\lambda=1+\sum_{s=1}^{n-m}\sigma_sk_s\rho_s^2$.

3.2. N-soliton solution

Theorem 2. *The N-soliton solution of the system of bilinear equations (3.2) is given in terms of the following determinants.*

$$f=|D|,\quad D=(d_{jk})_{1\le j,k\le N},\quad d_{jk}=\frac{z_jz_k^*-\mathrm{i}p_k^*c_{jk}}{p_j+p_k^*},\quad z_j=\exp\left(p_jx+\frac{\lambda}{p_j}t\right), \tag{3.3a}$$

$$h_j=-\frac{1}{\lambda}|D(\mathbf{a}_j^*;\mathbf{z}_t)|,\quad (j=1,2,...,m), \tag{3.3b}$$

$$g_j=|D|+\frac{\mathrm{i}}{\lambda}|D(\mathbf{z}_j^*;\mathbf{z}_t)|,\quad (j=1,2....,n-m), \tag{3.3c}$$

$$\mathbf{z}=(z_1,z_2,...,z_N),\ \mathbf{z}_t=\left(\frac{\lambda}{p_1}z_1,\frac{\lambda}{p_2}z_2,...,\frac{\lambda}{p_N}z_N\right), \tag{3.3d}$$

$$\mathbf{a}_j=(\alpha_{j1},\alpha_{j2},...,\alpha_{jN}),\quad (j=1,2,...,m), \tag{3.3e}$$

$$c_{jk}=\frac{\sum_{s=1}^{m}\sigma_s\alpha_{sj}\alpha_{sk}^*}{1+\sum_{s=1}^{n-m}\sigma_s(k_s\rho_s)^2\frac{\mathrm{i}(p_j-p_k^*)+k_s}{(p_j-\mathrm{i}k_s)(p_k^*+\mathrm{i}k_s)}},\quad (j,k=1,2,...,N), \tag{3.3f}$$

where p_j $(j=1,2,...,N)$ and α_{jk} $(j=1,2,...,m;k=1,2,...,N)$ are arbitrary complex parameters.

The components from (3.1a) take the form of the bright solitons with zero background whereas those of (3.1b) represent the dark solitons with plane-wave background. The properties of the bright soliton solutions of the FL equation have been explored in detail in [8]. It should be remarked that unlike purely plane-wave boundary conditions, no constraints are imposed on the parameters p_j. Consequently, the analysis of solutions becomes more easier than that of solutions for plane-wave boundary conditions.

Remark 3.

1) When compared with the soliton solutions with the pure plane-wave boundary conditions, the parameters p_j can be chosen arbitrary. Consequently, the explicit form of the N-soliton solution is available without solving algebraic equations like (2.7).

2) If we put $\rho_j=0,\ (j=1,2,...,n-m)$, then (3.1$a$) and (3.3) yield the bright N-soliton solution of the system (1.3) with the zero boundary conditions $u_j\to 0,|x|\to\infty$ [10].

Acknowledgement

This work was partially supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

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