Summary of thesis: Studies on generalizations of the classical orthogonal polynomials where gaps are allowed in their degree sequences

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An orthogonal polynomial sequence is a sequence of polynomials such that any two different polynomials in this sequence are orthogonal to each other under certain linear functional. Let $\mathcal{L}$ be a linear functional from $\mathbb{R}[x]$ to $\mathbb{R}$. The orthogonal polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ related to the linear functional $\mathcal{L}$ are defined upon the following conditions:

$$\mathcal{L}[p_n(x)p_m(x)] = h_n\delta_{m,n}, \quad h_n \neq 0, \tag{1a}$$

$$\deg(p_n(x)) = n, \tag{1b}$$

where $\delta_{m,n}$ is Kronecker's delta, $\delta_{n,n} = 1$ and $\delta_{m,n} = 0, \forall m \neq n$. The most widely used orthogonal polynomials are the classical orthogonal polynomials, which are characterized as eigenfunctions of certain second-order differential or difference operators. In this thesis we study several generalizations of the classical orthogonal polynomials. A common feature shared by these generalizations is that their degree sequences do not include all the non-negative integers, which means the condition (1b) is not satisfied. These generalizations are polynomial eigenfunctions of certain generalized operators which are self-adjoint. It was known that orthogonality can be implied by the self-adjoint condition. However, it is difficult to determined the related self-adjoint operator if it has polynomial eigenfunctions where gaps are allowed in the degree sequences.

In chapter 1, the history background and the definitions as well as the rich applications of the classical orthogonal polynomials are introduced. An important property called duality which reads as the equivalence of a three-term recurrence relation and an eigenvalue equation satisfied by the classical orthogonal polynomials is illustrated with a simple example. Generalizing the eigenvalue equation is a key in deriving the generalizations of the classical orthogonal polynomials.

In chapter 2, we give a brief review on the exceptional extensions of the very classical orthogonal polynomials, which are polynomial eigenfunction of a second-order differential operator. The exceptional orthogonal polynomials are good examples where gaps are allowed in the degree sequences. Extensive studies have been devoted to the exceptional orthogonal polynomials satisfying a second-order differential equation, which are classified as the Hermite, Laguerre, and Jacobi type. We clarify the relationship between an electrostatic model and the zeros of these exceptional orthogonal polynomials. We show that the energy function of the electrostatic model attains its maximum value at the zeros of the exceptional orthogonal polynomials under some sufficient conditions.

In chapter 3, we construct an exceptional extension of the Bannai-Ito polynomials, which are polynomial eigenfunction of the Dunkl-type difference operator. A powerful method in the construction of these exceptional polynomials is called the Darboux transformation. We generalize the Darboux transformation to a first-order Dunkl-type difference operator. From this generalized Darboux transformation we are able to derive the exceptional Bannai-Ito operators. We also give more general results on the intertwining relations regarding the multiple-step exceptional Bannai-Ito operator, and derived their eigenfunctions. We construct the 1-step exceptional Bannai-Ito polynomials and show that they are orthogonal with respect to a discrete measure on the exceptional Bannai-Ito grid. Interestingly enough, the degree sequences of the exceptional Bannai-Ito polynomials demonstrate different rules compared with all the known 1-step exceptional orthogonal polynomials. The positivity of the weight functions related to these 1-step exceptional Bannai-Ito polynomials is also considered, and we provide several sufficient conditions with respect to certain parameters.

In chapter 4, we introduced and characterized orthogonal functions that we have called Dunkl-supersymmetric. These functions are eigenfunctions of a class of Dunkl-type differential operators that can be cast within supersymmetric quantum mechanics. A significant feature of these orthogonal function families is that they do not involve polynomials of all degrees but are rather organized in pairs of polynomials both of the same degree (where the examples in terms of the Jacobi polynomials may be viewed as polynomials in some special variables). The connection with supersymmetric quantum mechanics has been exploited to obtain a number of Dunkl-SUSY orthogonal functions from exactly solvable problems. Informed by these results we could offer a general characterization of the Dunkl-SUSY orthogonal polynomials and could exhibit as well their recurrence relations.