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Templates forced by the Smale Horseshoe Trellises

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1. Introduction

Let $\varphi : D \to D$ be an orientation-preserving homeomorphism on the 2-disk D. Denote by $\mathcal{O}(\varphi)$ the set of all finite unions of periodic orbits of φ. Take an arbitrarily fixed isotopy $\Phi = \{\varphi_t\}_{0\leq t\leq 1}$ from id_D to φ. Each $P \in \mathcal{O}(\varphi)$ yields a braid $\beta_\Phi P$ as follows;

$$\beta_\Phi P := \bigcup_{0\leq t\leq 1} (\varphi_t(P) \times \{t\}).$$

The link $\mathcal{S}_\Phi P$ in the 3-sphere S^3, the closure of $\beta_\Phi P$, which is obtained by gluing the top and the bottom naturally. We say that φ *induces all link types* if there exists an isotopy Φ such that any link L in S^3 belongs to $\{\mathcal{S}_\Phi P | P \in \mathcal{O}(\varphi)\}$. In [9], it was shown that for the Smale horseshoe map $F : D \to D$ (Fig. 2), neither F nor F^2 induces all link types, and F^n induces all link types for any $n \geq 3$. This was proved by using *universal templates* (Section 4.2). In this note, we report that for any diffeomorphism $f : D \to D$ with some horseshoe homoclinic orbit, f^3 induces all link types (Theorem 5.1).

The note is organized as follows. In Section 2, we give some results on the trellis theory [5] which are our main machinery. In Section 3, we first define the trellises forced by horseshoe homoclinic orbits, and next we define the star homoclinic orbits which was originally studied in [3]. In Section 4, we review some results on the template theory [8]. In Section 5, we consider the relation between trellises and templates, and we give a result.

2. Trellises

Let $f : D \to D$ be a diffeomorphism with a hyperbolic fixed point p. Let $W^{S/U}(f,p)$ be the stable/unstable manifold for p.

Definition 2.1. A pair $T = (T^U, T^S)$ is a *trellis* for f if T satisfies the following conditions:

- $T^S \subset W^S(f,p)$ and $T^U \subset W^U(f,p)$,
- $T^{S/U}$ consists of finite many closed intervals with non-empty interiors,
- $f(T^S) \subset T^S$ and $f(T^U) \supset T^U$.

Let $T^V = T^U \cap T^S$. (Hence, T^V is the set of homoclinic points for p.) A *region* of T is the closure of a component of $D \setminus (T^S \cup T^U)$. A *segment* I of T is a closed subintervals of either T^U or T^S such that the interior of I is disjoint

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from T^V and the endpoints of I are in T^V. A *bigon* of T is a region bounded by one unstable segment and one stable segment.

Definition 2.2. Let f_0, f_1 be diffeomorphisms on D with the same trellis T. We say that (f_0, T_0) and (f_1, T_1) are *conjugate* if there is a homeomorphism h on D such that $h \circ f_0 = f_1 \circ h$ and $h(T_0^{U/S}) = T_1^{U/S}$. We call the conjugacy class of (f,T) the *trellis type* of (f,T), denoted by $[f,T]$.

For (f,T), the tree G, called the *topological tree representative* and the tree map $g : G \to G$ are defined [4],[5]. One of the advantage to study trellises is that one can obtain the lower bound of the topological entropy of f from the trellise. More precisely, the lower bound of the topological entropy of f is given by the topological entropy of g [5].

The second advantage is that one can study braid types of periodic orbits of f forced by the trellis (For the definition of the braid type, see [1]). To give this result more precisely, we fiest recall *thick trees* and *thick tree maps* [6, Section 2] (Fig. 1): Given a tree G, we "thicken" G to obtain the thick tree $\widehat{G}$. The thick tree $\widehat{G}$ is a set of the thick vertices $\widehat{V}$ and thick edges $\widehat{E}$ corresponding to the vertices V and edges E of G, and each thick edge $\widehat{E}$ has stable/unstable leaves. The thick tree map $\widehat{g} : \widehat{G} \to \widehat{G}$ is the automorphism which satisfies the following: $\widehat{g}(\widehat{V}) \subset Int(\widehat{g(V)})$, and $\widehat{g}(\widehat{E})$ crosses thick edges/thick vertices as the same order as $g(E)$ crosses edges/vertices. Moreover, $\widehat{g}$ is contractive on the set of the thick vertices, and $\widehat{g}$ expands unstable (resp. contracts stable) leaves in $\widehat{E}$ uniformly.

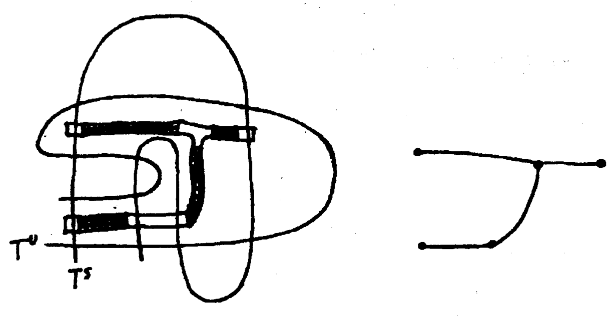

FIGURE 1. A trellis [left]; Its topological tree representative [right] and the thick tree [left].

Theorem 2.3. [4, Theorem 3.26] *Let $f : D \to D$ be a diffeomorphism with the trellis T, G the topological tree representative and $g : G \to G$ the tree map for (f,T). Suppose that P_i $(i = 1, \cdots, n)$ is a periodic orbit of $\widehat{g}$ which is not Nielsen equivalent to an attracting periodic orbit of $\widehat{g}$. Then there exists a* *finite union of periodic orbits $Q = Q_1 \cup \cdots \cup Q_n$ of f such that the braid type of $P = P_1 \cup \cdots \cup P_n$ for $\widehat{g}$ equals the braid type of Q for f.*

3. The Smale horseshoe map and pruning

3.1. **The Smale horseshoe map.** Let $F : D \to D$ be the Smale horseshoe map. The non-wandering set $\Omega(F)$ in R is a Cantor set $\Lambda = \{x \in D \mid F^m(x) \in R \text{ for all } m \in \mathbb{Z}\}$. Then there is an itinerary homeomorphism $k : \Lambda \to \Sigma_2 = \{0,1\}^{\mathbf{Z}}$ such that $\sigma(k(x)) = k(F(x))$, where $\sigma : \Sigma_2 \to \Sigma_2$ is the shift map. By using this identification k, the unique fixed point p in the "0" rectangle is $\bar{0} = \cdots 000 \cdots$, and the homoclinic orbits for p are of the form $\bar{0}w\bar{0} = \cdots 000w000 \cdots$ for some word w.

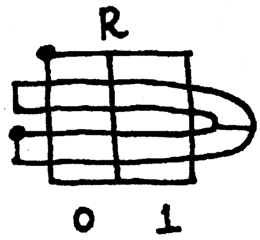

Figure 2

Definition 3.1. Let q_0 be the homoclinic point $\bar{0}.10\bar{0}$, and let $q_i = F^i(q_0)$. For $n > 0$, we call (T^U_n, T^S_n) the *full horseshoe trellis of signature* n, where $T^S_n = [p, q_j]_S$ and $T^U_n = [p, q_{j+n}]_U$ for some $j \leq 0$.

Definition 3.1 is well defined. That is

$$[F, ([p, q_{j+n}]_U, [p, q_j]_S)] = [F, ([p, q_{k+n}]_U, [p, q_k]_S)]$$

for any $j, k \leq 0$.

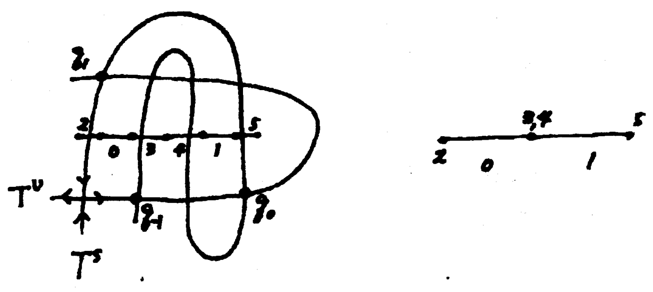

Figure 3. The full horseshoe trellis with signature 2 [left]; Its topological tree representative [right].

Definition 3.2. Let H be a horseshoe homoclinic orbit for p. The *signature of* H is the least integer n such that H contains some point $q \in T_n^U \cap T_n^S$ for the full horseshoe trellis of signature n.

For example, the signatures of $\bar{0}1010\bar{0}$, $\bar{0}1011010\bar{0}$ and $\bar{0}10010\bar{0}$ are $2, 5, 3$ respectively.

3.2. **pruning.** Given (f, T), we consider an isotopy from f to f' so that f' has a trellis T' obtained from T by removing some intersections of bigons of T. First of all, we consider the following example:

Example 3.3. Let T be the full horseshoe trellis of signature 3. Let b be the bigon as in Fig. 4[left]. Take a diffeomorphism h supported on a small neighborhood N of b so that $T^U \cap h(T^S) \cap N = \emptyset$. Let $\widetilde{F} = F \circ h^{-1}$, and $\widetilde{T}$ be the trellis for $\widetilde{F}$ corresponding T for F. Then $\widetilde{T}$ has fewer intersections than T (Fig. 4[right]).

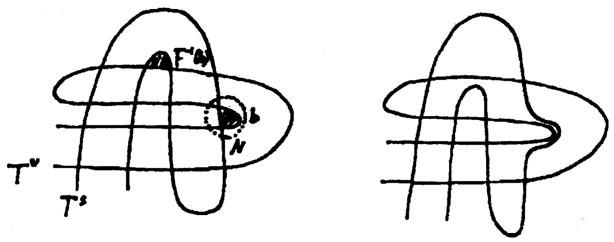

FIGURE 4. (F,T) [left]; $(\widetilde{F},\widetilde{T})$ [right].

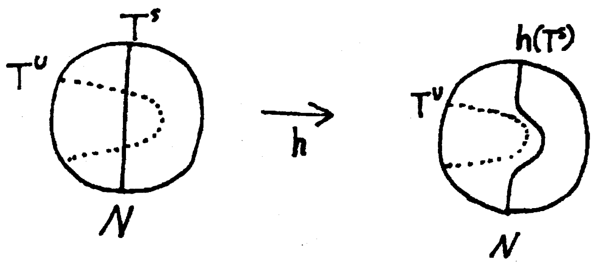

FIGURE 5

In general, we have:

Lemma 3.4. *Let $f : D \to D$ be a diffeomorphism with the trellis T. For a bigon b of T with vertices v, w, take the least integers $n_0 \geq 0$ and $m_0 \geq 0$ such that:*

- $f^{-(n_0+1)}(T^S[v,w]) \cap T^S = \emptyset$,
- $f^{m_0+1}(T^U[v,w]) \cap T^U = \emptyset$.

Suppose that $f^n(b)$ is a bigon of T for each $-n_0 \leq n \leq m_0$. Then take a diffeomorphism h supported on a small neighborhood N of $f^{m_0}(b)$ so that T^U and $h(T^S)$ has the fewest intersection in N. Let $\tilde{f} = f \circ h^{-1}$, and $\tilde{T}$ the trellis for $\tilde{f}$ corresponding T for f. Then $\tilde{T}$ has fewer intersections than T.

We say that $\tilde{T}$ *is obtained by pruning away bigons of* T.

Let H be a horseshoe homoclinic orbit of signature n. The *trellis forced by H* is the trellis obtained from the full horseshoe trellis of signature n by pruning away as many bigons as possible which do not contain a point of H.

We will define the star homoclinic orbits and give examples of the trellis forced by the homoclinic orbits.

Definition 3.5. Given $q = m/n \in \mathbb{Q} \cap (0, 1/2]$, let L_q be the straight line in $\mathbb{R}^2$ from $(0,0)$ to (n,m). For $0 \leq i \leq n$, let $s_i = 1$ if L_q crosses some line $y =$ integer for $x \in (i-1, i+1)$, and $s_i = 0$ otherwise. We call the homoclinic orbit of the form $H_q = \bar{0}s_0s_1 \cdots s_n\bar{0}$ the *star homoclinic orbit.*

Example 3.6. The trellises forced by $H_{1/2} = \bar{0}1010\bar{0}$, $H_{2/5} = \bar{0}1011010\bar{0}$ and $H_{1/3} = \bar{0}10010\bar{0}$ and the topological tree representatives are as follows:

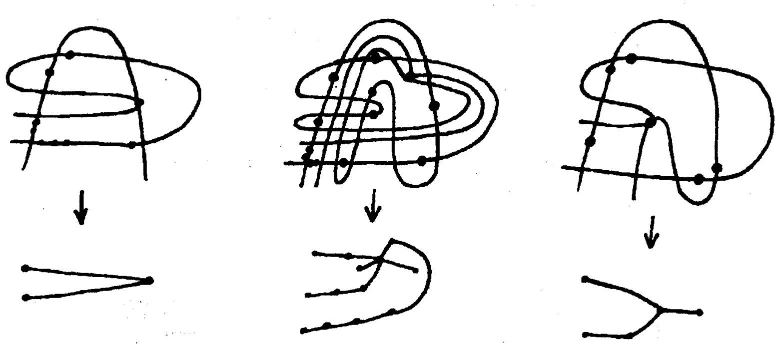

FIGURE 6. $H_{1/2}$ [left]; $H_{2/5}$ [center]; $H_{1/3}$ [right].

4. TEMPLATES

4.1. **Template Theorem by Birman and Williams.** *A template* is a compact branched 2-manifold with boundary and with smooth expansive semiflow built from a finite number of *joining charts* and *splitting charts* as in Fig. 7.

The Template Theorem of Birman and Williams shows us that if a 3-dimensional flow has a non-trivial hyperbolic chain recurrent set, then the set of links of closed orbits of the flow is captured by a template. In particular, the flow has a 1-dimensional hyperbolic chain recurrent set, then the theorem is described as follows.

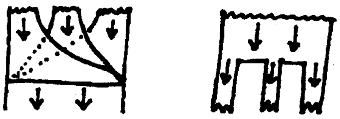

FIGURE 7. joining chart [left]; splitting chart [right].

Theorem 4.1. [2, Theorem 2.1] *Let $f_t : M \to M$ ($t \in \mathbb{R}$) be a smooth flow on a 3-manifold M. Suppose f_t has a 1-dimensional hyperbolic chain-recurrent set. Then there exists a template $\mathcal{W}$ embedded in M such that the links of the periodic orbits $L_{\mathcal{W}}$ on $\mathcal{W}$ correspond one-to-one to the links of the periodic orbits L_f of f_t. For any finite union of the periodic orbits, the correspondence is via ambient isotopy.*

4.2. **Universal template.** For $i_1, i_2 \in \mathbb{Z}$, $\mathcal{L}(i_1, i_2)$ denotes the template as in Fig. 8 embedded in S^3 having a single branch line and having two unknotted, unlinked strips with i_1 and i_2 half-twists respectively. We call it a *Lorenz-like template.*

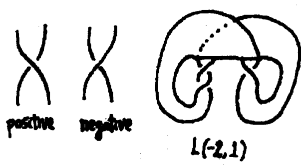

FIGURE 8

We say that a template $\mathcal{W}$ embedded in S^3 is *universal* if for each link L in S^3, there exists a finite union of periodic orbits P_L of the semiflow on $\mathcal{W}$ such that $L = P_L$. It is known that the universal template exists:

Theorem 4.2. [7, Corollary 3] *For each $i_2 < 0$, the Lorenz-like template $\mathcal{L}(0, i_2)$ is universal.*

5. TRELLISES AND TEMPLATES

Let $f : D \to D$ be a diffeomorphism of D with the trellis T. Let G be the topological tree representative and $g : G \to G$ the tree map for (f, T). Since a suspension flow of the thick tree map $\widehat{g}$ satisfies the assumption of Theorem 4.1, one can obtain the template $\mathcal{U}(\widehat{g})$ for a suspension flow of $\widehat{g}$. For example, in the case of Example 3.6(3), $g : G \to G$ is given by Fig. 9 and $\mathcal{U}(\widehat{g})$ is as in Fig. 10.

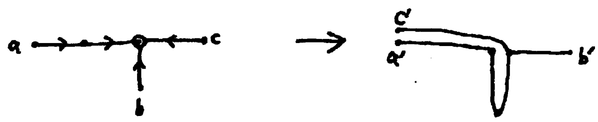

FIGURE 9

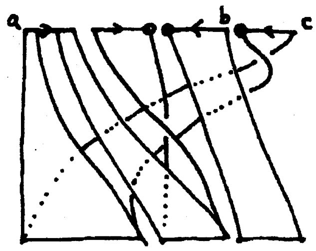

FIGURE 10

Theorem 5.1. *Let T_q be the trellis forced by the star homoclinic orbit H_q. If $q > 1/3$, then for any diffeomorphism $f : D \to D$ with T_q, f^3 induces all link types.*

Theorem 5.1 shows that if f has a homoclinic orbit of the braid type H_q ($q > 1/3$), then f^3 induces all link types.

Outline of the proof of Theorem 5.1. Let G_q be the the topological tree representative and $g_q : G_q \to G_q$ the tree map for T_q. By Theorem 2.3, we study whether $\widehat{g_q^3}$ induces all link types. Let $\mathcal{U}(\widehat{g_q^3})$ be the template for a suspension flow of $\widehat{g_q^3}$. Then it is shown that if $q > 1/3$, then $\mathcal{U}(\widehat{g_q^3})$ contains some universal Lorenz-like template. This shows that $\widehat{g_q^3}$ induces all link types. By using this, one can show that for any diffeomorphism $f : D \to D$ with T_q, f^3 induces all link types.

REFERENCES

[1] P. Boyland, *Topological methods in surface dynamics*, Topology Appl. 58 (3) (1994) 223-298.

[2] J. Birman and R. Williams, *Knotted periodic orbits in dynamical systems-II: Knot holders for fibered knots*, Contemp. Math. 20 (1983) 1-60.

[3] A. de Carvalho and T. Hall, *Braid forcing and star-shaped train tracks*, Topology 43 (2004) 247-287.

[4] P. Collins, *Forcing relations for homoclinic and periodic orbits of the Smale horseshoe map*, preprint.

[5] P. Collins, *Dynamics of surface maps with homoclinic and heteroclinic tangles*, PhD Thesis.

[6] J. Franks and M. Misiurewicz, *Cycles for disk homeomorphisms and thick trees*, Contemp. Math. 152 (1993) 69-139.

[7] R. Ghrist, *Branched two-manifolds supporting all links*, Topology 36 (2) (1997) 423-448

[8] R. Ghrist, P. Holmes, and M. Sullivan, Knots and Links in Three-Dimensional Flows, Lect. Notes in Math. 1654, Springer-Verlag (1997).

[9] E. Kin, *The third power of the Smale horseshoe induces all link types*, J. Knot Theory Ramifications 9 (7) (2000) 939-953.