ON A SIMPLE PROOF OF SLIGHTLY CURVED SEQUENCES CONTAINING ARBITRARILY LONG ARITHMETIC PROGRESSIONS

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Abstract. The author and Yoshida proved that a strictly increasing sequence $\{a(n)\}_{n\in A}$ of positive integers, which can be written as $a(n) = f(n) + O(1)$ for some function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f''(x) = O(1/x^\alpha)$ for some $\alpha > 0$, must contain arbitrarily long arithmetic progressions for all $A \subset \mathbb{N}$ with positive upper Banach density. In this article, we get a simple proof and the same conclusion if we replace the condition $f''(x) = O(1/x^\alpha)$ to $f''(x) = o(1)$.

1. Introduction

In this article, we consider problems involving arithmetic progressions. Let $d \geq 1$ and $k \geq 3$ be integers. A sequence $\{a(j)\}_{j=0}^{k-1} \subset \mathbb{N}^d$ is called an *arithmetic progression (AP) of length k* if there exists $D \in \mathbb{N}^d$ such that

$$a(j) = a(0) + jD$$

for all $j = 0, 1, \ldots, k-1$. Here $\mathbb{N}$ denotes the set of all positive integers. APs are taken interests from researchers studying number theory, arithmetic combinatorics, geometric measure theory, and fractal geometry. The author and Yoshida have found a new class of sets containing arbitrarily long APs, which is named a *slightly curved sequence*. Let $g : \mathbb{N} \to \mathbb{R}$ be an eventually positive function, and let $\mathbb{R}^+ = (0, \infty)$. A strictly increasing sequence $\{a(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ is called a *slightly curved sequence with error $O(g(n))$* if there exists a twice differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f''(x) = O(1/x^\alpha) \tag{1.1}$$

for some $\alpha > 0$, and

$$a(n) = f(n) + O(g(n)).$$

We define the *graph* of sequence $\{a(n)\}_{n\in A}$ as the set $\{(n, a(n)) : n \in A\}$. The author and Yoshida proved that if $\{a(n)\}_{n=1}^{\infty}$ is a slightly curved sequence with error $O(1)$ and $A \subset \mathbb{N}$ has positive upper Banach density, then $\{a(n)\}_{n\in A}$ contains arbitrarily long APs. Here we say that a set $A \subset \mathbb{N}$ has *positive upper Banach density* if the condition

$$\varlimsup_{N\to\infty} \frac{\max_{M\in\mathbb{N}} |A \cap [M, M+N-1]|}{N} > 0$$

holds. This result is contained Szeméredi's cerebrated theorem:

Proposition 1.1 (Szemerédi [S]). For every $k \geq 3$ and $0 < \delta \leq 1$ there exists an integer $N(k, \delta) > 0$ such that if $N \geq N(k, \delta)$, then every set $A \subset \{1, 2, \ldots, N\}$ with $|A| \geq \delta N$ contains an AP of length k.

Here $|X|$ denotes the cardinality of a finite set X. Note that the author and Yoshida obtained their result by using Szemerédi's theorem. Thus they do not give another proof of Szemeredi's theorem. As an application, the following result holds:

Proposition 1.2 ([SY, Corollary 1.5]). If a set $A \subset \mathbb{N}$ has positive upper Banach density, then the graph of $\{\lfloor n^a \rfloor\}_{n\in A}$ contains arbitrarily long APs for every $1 \leq a < 2$.

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We refer [SY] to the reader for more details. The goal of this article is to give a simple proof and to extend the condition (1.1). More precisely, we prove the following result:

Theorem 1.3. Suppose that a strictly increasing sequence $\{a(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfies that there exists a twice differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f''(x) = o(1) \tag{1.2}$$

and

$$a(n) = f(n) + O(1). \tag{1.3}$$

Then the graph of $\{a(n)\}_{n\in A}$ contains arbitrarily long arithmetic progressions for every $A \subset \mathbb{N}$ with positive upper Banach density.

2. Preparation

In order to prove our theorem, let us define a semi-norm on the vector space $\mathcal{F} = \{f \mid f : [0,\infty) \to \mathbb{R}\}$. Let $k \geq 3$ be an integer and $P = \{b(j)\}_{j=0}^{k-1} \subset [0,\infty)$ be a strictly increasing sequence. We define

$$N_P(f) = \sum_{j=0}^{k-3} |\Delta^2[f \circ b](j)|,$$

for every $f \in \mathcal{F}$, where Δ denotes the difference operator, that is,

$$\Delta[f](x) = f(x+1) - f(x),$$

and $\Delta^2 := \Delta \circ \Delta$. We can find that N_P satisfies the following properties:

(N1) for every strictly increasing function $f \in \mathcal{F}$,

$$N_P(f) = 0 \text{ if and only if } f(P) \text{ is an AP of length } k;$$

(N2) $N_P(f) \geq 0$ for all $f \in \mathcal{F}$;

(N3) $N_P(f+g) \leq N_P(f) + N_P(g)$ for all $f, g \in \mathcal{F}$.

We omit the proof of all the properties (N1), (N2), and (N3) because they are trivial. The semi-norm $N_P(\cdot)$ first appeared in [SY].

3. Proof

Proof of Theorem 1.3. Fix a set $A \subset \mathbb{N}$ with positive Banach upper density and $k \geq 3$. We show that $N_P(a) = 0$ for some arithmetic progression $P = \{b_j\}_{j=0}^{k-1} \subset A$ of length k. Let $R(x) := a(x) - f(x)$. Then there exists a positive integer $M > 0$ such that $|R(x)| < M$ for every $x \in \mathbb{N}$ since $a(x) = f(x) + O(1)$. Let

$$\delta := \varlimsup_{N\to\infty} \frac{\max_{M\in\mathbb{N}} |A \cap [M, M+N-1]|}{N},$$

and let

$$L := N\left(\frac{\delta}{2}, N\left(\frac{1}{4kM}, k\right)\right).$$

Assume that there exists $j_0 > 0$ such that for every $m \geq m_0$ we have

$$|A \cap [1 + (m-1)L, mL]| < L\delta/2.$$

Let M be a parameter of a positive integer. If $M < 1 + (m_0 - 1)L$ holds, then we obtain that

$$\begin{aligned} |A \cap [M, M+N-1]| &\leq |A \cap [1, (m_0-1)L]| + |A \cap [1 + (m_0-1)L, (m_0-1)L + N - 1]| \\ &\leq N\delta/2 + O_{m_0,L}(1) \end{aligned}$$

On the other hand, if $M \geq 1 + (m_0 - 1)L$ holds, then we obtain that

$$|A \cap [M, M+N-1]| \leq N\delta/2 + O_{m_0,L}(1).$$

Therefore we have

$$\varlimsup_{N\to\infty} \frac{\max_{M\in\mathbb{N}} |A\cap[M, M+N-1]|}{N} \leq \delta/2,$$

which is a contradiction. Hence there exists a infinite sequence $0 < m_1 < m_2 < \cdots$ of integers such that for every $s = 1, 2, \ldots$

$$|A \cap [1 + (m_s - 1)L, m_s L]| \geq L\delta/2$$

holds. Let $I_s := [1+(m_s-1)L, m_s L]$ for every $s = 1, 2, \ldots$. We can find an arithmetic progression $P' \subseteq A \cap I_s$ of length $N(1/(4kM), k)$ by Szemerédi's theorem (Proposition 1.1). Let

$$S_j := \left[-M + \frac{j-1}{2k}, -M + \frac{j}{2k}\right), \quad B_j = \{x \in P' \mid R(x) \in S_j\}$$

for every $j = 1, 2, \ldots, 4kM$. We partition the arithmetic progression P' into small $4kM$ sets B_j. Since at least one B_j satisfies $|B_j| \geq |P'|/(4kM)$, there exists an integer $q \in \{1, 2, \ldots, 4kM\}$ such that B_q contains at least one arithmetic progression P of length k by Szemerédi's theorem (Proposition 1.1). Let $P = \{b(j)\}_{j=0}^{k-1}$. From the triangle inequality (N3), it follows that

$$N_P(a) = N_P(f - R) \leq N_P(f) + N_P(R). \tag{3.1}$$

From $P \subseteq B_q$, the inequality $|\Delta[R \circ b](j)| \leq 1/2k$ holds for all $j = 0, 1, \ldots, k-1$. Thus the second term can be bounded as follows:

$$N_P(R) = \sum_{j=0}^{k-3} |\Delta^2[R\circ b](j)| \leq \sum_{j=0}^{k-3}(|\Delta[R\circ b](j+1)| + |\Delta[R \circ b](j)|) \leq (k-2)\frac{1}{k} = 1 - \frac{2}{k}.$$

The remaining part is to estimate the first term on the right hand side of (3.1). Let $b(j) = dj+e$ for some $d, e \in \mathbb{N}$. By the mean value theorem, for every $j = 0, 1, \ldots k-1$ there exists $\eta_j, \theta_j \in [0, 1)$ such that

$$\begin{aligned}
\Delta^2[f\circ b](j) &= (f\circ b(j+2) - f \circ b(j+1)) - (f\circ b(j+1) - f\circ b(j)) \\
&= d(f'\circ b(j+\eta_j+1) - f'\circ b(j+\eta_j)) \\
&= d^2 f''\circ b(j+\eta_j+\theta_j).
\end{aligned}$$

Since $b(j) \in P \subseteq P' \subseteq A \cap I_s$ holds, we obtain $e \geq m_s$ and $d \leq L$. Hence we have

$$N_P(f) = \sum_{j=0}^{k-3} |\Delta^2[f\circ b](j)| = \sum_{j=0}^{k-3} d^2 f''\circ b(j+\eta_j+\theta_j) \leq L^2(k-2)\times o(1) \to 0$$

as $s \to \infty$. Therefore if s is sufficiently large, then the following inequality holds:

$$0 \leq N_P(a) \leq N_P(f) + N_P(R) \leq L^2(k-2)\times o(1) + 1 - \frac{2}{k} < 1.$$

Therefore $N_P(a) = 0$. Hence $a(P)$ is an arithmetic progression of length k. □

References

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