# NOTES ON CHOW RINGS OF $G/B$ AND $BG$

NOBUAKI YAGITA

## 1. Introduction

Let $p$ be a prime number. Let $G$ and $T$ be a connected compact Lie group and its maximal torus such that $H^*(G)$ has $p$-torsion. Given a field $k$ with $ch(k)=0$, let $G_k$ and $T_k$ be a split reductive group and a split maximal torus over the field $k$, corresponding to $G$ and $T$. Let us write by $BG_k$ its classifying space defined by Totaro [To1,3]. Let $B_k$ be the Borel subgroup containing $T_k$. Let $\mathbb{G}$ be a $G_k$-torsor. Then $\mathbb{F}=\mathbb{G}/B_k$ is a (twisted) form of the flag variety $G_k/B_k$.

For a smooth algebraic variety $X$ over $k$, let $CH^*(X)=CH^*(X)_{(p)}$ mean the $p$-localized Chow ring generated by algebraic cycles modulo rational relations. The cofibering $G/T \stackrel{j}{\to} BT \stackrel{i}{\to} BG$ ([Tod1,2]) induces the maps

$$(1.1)\quad CH^*(BG_k) \stackrel{i^*}{\to} CH^*(BB_k) \stackrel{j^*}{\to} CH^*(\mathbb{G}/B_k),$$

whose composition $j^*i^*=0$ for $*>0$. But it is far from exact when $\mathbb{G}\cong G_k$. (Here exact means $Ker(j^+)=Ideal(Im(i^+))$.) However, we observe that it becomes near exact when $\mathbb{G}$ is sufficient twisted, while it is still not exact for most cases.

The author thanks Maski Kameko for pointing out errors in the first version of this paper.

## 2. $CH^*(\mathbb{G}/B_k)$

Recall that $G_k$ and $T_k$ are the split reductive group and split maximal torus over a field $k$ with $ch(k)=0$, corresponding to Lie groups $G$ and $T$. Let $B_k$ be the Borel subgroup containing $T_k$. Recall that $\mathbb{G}$ is a $G_k$-torsor, and let us write $\mathbb{F}=\mathbb{G}/B_k$ in this section.

By Petrov-Semenov-Zainoulline ([Pe-Se-Za], [Se-Zh]), it is known that the $p$-localized motive $M(\mathbb{F})_{(p)}$ of $\mathbb{F}$ is decomposed as

$$(2.1)\quad M(\mathbb{F})_{(p)}=M(\mathbb{G}/B_k)_{(p)}\cong R(\mathbb{G})\otimes(\oplus_i\tilde{\mathbb{T}}^{\otimes s_i})$$

where $\tilde{\mathbb{T}}$ is the reduced Tate motive and $R(\mathbb{G})$ is some motive called generalized Rost motive. (It is the original Rost motive([Ro], [Vo1,2], [Pe-Se-Za], [Ya2]) when $G$ is of type $(I)$ as explained below). Hence we have maps

$$(2.2)\quad CH^*(BB_k)\to CH^*(\mathbb{F}) \stackrel{split\ surj.}{\to} CH^*(R(\mathbb{G}))$$

where $BB_k$ is the classifying space for $B_k$-bundles. From Merkurjev and Karpenko [Me-Ne-Za], [Kar], we know that the first map is also surjective when $\mathbb{G}$ is a versal $G_k$-torsor. (For the definition of *versal torsor* see [Ga-Me-Se], [ Me-Ne-Za], [Kar],

---

2010 *Mathematics Subject Classification.* Primary 57T15, 20G15, 14C15.
*Key words and phrases.* Chow rings, classifying spaces, complete flag varieties.

[To2].) In particular, when $G$ is of type $(I)$, if $\mathbb{G}$ is a non-trivial $G_k$-torsor, then it is versal.

To explain groups of type $(I)$, we recall arguments for $H^*(G/T)$ in algebraic topology. By Borel, its $mod(p)$ cohomology is (for $p$ odd)

$$H^*(G;\mathbb{Z}/p) \cong P(y)/p \otimes \Lambda(x_1,...,x_\ell), \quad |x_i| = odd$$

where $P(y)$ is a truncated polynomial ring generated by *even* dimensional elements $y_i$, and $\Lambda(x_1,...,x_\ell)$ is the $\mathbb{Z}/p$-exterior algebra generated by $x_1,...,x_\ell$. When $p=2$, we consider the graded ring $grH^*(G;\mathbb{Z}/2)$ which is isomorphic to the right hand side ring above.

When $G$ is simply connected and $P(y)$ is generated by just one generator, we say that $G$ is of type $(I)$. Except for $(E_7, p=2)$ and $(E_8, p=2,3)$, all exceptional (simple) Lie groups are of type $(I)$. Note that in these cases, it is known $rank(G) = \ell \geq 2p-2$.

We consider the fibering ([Tod2], [Mi-Ni]) $G \stackrel{\pi}{\to} G/T \stackrel{i}{\to} BT$ and the induced spectral sequence

$$E_2^{*,*} = H^*(BT; H^*(G;\mathbb{Z}/p)) \Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

Here we can write $H^*(BT) \cong S(t) = \mathbb{Z}[t_1,...,t_\ell]$ with $|t_i| = 2$.

It is well known that $y_i \in P(y)$ are permanent cycles and that there is a regular sequence $(\bar{b}_1,...,\bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i) = \bar{b}_i$ ([Tod2], [Mi-Ni]).

We know that $G/T$ is a manifold such that $H^*(G/T) = H^{even}(G/T)$ and $H^*(G/T)$ is torsion free. We also see that there is a filtration in $H^*(G/T)_{(p)}$ such that

$$grH^*(G/T)_{(p)} \cong P(y) \otimes S(t)/(b_1,...,b_\ell)$$

where $b_i \in S(t)$ with $b_i = \bar{b}_i \ mod(p)$.

For the algebraic closure $\bar{k}$ of $k$, let us write $\bar{X} = X|_{\bar{k}}$. Then considering (2.1) over $\bar{k}$, we see

$$CH^*(\bar{R}(\mathbb{G}))/p \cong P(y), \quad CH^*(\oplus_i \tilde{\mathbb{T}}^{\otimes s_i}) \cong S(t)/(b_1,...,b_\ell).$$

Moreover when $\mathbb{G}$ is versal, we can see ([Ya2]) that $CH^*(R(\mathbb{G}))$ is additively generated by products of $b_1,...,b_\ell$ in (2.2). Hence we have surjections $CH^*(BB_k) \to CH^*(\mathbb{F}) \stackrel{pr.}{\to} CH^*(R(\mathbb{G}))$.

By giving the filtration on $S(t)$ by $b_i$, we can write (additively)

$$grS(t)/p \cong A \otimes S(t)/(b_1,...,b_\ell) \quad for \ A = \mathbb{Z}/p[b_1,...,b_\ell].$$

In particular, we have maps $A \stackrel{i_A}{\to} CH^*(\mathbb{F})/p \to CH^*(R(\mathbb{G}))/p$. We also see that the above composition map is surjective.

**Lemma 2.1.** *([Ya2]) Suppose that there are $f_1(b),...,f_s(b) \in A$ such that $CH^*(R(\mathbb{G}))/p \cong A/(f_1(b),...,f_s(b))$. Moreover if $f_i(b) = 0$ for $1 \leq i \leq s$ also in $CH^*(\mathbb{F})/p$, we have the isomorphism*

$$CH^*(\mathbb{F})/p \cong S(t)/(p, f_1(b),...,f_s(b)).$$

For $N > 0$, let us write $A_N = \mathbb{Z}/p\{b_{i_1}...b_{i_k} | |b_{i_1}| + ... + |b_{i_k}| \leq N\}$.

**Lemma 2.2.** *Let $pr : A_N \to CH^*(\mathbb{F})/p \to CH^*(R(\mathbb{G}))/p$, and $b \in Ker(pr)$. Then $b = \sum b'u'$ with $b' \in A_N$, $u' \in S(t)^+/(p, b_1,...,b_\ell)$ i.e., $|u'| > 0$.*

Using these, we can prove

**Theorem 2.3.** *([Ya2]) Let $G$ be of type $(I)$ and $rank(G) = \ell$. Let $\mathbb{G}$ be a non-trivial $G_k$-torsor. Then $2p-2 \le \ell$, and we can take $b_i \in S(t) = CH^*(BB_k)$ for $1 \le i \le \ell$ such that there are isomorphisms*

$$CH^*(R(\mathbb{G}))/p \cong \mathbb{Z}/p\{1, b_1, ..., b_{2p-2}\},$$

$$CH^*(X)/p \cong S(t)/(p, b_ib_j, b_k | 0 \le i, j \le 2p-2 < k \le \ell)$$

*where $\mathbb{Z}/p\{a, b, ...\}$ is the $\mathbb{Z}/p$-free module generated by $a, b, ...$*

## 3. RELATION $\mathbb{G}/B_k$ AND $BG$

Let $h^*(X) = CH^*(X)/I(h)$ for some ideal $I(h)$ (e.g., $CH^*(X)/p$). We note here the following lemma for each $G_k$-torsor $\mathbb{G}$ (not assumed twisted).

**Lemma 3.1.** *For the above $h^*(X)$, the composition of the following maps is zero for $* > 0$*

$$h^*(BG_k) \to h^*(BB_k) \to h^*(\mathbb{G}/B_k).$$

*Proof.* Take $U$ (e.g., $GL_N$ for a large $N$) such that $U/G_k$ approximates the classifying space $BG_k$ [To3]. Namely, we can take $\mathbb{G} = f^*U$ for the classifying map $f : \mathbb{G}/G_k \to U/G_k$. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{F} = \mathbb{G}/B_k & \longrightarrow & U/B_k \\ \downarrow & & \downarrow \\ Spec(k) \cong \mathbb{G}/G_k & \longrightarrow & U/G_k \end{array}$$

where $U/B_k$ (resp. $U/G_k$) approximates $BB_k$ (resp. $BG_k$). Since $h^*(Spec(k)) = CH^*(Spec(k))/I(h) = 0$ for $* > 0$, we have the lemma. □

The above sequences of maps in the lemma is not exact, in general. However we get some informations from $h^*(\mathbb{F})$ to $h^*(BG_k)$. For example, we get much informations of $h^*(BG_k)$ from $h^*(\mathbb{F})$ than from $h^*(G_k/B_k)$ when $\mathbb{G}$ is versal.

Let us write the induced maps

$$h^+(BG_k) \xrightarrow{i^+} h^+(BT) \xrightarrow{j(\mathbb{G})^+} h^+(\mathbb{G}/B_k)$$

where $h^+(-)$ is the ideal of the positive degree parts. Let us define

$$D_h(\mathbb{G}) = Ker(j^+)/(Ideal(Im(i^+)).$$

Let $\mathbb{G}$ be versal and $k'$ is some extension of $k$. Then

$$D_h(\mathbb{G}) \subset D_h(\mathbb{G}|_{k'}) \subset D_h(G|_{\bar{k}}) \cong D_h(G_k).$$

For ease of arguments we mainly consider the case $h^*(X) = CH^*(G)/p$, and write $D_h(\mathbb{G})$ simply by $D(\mathbb{G})$.

**Theorem 3.2.** *Let $\mathbb{G}$ be versal. Then additively*

$$D(G_k)/D(\mathbb{G}) \cong CH^+(R(\mathbb{G}))/p \otimes S(t)/(b_1, ..., b_\ell).$$

*Proof.* We consider the map $S(t) \cong CH^*(BB_k) \xrightarrow{j^*} CH^*(\mathbb{G}/B_k)$. Recall that

$$CH^*(G_k/B_k)/p \cong P(y) \otimes S(t)/(b) \quad (b) = Ideal(b_1, ..., b_\ell).$$

So $Ker(j(G_k)) = (b)$. Hence

$$D(G_k)/(D(\mathbb{G})) \cong (Ker(j(G_k)/Im(i^+))/(Ker(j(\mathbb{G}))/Im(i^+))$$

$$\cong Ker(j(G_k))/Ker(j(\mathbb{G})) \subset CH^*(\mathbb{F})/p \stackrel{pr.}{\to} CH^+(R(\mathbb{G}))/p.$$

This composition map is a surjection. Because each element

$$x \in Ker(j(G_k)) = (b_1, ..., b_\ell) \subset S(t)/p$$

can be written using $A(b)^+ = \mathbb{Z}/p[b_1, ..., b_\ell]^+$

$$x = \sum b_I t(i) \quad b_I \in A(b)^+, \quad 0 \neq t(I) \in S(t)/(b_1, ..., b_\ell).$$

This also means that the ideal $Ker(j(G)) \cong A(b)^+ \otimes S(t)/(b)$, which implies

$$Ker(j(G))/Ker(j(\mathbb{G})) \cong CH^+(R(\mathbb{G}))/p \otimes S(t)/(b).$$

□

**Corollary 3.3.** *There is a surjection $D(G_k) \to CH^+(R(\mathbb{G}))/2$.*

Thus we have a very weak version of the decomposition theorem by Petrov-Semenov-Zainoulline [Pe-Se-Za], without using deep motive theories.

**Corollary 3.4.** *Let $\mathbb{G}$ be versal. Then we have an additive decomposition of the mod(p) Chow ring*

$$CH^*(\mathbb{G}/B_k)/p \cong S(t)/(p, b_1, ..., b_\ell) \oplus D(G_k)/D(\mathbb{G})$$

$$\cong (\mathbb{Z}/p\{1\} \oplus CH^+(R(\mathbb{G})/p) \otimes S(t)/(b_1, ..., b_\ell).$$

## 4. $SO(2\ell+1)$

At first we consider the orthogonal groups $G = SO(m)$ and $p = 2$. The $mod(2)$-cohomology is written as ( see for example [Tod-Wa], [Ni])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, ..., x_{m-1})$$

where $|x_i| = i$, and the multiplications are given by $x_s^2 = x_{2s}$.

For ease of argument, we only consider the case $m = 2\ell + 1$ so that

$$H^*(G; \mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_1, x_3, ..., x_{2\ell-1})$$

$$grP(y)/2 \cong \Lambda(y_2, ..., y_{2\ell}), \quad \textit{letting } y_{2i} = x_{2i} \ \ (\textit{hence } y_{4i} = y_{2i}^2).$$

The Steenrod operation is given as $Sq^k(x_i) = \binom{i}{k}(x_{i+k})$. The $Q_i$-operations are given by Nishimoto [Ni]

$$Q_n x_{2i-1} = y_{2i+2^{n+1}-2}, \qquad Q_n y_{2i} = 0.$$

In particular, $Q_0(x_{2i-1}) = y_{2i}$ in $H^*(G; \mathbb{Z}/2)$. It is well known that the transgression $b_i = d_{2i}(x_{2i-1}) = c_i$ is the $i$-th elementary symmetric function on $S(t)$. Hence we have

**Lemma 4.1.** *We have an isomorphism*

$$grH^*(G/T) \cong P(y) \otimes S(t)/(c_1, ..., c_\ell).$$

Moreover, the cohomology $H^*(G/T)$ is computed completely by Toda-Watanabe [Tod-Wa] (e.g. $2y_{2i} = c_i \ mod(4)$).

Let $T$ be a maximal Torus of $SO(m)$ and $W = W_{SO(m)}(T)$ its Weyl group. Then $W \cong S_\ell^{\pm}$ is generated by permutations and change of signs so that $|S_k^{\pm}| = 2^k k!$. Hence we have

$$H^*(BT)^{W'} \cong \mathbb{Z}_{(2)}[p_1, ..., p_\ell] \subset H^*(BT) \cong \mathbb{Z}_{(2)}[t_1, ..., t_\ell], \ |t_i| = 2$$

where the Pontriyagin class $p_i$ is defined by $\Pi_i(1 + t_i^2) = \sum_i p_i$.

Here we recall

$$H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, ..., w_{2\ell+1}], \quad Q_0(w_{2i}) = w_{2i+1} \ mod(w_s w_t).$$

It is known $H^*(BG)$ has no higher 2-torsion and

$$H(H^*(BG; \mathbb{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbb{Z}/2$$

where $H(A; Q_0)$ is the homology of $A$ with the differential $Q_0$ and $Tor$ is the torsion ideal in $H^*(BG)$. Hence we have

$$H^*(BG)/Tor \cong D \quad where \ \ D = \mathbb{Z}_{(2)}[c_2, c_4, ..., c_{2\ell}].$$

The isomorphism $j : H^*(BG)/Tor \to H^*(BT)^W$ is given by $c_{2i} \mapsto p_i$.

Now we consider the $mod(2)$ Chow ring and the case that $\mathbb{G}$ is the split group $G_k$.

**Lemma 4.2.** *We have additive isomorphism*

$$D(G_k) \cong \Lambda(c_1, .., c_\ell)^+ \otimes S(t, c) \quad with \ S(t, c) \cong S(t)/(c_1, ..., c_\ell),$$

*namely, each element $x \in D(G_k)$ is written as $x = \sum c_I t(I)$ with $c_I \in \Lambda(c_1, ..., c_\ell)^+$ and $t(I) \neq 0 \in S(t)/(c_1, ..., c_\ell)$.*

*Proof.* Recall that

$$CH^*(G_k/B_k)/2 \cong H^*(G/T)/2 \cong P(y)/2 \otimes S(t)/(c_1, ..., c_\ell).$$

Hence we see

$$Ker(j) \cong (c_1, ..., c_\ell) \subset CH^*(BB_k)/2 \cong H^*(BT)/2.$$

Here $j : p_i \mapsto c_i^2 \ mod(2)$ by definition of the Pontryagin class $p_i$.

On the other hand, we know by Totaro [To1]

$$CH^*(BG_k) \cong \mathbb{Z}[c_2, ..., c_{2\ell+1}]/(2c_{odd}).$$

Hence $CH^*(BG_k)/Tor \cong D \cong H^*(BT)^W$ by $i : c_{2i} \mapsto p_i$. Thus the ideal generated by the image is $(Im(i)) \cong (c_2, c_4, ..., c_{2\ell}) \subset S(t)$. Since $j : p_i \mapsto c_i^2$, we have

$$Ker(j)/(Im(i)) \cong (c_1, ..., c_\ell)/(c_1^2, ..., c_\ell^2) \subset S(t)/(c_1^2, ..., c_\ell^2)$$

which is additively isomorphic to $\Lambda(c_1, ..., c_\ell)^+ \otimes S(t)/(c_1, ..., c_\ell)$. □

Recall that there is a surjection $D(G_k) \to CH^+(R(\mathbb{G}))/p$ from Lemma 2.1. We can see $c_1 ... c_\ell \neq 0$ in $CH^*(R(\mathbb{G}))/2$ (for example using the torsion index $t(G) = 2^\ell$ (for the torsion index, see [To2]).

**Theorem 4.3.** *(Petrov* [Pe], [Ya2]*) Let $(G, p) = (SO(2\ell + 1), 2)$ and $\mathbb{F} = \mathbb{G}/B_k$ be versal. Then $CH^*(\mathbb{F})$ is torsion free, and*

$$CH^*(\mathbb{F})/2 \cong S(t)/(2, c_1^2, ..., c_\ell^2), \quad CH^*(R(\mathbb{G}))/2 \cong \Lambda(c_1, ..., c_\ell).$$

**Corollary 4.4.** *Let $(G, p) = (SO(2\ell + 1), 2)$ and $\mathbb{G}$ be versal.*
*Then we have $D(\mathbb{G}) \cong 0$.*

## 5. $Spin(7)$ FOR $p = 2$

Hereafter this section, we assume $G = Spin(7)$ and $p = 2$. It is well known

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8]$$

where $w_i$ for $i \le 7$ (resp. $i = 8$) are the Stiefel-Whitney classes for the representation induced from $Spin(7) \to SO(7)$ (resp. the spin representation $\Delta$).

Thus the integral cohomology is written as (using $Q_0 w_6 = w_7$)

$$H^*(BG) \cong \mathbb{Z}_{(2)}[w_4, c_6, w_8] \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\})$$
$$\cong D \otimes \Lambda_{\mathbb{Z}}(w_4, w_8) \otimes (\mathbb{Z}_{(2)}\{1\} \oplus \mathbb{Z}/2[w_7]\{w_7\})$$

where $D = \mathbb{Z}_{(2)}[c_4, c_6, c_8]$ with $c_i = w_i^2$.

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \Longrightarrow BP^*(BG).$$

We can compute the spectral sequence

$$grBP^*(BG) \cong D \otimes (BP^*\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\}$$
$$\oplus BP^*/(2, v_1, v_2)[c_7]\{c_7\}/(v_3c_7c_8)).$$

Then $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)}$ is isomorphic to ([Ko-Ya])

$$D\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\}/(2v_1w_8) \oplus D/2[c_7]\{c_7\}.$$

On the other hand, the Chow ring of $BG_{\mathbb{C}}$ is given by Guillot ([Gu],[Ya1])

**Theorem 5.1.** *Let $k = \bar{k}$. Then we have isomorphisms*

$$CH^*(BG_k) \cong BP^*(BG_k) \otimes_{BP^*} \mathbb{Z}_{(2)}$$
$$\cong D \otimes (\mathbb{Z}_{(2)}\{1, c_2', c_4'.c_6'\} \oplus \mathbb{Z}/2\{\xi_3\} \oplus \mathbb{Z}/2[c_7]\{c_7\})$$

*where $cl(c_i) = w_i^2$, $cl(c_2') = 2w_4$, $cl(c_4') = 2w_8$, $cl(c_6') = 2w_4w_8$, and $cl(\xi_3) = 0$, $|\xi_3| = 6$. However $cl_\Omega(\xi_3) = v_1w_8$ in $BP^*(BT)^W$, for the cycle map $cl_\Omega$ of the algebraic cobordism.*

Now we consider $CH^*(\mathbb{G}/B_k)$. Let $G = Spin(7)$ and $\mathbb{G}$ be versal. The group $G$ is of type $(I)$ and we can take $b_1 = c_2, b_2 = c_3, b_3 = e_4$ with $|e_4| = 8$ (for details see [Ya2]). The Chow ring $CH^*(\mathbb{G}/B_k)$ is given in Theorem 2.3 (in fact, $G$ is of type $(I)$)

$$CH^*(\mathbb{G}/B_k) \cong S(t)/((2c_2, c_2^2, c_2c_3, c_3^2, e_4), \quad S(t) = \mathbb{Z}_{(2)}[t_1, t_2, t_3].$$

Hence we have $Ker(j(\mathbb{G})) \cong (2c_2, c_2^2, c_2c_3, c_3^2, e_4)$. Recall

$$CH^*(BG_{\bar{k}})/(Tor) \cong CH^*(BB_k)^W \cong D\{1, c_2'', c_4'', c_6''\}$$

where $c_i''$ is a Chern class of the (complex) spin representation. Since $i(c_2'') = 2w_4, ...,$ we see

$$D/2 \cong Im(i^*/2 : CH^*(BG_k) \to CH^*(BT)/2).$$

We can see that the map $i^*$ is given $c_4 \mapsto c_2^2$, $c_6 \mapsto c_3^2$, $c_8'' \mapsto e_4^2$, and

$$c_2'' \mapsto 2c_2, \quad c_4'' \mapsto 2e_4, \quad c_6'' \mapsto 2c_2e_4.$$

In particular $i^*CH^*(BG_k) = i^*CH^*(BG_{\bar{k}})$. Thus we see

**Proposition 5.2.** *Let $G = Spin(7)$ and $\mathbb{G}$ be versal. Then we have additively*

$$D(\mathbb{G}) \cong \Lambda(c_2c_3, e_4)^+ \otimes S(t, c) \quad \textit{for } S(t, c) \cong S(t)/(c_2, c_3, e_4).$$

## References


[Ga-Me-Se] S. Garibaldi, A. Merkurjev and J-P Serre. Cohomological invariants in Galois cohomology. *University Lecture Series 28, Amer. Math. Soc. Providence, RI* (2003), viii+168pp.

[Gu] P. Guillot, The Chow rings of $G_2$ and $Spin(7)$, *J. reine angew. Math.* **604** (2007), 137-158.

[Ha] M.Hazewinkel, Formal groups and applications, *Pure and Applied Math. 78, Academic Press Inc.* (1978), xxii+573pp.

[Ka-Ya] M. Kameko and N. Yagita. Chern subrings. *Proc. Amer. Math. Soc.* **138** (2010), 367-373.

[Kar] N. Karpenko. Chow groups of some generically twisted flag varieties. *Ann K-theory* **2** (2017), 341-356.

[Ko-Ya] A. Kono and N.Yagita, Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups, *Trans. Amer. Math. Soc.* **339** (1993), 781-798.

[Me-Ne-Za] A. Merkurjev, A. Neshitov and K. Zainoulline. *Invariants o degree* 3 *and torsion in the Chow group of a versal flag.* Composito Math. **151** (2015), 11416-1432.

[Mi-Ni] M. Mimura and T. Nishimoto. Hopf algebra structure of Morava K-theory of exceptional Lie groups. *Contem. Math.* **293** (2002), 195-231.

[Ni] T. Nishimoto. Higher torsion in Morava K-thoeory of $SO(m)$ and $Spin(m)$. *J. Math. Soc. Japan.* **52** (2001), 383-394.

[Pe] V. Petrov. Chow ring of generic maximal orthogonal Grassmannians. *Zap. Nauchn, Sem. S.-Peterburg. Otdel. Mat. Inst. Skelov. (POMI) 443* (2016), 147-150.

[Pe-Se-Za] V.Petrov, N.Semenov and K.Zainoulline. J-Invariant of linear algebraic groups. *Ann. Scient. Ec. Norm. Sup.* **41**, (2008) 1023-1053.

[Ra] D.Ravenel. Complex cobordism and stable homotopy groups of spheres. *Pure and Applied Mathematics, 121. Academic Press* (1986).

[Ro] M.Rost. Some new results on Chowgroups of quadrics. *preprint* (1990).

[Se-Zh] N. Semenov and M Zhykhovich. Integral motives, relative Krull-Schumidt principle, and Maranda-type theorems. *Math. Ann. 363* (2015) 61-75.

[Tod1] H. Toda, Cohomology $mod(3)$ of the classifying space $BF_4$ of the exceptional group $F_4$. *J. Math. Kyoto Univ.* **13** (1973) 97-115.

[Tod2] H.Toda. On the cohomolgy ring of some homogeneous spaces. *J. Math. Kyoto Univ.* **15** (1975), 185-199.

[Tod-Wa] H.Toda and T.Watanabe. The integral cohomology ring of $F_4/T$ and $E_4/T$. *J. Math. Kyoto Univ.* **14** (1974), 257-286.

[To1] B. Totaro. The Chow ring of classifying spaces. *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington,Seattle)* **67** (1999), 248-281.

[To2] B. Totaro. The torsion index of the spin groups. *Duke Math. J.* **299** (2005), 249-290.

[To3] B. Totaro, Group cohomology and algebraic cycles, *Cambridge tracts in Math. (Cambridge Univ. Press)* **204** (2014).

[Vo1] V. Voevodsky. Motivic cohomology with $\mathbb{Z}/2$ coefficient. *Publ. Math. IHES* **98** (2003), 59-104.

[Vo2] V.Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. *Ann. of Math.* **174** (2011), 401-438.

[Ya1] N. Yagita, The image of the cycle map of classifying space of the exceptional Lie group $F_4$, *J. Math. Kyoto Univ.* **44** (2004), 181-191.

[Ya2] N. Yagita. Chow rings of versal complete flag varities. *arXiv:1609.08721vl. (math. KT).* (2016)



[N. Yagita] Department of Mathematics, Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan

*E-mail address*: `nobuaki.yagita.math@vc.ibaraki.ac.jp`