Some results on countably based consonant spaces

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1 Introduction

Consonance was introduced by S. Dolecki, G.H. Greco and A. Lechicki in [9], and further investigated by T. Nogura and D. Shakhmatov in [12] under the name *uK-trivial*. A. Bouziad showed in [1] that a metrizable separable co-analytic space is consonant if and only if it is Polish, and later showed that it is independent of ZFC whether every metrizable analytic consonant space is completely metrizable [3].

So far there is no characterization of non-metrizable consonant spaces that is analogous to A. Bouziad's result, and the purpose of this paper is to make some progress in this direction by focusing on countably based (possibly non-metrizable) spaces. In particular, we will prove the following theorem, which shows that there is a close connection between countably based consonant spaces and a class of countably based T_0-spaces called quasi-Polish spaces [4].

Theorem 1 *If X is a co-analytic countably based sober space, then every $\mathbf{\Pi}^0_2$-subspace of X is consonant if and only if X is quasi-Polish.* □

Consonance is hereditary with respect to both closed subspaces and open subspaces, but there are examples of non-countably based consonant spaces which have dissonant $\mathbf{\Pi}^0_2$-subspaces (see [9] and [12]). To the best of our knowledge, it is still open whether or not consonance is preserved by $\mathbf{\Pi}^0_2$-subspaces of *countably based* consonant spaces. If such a preservation result does hold, then together with Theorem 1 we would obtain a complete Bouziad-like characterization of consonance for countably based sober spaces. Unfortunately, we have not yet been able to prove such a preservation result, but we recommend the paper [2] to any readers that are interested in pursuing this strategy.

A major part of the proof of Theorem 1 depends on a Hurewicz-like result for quasi-Polish spaces that was obtained in [5]. There it was shown that a countably based co-analytic sober space is *not* quasi-Polish if and only if it has a $\mathbf{\Pi}^0_2$-subspace homeomorphic to either the rationals $\mathbb{Q}$ or a certain countable space called S_0. Since it is known from [1] that the rationals are dissonant (i.e., *not* consonant), in order to prove Therorem 1 it only remains to show that the space S_0 is dissonant.

Therefore, the majority of this paper is dedicated to proving that S_0 is dissonant. In Section 3 we will define a two-player infinite game $\mathcal{G}(X)$ for each topological space X, and prove that if X is consonant then the second player has a winning strategy for $\mathcal{G}(X)$

(Theorem 4). Then in Section 4 we will prove that S_0 is dissonant (Lemma 5) by showing that the first player has a winning strategy for $\mathcal{G}(S_0)$.

The game we introduce is actually a restricted version of a game introduced by T. Plewe in [13] that characterizes the spatiality of localic products of spaces. The core of the proof of our game theoretic characterization (Theorem 4) comes from T. Plewe's proof of Theorem 1.2 in [13]. The game strategy used in our proof that S_0 is dissonant (Lemma 5) was first announced at the CCC 2019 workshop as part of our proof that the localic product $S_0 \times_{loc} S_0$ is not spatial [6]. The main original technical contribution of this paper is Lemma 3, which implicitly provides the link between consonance and localic products needed to show that T. Plewe's game partially characterizes consonance.

Although this paper makes some implicit connections between consonance and spatial localic products, to ensure that the results are accessible to general topologists interested in consonance, we have structured this paper so that it is self-contained and does not require any knowledge of locale theory or localic products.

However, the implicit connections we make in this paper are not as strong as we expect they really are. For readers interested in further clarifying the relationship between consonance and the spatiality of localic products, we recommend the references [11] and [10] for two different approaches to constructing localic products. We originally found the proof of Lemma 3 using the construction in terms of antitone Galois connections that is described in [10]. However, the approach we use in this paper is the one described in Section 1 of [13], and is perhaps closer related to the construction using C-ideals in [11]. This approach makes it easier to apply T. Plewe's proof to our Theorem 4, so that we can include that proof and obtain a self-contained paper. The connections with C-ideals should make it an easy exercise for the reader to convert P. Johnstone's proof in [11] of the non-spatiality of the localic product $\mathbb{Q} \times_{loc} \mathbb{Q}$ into a winning strategy for the first player in $\mathcal{G}(\mathbb{Q})$, thereby obtaining an alternative proof that $\mathbb{Q}$ is dissonant.

2 Preliminaries

We do not implicitly assume any separation axioms for the topological spaces in this paper. Given a topological space X, we write $\mathbf{O}(X)$ for the complete lattice of open subsets of X. $\mathbf{K}(X)$ denotes the set of compact subsets of X, and for $K \in \mathbf{K}(X)$ we define $\nabla K = \{U \in \mathbf{O}(X) \mid K \subseteq U\}$. A subset $\mathcal{D} \subseteq \mathbf{O}(X)$ is *directed* if it is non-empty and for every $U, V \in \mathcal{D}$ there is $W \in \mathcal{D}$ with $U \subseteq W$ and $V \subseteq W$. A subset $\mathcal{H} \subseteq \mathbf{O}(X)$ is *Scott-open* if and only if $\mathcal{H}$ is an upper set and $\bigcup \mathcal{D} \in \mathcal{H}$ implies $\mathcal{D} \cap \mathcal{H} \neq \emptyset$ for each directed subset $\mathcal{D} \subseteq \mathbf{O}(X)$.

A topological space X is *consonant* if and only if for every Scott-open $\mathcal{H} \subseteq \mathbf{O}(X)$ and $U \in \mathcal{H}$, there is $K \in \mathbf{K}(X)$ such that $U \in \nabla K \subseteq \mathcal{H}$. A space is *dissonant* if it is not consonant.

The underlying set of S_0 is the set $\mathbb{N}^{<\mathbb{N}}$ of finite sequences of natural numbers, and the topology of S_0 is generated by subbasic open sets of the form $\{\tau \in S_0 \mid \sigma \not\preceq \tau\}$ for $\sigma \in S_0$, where $\preceq$ is the usual prefix relation. S_0 is a sober space, but it does not satisfy the T_1-axiom. Also note that S_0 has uncountably many distinct open sets, and every closed subspace of S_0 is a Baire space (i.e., a space in which the intersection of countably many dense open sets is dense) [5].

The remaining definitions in this section are needed to understand the statement of Theorem 1. However, they will not be used directly in the proof that S_0 is dissonant, so the reader can simply skim over them on a first reading.

A non-empty closed set is *irreducible* if it is not equal to the union of any two proper closed subsets. A space is *sober* if every irreducible closed subset is the closure of a unique point. Every sober space is a T_0-space, and every Hausdorff space is sober, but sobriety is incomparable with the T_1-axiom. Sobriety is a kind of completeness property which is precisely what is needed to recover a space from its lattice of open subsets.

A subset A of a topological space X is $\mathbf{\Pi}^0_2$ if and only if there are sequences $(U_i)_{i\in\mathbb{N}}$ and $(V_i)_{i\in\mathbb{N}}$ of open subsets of X such that

$$x \in A \iff (\forall i \in \mathbb{N})\, [x \in U_i \Rightarrow x \in V_i].$$

Every G_δ-set (i.e. a countable intersection of open sets) is a $\mathbf{\Pi}^0_2$-set, and every $\mathbf{\Pi}^0_2$-subset of a separable metrizable set is G_δ, but it is common for non-Hausdorff spaces to have $\mathbf{\Pi}^0_2$-subsets which are not G_δ.

A subset of a space is *analytic* ($\mathbf{\Sigma}^1_1$) if it is empty or else equal to the continuous image of a Polish space, and a subset is *co-analytic* ($\mathbf{\Pi}^1_1$) if its complement is analytic. A countably based space is co-analytic if it is homeomorphic to a co-analytic subspace of $\mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ is the powerset of the natural numbers with the topology generated by subbasic opens of the form $\{S \in \mathcal{P}(\mathbb{N}) \mid n \in S\}$ for $n \in \mathbb{N}$. Note that every countably based T_0-space can be embedded into $\mathcal{P}(\mathbb{N})$.

A space is *quasi-Polish* if and only if it is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathcal{P}(\mathbb{N})$ (see [4] for several other equivalent characterizations). Every Polish space is quasi-Polish, and conversely every metrizable quasi-Polish space is Polish. It was shown in [8] that every quasi-Polish space is consonant, which turned out to be an important property for the study of various powerspaces in [7].

3 A game for consonance

In this section, we introduce a two-player infinite game $\mathcal{G}(X)$ for each topological space X, and prove that the second player has a winning strategy for the game whenever X is consonant. The game $\mathcal{G}(X)$ defined here is a restricted version of the game introduced by T. Plewe to characterize the spatiality of localic products of spaces [13].

Definition 2 *The game $\mathcal{G}(X)$ is defined as follows for any topological space X. Player I begins the game by choosing a non-empty covering $\mathcal{U}$ of $X \times X$ by open rectangles (i.e. open subsets of $X \times X$ of the form $V \times W$). Set $V_0 = W_0 = X$. The $(i+1)$-th round $(i \geq 0)$ proceeds as follows:*

- *Player I chooses a point $x_{i+1} \in V_i$.*
- *Player II responds with an open set $V_{i+1} \subseteq V_i$ containing x_{i+1}.*
- *Player I then chooses a point $y_{i+1} \in W_i$.*

- *Player II finishes the round by playing an open set $W_{i+1} \subseteq W_i$ containing y_{i+1}.*

The game then continues on to the next round. Player II wins the game if there is $i \in \mathbb{N}$ and $V \times W \in \mathcal{U}$ such that $V_i \times W_i \subseteq V \times W$. Otherwise, Player I wins. □

The rest of this section is dedicated to proving Theorem 4 below, where it is shown that if X is consonant then Player II has a winning strategy in the game $\mathcal{G}(X)$.

Fix a topological space X and let $\mathcal{U}$ be a non-empty covering of $X \times X$ by open rectangles. Define a subset $\mathcal{A}^{(\beta)}$ of $\mathbf{O}(X) \times \mathbf{O}(X)$ for each ordinal β as follows:

$$\begin{aligned} \mathcal{A}^{(0)} &= \{\langle V, W\rangle \mid V, W \in \mathbf{O}(X) \;\&\; (\exists V' \times W' \in \mathcal{U})\, V \times W \subseteq V' \times W'\} \\ \mathcal{A}^{(\beta+1)} &= \{\langle \bigcup \mathcal{S}, W\rangle \mid \mathcal{S} \subseteq \mathbf{O}(X) \;\&\; (\forall V \in \mathcal{S})\, \langle V, W\rangle \in \mathcal{A}^{(\beta)}\} \\ &\quad \cup \{\langle V, \bigcup \mathcal{S}\rangle \mid \mathcal{S} \subseteq \mathbf{O}(X) \;\&\; (\forall W \in \mathcal{S})\, \langle V, W\rangle \in \mathcal{A}^{(\beta)}\}, \\ \mathcal{A}^{(\lambda)} &= \bigcup_{\beta<\lambda} \mathcal{A}^{(\beta)} \text{ for limit ordinal } \lambda. \end{aligned}$$

For cardinality reasons, there is some ordinal β such that $\mathcal{A}^{(\beta)} = \mathcal{A}^{(\beta+1)}$, and we will use ∞ to denote the least such ordinal.

Lemma 3 *Assume X is a consonant space and $\mathcal{U}$ is a non-empty covering of $X \times X$ by open rectangles. Then $\langle X, X\rangle \in \mathcal{A}^{(\infty)}$.*

Proof: We first show that for each ordinal β, if $\langle V', W'\rangle \in \mathcal{A}^{(\beta)}$ and $V \subseteq V'$ and $W \subseteq W'$ are open, then $\langle V, W\rangle \in \mathcal{A}^{(\beta)}$. This clearly holds for $\beta = 0$, and if it holds for all ordinals less than a limit ordinal λ then it clearly holds for λ. So it only remains to prove it for successor ordinals. Assume it holds for β and we will show it holds for $\beta + 1$. Fix $\langle V', W'\rangle \in \mathcal{A}^{(\beta+1)}$ and opens $V \subseteq V'$ and $W \subseteq W'$. If $\langle V', W'\rangle \in \mathcal{A}^{(\beta)}$ then $\langle V, W\rangle \in \mathcal{A}^{(\beta)}$ by the induction hypothesis for β. Otherwise, there are two cases:

1. $V' = \bigcup \mathcal{S}$ for some $\mathcal{S} \subseteq \mathbf{O}(X)$ satisfying $(\forall U \in \mathcal{S})\, \langle U, W'\rangle \in \mathcal{A}^{(\beta)}$. Then by the induction hypothesis for β we have $\langle V \cap U, W\rangle \in \mathcal{A}^{(\beta)}$ for each $U \in \mathcal{S}$. By defining $\mathcal{S}' = \{U \cap V \mid U \in \mathcal{S}\}$ we have $\langle V, W\rangle = \langle \bigcup \mathcal{S}', W\rangle \in \mathcal{A}^{(\beta+1)}$.

2. If the first case does not hold, then $W' = \bigcup \mathcal{S}$ for some $\mathcal{S} \subseteq \mathbf{O}(X)$ satisfying $(\forall U \in \mathcal{S})\, \langle V', U\rangle \in \mathcal{A}^{(\beta)}$. Then an argument that is symmetric to the first case yields $\langle V, W\rangle \in \mathcal{A}^{(\beta+1)}$, which completes the inductive proof.

It follows that $\mathcal{A}^{(\infty)}$ is a lower set with respect to the pairwise ordering on $\mathbf{O}(X) \times \mathbf{O}(X)$. This implies that

$$\mathcal{H} = \{U \in \mathbf{O}(X) \mid \langle U, U\rangle \notin \mathcal{A}^{(\infty)}\}$$

is an upper subset of $\mathbf{O}(X)$. We next show that $\mathcal{H}$ is Scott-open. Fix any directed subset $\mathcal{D} \subseteq \mathbf{O}(X)$ with $\mathcal{D} \cap \mathcal{H} = \emptyset$. Fix any $U \in \mathcal{D}$. For any other $V \in \mathcal{D}$, since $\mathcal{D}$ is directed there is $W \in \mathcal{D}$ containing both U and V. The assumption $\mathcal{D} \cap \mathcal{H} = \emptyset$ implies $\langle W, W\rangle \in \mathcal{A}^{(\infty)}$, and since $\mathcal{A}^{(\infty)}$ is a lower set we obtain $\langle U, V\rangle \in \mathcal{A}^{(\infty)}$. Since $V \in \mathcal{D}$ was arbitrary, we have $\langle U, \bigcup \mathcal{D}\rangle \in \mathcal{A}^{(\infty+1)} = \mathcal{A}^{(\infty)}$, and since $U \in \mathcal{D}$ was arbitrary it follows that $\langle \bigcup \mathcal{D}, \bigcup \mathcal{D}\rangle \in \mathcal{A}^{(\infty+1)} = \mathcal{A}^{(\infty)}$. Therefore, $\bigcup \mathcal{D} \notin \mathcal{H}$, which completes the proof that $\mathcal{H}$ is Scott-open.

Now assume for a contradiction that $\langle X, X\rangle \notin \mathcal{A}^{(\infty)}$. Then $X \in \mathcal{H}$, and the consonance of X implies there is $K \in \mathbf{K}(X)$ such that $X \in \nabla K \subseteq \mathcal{H}$.

For each $x \in K$, we find some $\langle V_x, W_x\rangle \in \mathcal{A}^{(\infty)}$ satisfying $x \in V_x$ and $K \subseteq W_x$ as follows. For every other $y \in K$, there is $\langle V_{x,y}, W_{x,y}\rangle \in \mathcal{A}^{(\infty)}$ with $x \in V_{x,y}$ and $y \in W_{x,y}$ because of the definition of $\mathcal{A}^{(0)} \subseteq \mathcal{A}^{(\infty)}$ and the fact that $\mathcal{U}$ covers $X \times X$. From the compactness of K, there is finite $F \subseteq K$ such that $K \subseteq \bigcup_{y\in F} W_{x,y}$. Define $V_x = \bigcap_{y\in F} V_{x,y}$ and $W_x = \bigcup_{y\in F} W_{x,y}$. Then $x \in V_x$ and $K \subseteq W_x$. Furthermore, $\langle V_x, W_{x,y}\rangle \in \mathcal{A}^{(\infty)}$ holds for each $y \in F$ because $\mathcal{A}^{(\infty)}$ is a lower set, hence $\langle V_x, W_x\rangle = \langle V_x, \bigcup_{y\in F} W_x\rangle \in \mathcal{A}^{(\infty)}$.

Again using the compactness of K, there is finite $G \subseteq K$ such that $K \subseteq \bigcup_{x\in G} V_x$. Define $V = \bigcup_{x\in G} V_x$ and $W = \bigcap_{x\in G} W_x$. Then repeating the argument at the end of the previous paragraph yields $\langle V, W\rangle \in \mathcal{A}^{(\infty)}$. By defining $U = V \cap W$, we have $\langle U, U\rangle \in \mathcal{A}^{(\infty)}$ because $\mathcal{A}^{(\infty)}$ is a lower set.

Thus $K \subseteq U$ and $U \notin \mathcal{H}$, which contradicts $\nabla K \subseteq \mathcal{H}$. □

The core of the following proof is from T. Plewe's proof of a game theoretic characterization of spatial localic products (Theorem 1.2 in [13]).

Theorem 4 *If X is consonant, then Player II has a winning strategy in the game $\mathcal{G}(X)$.*

Proof: Player I initializes the game by choosing a non-empty cover $\mathcal{U}$ of $X \times X$ by open rectangles. Let β be the least ordinal such that $\langle X, X\rangle \in \mathcal{A}^{(\beta)}$, which exists by Lemma 3. Clearly, either $\beta = 0$ or else β is a successor ordinal. If $\beta = 0$, then $X \times X \in \mathcal{U}$, hence any valid play by Player II in the first round will be winning. So assume $\beta = \beta_0 + 1$ for some ordinal β_0, and set $V_0 = W_0 = X$.

In the $(i+1)$-th round $(i \geq 0)$, we can assume $\langle V_i, W_i\rangle \in \mathcal{A}^{(\beta_i+1)} \setminus \mathcal{A}^{(\beta_i)}$. Player II's strategy for this round will depend on the two possible ways in which $\langle V_i, W_i\rangle$ could have been added to $\mathcal{A}^{(\beta_i+1)}$:

1. The first possibility is $V_i = \bigcup \mathcal{S}$ for some $\mathcal{S} \subseteq \mathbf{O}(X)$ satisfying $(\forall V \in \mathcal{S})\, \langle V, W_i\rangle \in \mathcal{A}^{(\beta_i)}$. In this case, for any point $x_{i+1} \in V_i$ chosen by Player I, there is some $\langle V, W_i\rangle \in \mathcal{A}^{(\beta_i)}$ with $x_{i+1} \in V$. Player II plays $V_{i+1} = V$ for any such V, and in the second half of the round plays $W_{i+1} = W_i$ in response to any $y_{i+1} \in W_i$ played by Player I.

2. If the first case does not hold, then $W_i = \bigcup \mathcal{S}$ for some $\mathcal{S} \subseteq \mathbf{O}(X)$ satisfying $(\forall W \in \mathcal{S})\, \langle V_i, W\rangle \in \mathcal{A}^{(\beta_i)}$. In this case, Player II plays $V_{i+1} = V_i$ in response to any point $x_{i+1} \in V_i$ chosen by Player I. In the second half of the round, for any point $y_{i+1} \in W_i$ chosen by Player I, there is some $\langle V_i, W\rangle \in \mathcal{A}^{(\beta_i)}$ with $y_{i+1} \in W$. Player II chooses any such W and plays $W_{i+1} = W$.

At the end of round $(i+1)$, Player II has played some $\langle V_{i+1}, W_{i+1}\rangle \in \mathcal{A}^{(\beta_i)}$. Let β be the least ordinal such that $\langle V_{i+1}, W_{i+1}\rangle \in \mathcal{A}^{(\beta)}$. If $\beta = 0$, then Player II has won the game. Otherwise, $\beta = \beta_{i+1} + 1$ for some ordinal β_{i+1}, and we have $\langle V_{i+1}, W_{i+1}\rangle \in \mathcal{A}^{(\beta_{i+1}+1)} \setminus \mathcal{A}^{(\beta_{i+1})}$. The game then continues to the next round.

Since $\beta_{i+1} < \beta_i$, and every strictly decreasing sequence of ordinals is finite, the strategy defined above is winning for Player II. □

4 S_0 is dissonant

Using the game defined in the previous section, we can now prove that S_0 is dissonant by showing that Player I has a winning strategy in the game $\mathcal{G}(S_0)$. We first announced the game strategy used in the following proof at the CCC 2019 workshop in order to prove that the localic product $S_0 \times_{loc} S_0$ is not spatial [6].

Lemma 5 *S_0 is dissonant.*

Proof: Denote the length of $\sigma \in \mathbb{N}^{<\mathbb{N}}$ by $|\sigma|$. The empty string is denoted as ε, and the string consisting of m zeros is written $0^{(m)}$. The string obtained by appending $n \in \mathbb{N}$ to $\sigma \in \mathbb{N}^{<\mathbb{N}}$ is written $\sigma \diamond n$. We also write $\sigma \diamond \tau$ for the concatenation of strings. For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$, define

$$F_{\sigma,\tau} = \{s \diamond n \mid s \in \mathbb{N}^{<\mathbb{N}} \ \& \ s \preceq \sigma \ \& \ s \diamond n \not\preceq \sigma \ \& \ n \leq |\sigma| + |\tau|\},$$

$$U_{\sigma,\tau} = \{t \in \mathbb{N}^{<\mathbb{N}} \mid (\forall s \in F_{\sigma,\tau})\, s \not\preceq t\}.$$

Then $\sigma \in U_{\sigma,\tau}$, hence

$$\mathcal{U} = \{U_{\sigma,\tau} \times U_{\tau,\sigma} \mid \sigma, \tau \in \mathbb{N}^{<\mathbb{N}}\}$$

is an open cover of $S_0 \times S_0$. Observe that if $s \in U_{\sigma,\tau}$ and every element of s is less than or equal to $|\sigma| + |\tau|$ then $s \preceq \sigma$.

Also note that if $U \subseteq S_0$ is open and $\sigma \in U$, then there exist infinitely many $n \in \mathbb{N}$ such that every string that has $\sigma \diamond n$ as a prefix is also in U. This is because there is a basic open $U_0 \subseteq S_0$ (i.e., an open subset of the form $U_0 = \{\tau \in \mathbb{N}^{<\mathbb{N}} \mid (\forall s \in F)\, s \not\preceq \tau\}$ for some finite $F \subseteq S_0$) such that $\sigma \in U_0 \subseteq U$. One can then choose any $n \in \mathbb{N}$ that is larger than any element contained in any of the strings $s \in F$. Then for each $s \in F$, we have that $s \not\preceq \sigma \diamond n$ and $\sigma \diamond n \not\preceq s$, hence no extension of $\sigma \diamond n$ will have s as a prefix.

We now define a winning strategy for Player I in the game $\mathcal{G}(S_0)$. Player I initiates the game by choosing the open covering $\mathcal{U}$ of $S_0 \times S_0$. The game begins with round 1. For convenience, define $V_0 = W_0 = S_0$, and $x_0 = y_0 = \varepsilon$, and $m_0 = n_0 = 0$. Player I's strategy for the i-th round ($i \geq 1$) proceeds as follows:

- Player I chooses $m_i \in \mathbb{N}$ such that every sequence extending $y_{i-1} \diamond m_i$ is in W_{i-1}. Player I then plays $x_i = x_{i-1} \diamond n_{i-1} \diamond 0^{(m_i)}$.
- Player II must respond with an open subset $V_i \subseteq V_{i-1}$ containing x_i.
- Next, Player I finds distinct n_i and n'_i in $\mathbb{N}$ such that any sequence that has either $x_i \diamond n_i$ or $x_i \diamond n'_i$ as a prefix is in V_i. Player I plays $y_i = y_{i-1} \diamond m_i \diamond 0^{(n_i + n'_i)}$.
- Player II must respond with an open subset $W_i \subseteq W_{i-1}$ containing y_i.

The game then continues on to round $i + 1$.

We show that at the end of each round $i \geq 1$, the open rectangle $V_i \times W_i$ chosen by Player II is not a subset of any open rectangle in $\mathcal{U}$. Fix any $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ with $x_i \in U_{\sigma,\tau}$ and $y_i \in U_{\tau,\sigma}$. Since $y_{i-1} \preceq y_i$ we have $y_{i-1} \in U_{\tau,\sigma}$, and an inductive argument (keep reading) yields $y_{i-1} \preceq \tau$. Using the fact that $|y_{i-1}| \geq n_{i-1}$ it can be shown that every element occurring in x_i is less than or equal to $|y_{i-1}| \leq |\sigma| + |\tau|$, hence the assumption

$x_i \in U_{\sigma,\tau}$ and the observation at the end of the first paragraph of this proof implies $x_i \preceq \sigma$. Similarly, every element of y_i is less than or equal to $|x_i| \leq |\tau|+|\sigma|$, hence $y_i \in U_{\tau,\sigma}$ implies $y_i \preceq \tau$ (thereby completing the inductive argument). Either $x_i \diamond n_i \not\preceq \sigma$ or $x_i \diamond n_i' \not\preceq \sigma$, and $n_i, n_i' \leq |y_i| \leq |\sigma| + |\tau|$, thus $x_i \diamond n_i \notin U_{\sigma,\tau}$ or $x_i \diamond n_i' \notin U_{\sigma,\tau}$, but both $x_i \diamond n_i$ and $x_i \diamond n_i'$ are in V_i, so we conclude that $V_i \times W_i \not\subseteq U_{\sigma,\tau} \times U_{\tau,\sigma}$. Therefore, the above strategy is winning for Player I, hence S_0 is not consonant. ☐

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