Isomorphism between the stabilizers of finite sets of numbers in the R. Thompson group F

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1. Introduction

This article is an extended version of the talk given at the RIMS Meeting on Set Theoretic and Geometric Topology held in Kyoto University from June 5 to June 7, 2019.

Thompson's group F was discovered by Richard Thompson in 1965. It is usually defined as a group of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2. The group F has many fascinating properties which are surveyed in [1]. It is a finitely presented torsion free group and does not contain a nonabelian free subgroup. One of the most interesting open problems about this group is whether F is amenable. From a motivation to study the amenability of F, in [3, 4], D. Savchuk constructed Schreier graphs of subgroups H_U of F, which are the stabilizers of finite sets of real numbers $U \subset (0, 1)$. He proved that the Schreier graphs of H_U are amenable and also showed that if U consists of one number, then H_U is an infinite index maximal subgroup of F. In [2], G. Golan and M. Sapir studied the subgroups H_U for arbitrary finite U. Let $U = \{\alpha_1, \ldots, \alpha_n\} \subset [0, 1]$, where $\alpha_j < \alpha_{j+1}$, and define a type $\tau(U)$ as the word of length n in the alphabet $\{1, 2, 3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a dyadic rational, 2 if α_i is rational but not a dyadic rational, and 3 if α_i is irrational. They described the algebraic structure of H_U for finite $U \subset (0, 1)$ and also proved that H_U is finitely generated if and only if U does not contain irrational numbers. Moreover, it was proved that if $\tau(U) \equiv \tau(V)$ for finite sets U, $V \subset (0, 1)$, then H_U and H_V are isomorphic ($p \equiv q$ denotes letter-by-letter equality of words p, q) and that the converse of the statement does not hold. They stated [2, Subsection 4.1] that finding a necessary and sufficient condition for H_U and H_V to be isomorphic is still an open problem.

In this article we are going to summarize the problem and give a necessary and sufficient condition for H_U and H_V to be isomorphic.

2. Notation, Terminology, and Previous Results

Recall that the R. Thompson group F is the group of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2. The group operation is composition of homeomorphisms. Composition and evaluation of functions in F will be in word order. That is, for any two elements f, g in F and any $t \in [0,1]$, $tf = f(t)$ and $fg = g \circ f$. Basic facts about this group can be found in [1]. In particular, it is known that the commutator subgroup $[F,F]$ is simple, and also proved that F is generated by two homeomorphisms x_0 and x_1 given by

$$ax_0 = \begin{cases} 2a & \text{if} \quad 0 \le a \le \frac{1}{4}, \\ a + \frac{1}{4} & \text{if} \quad \frac{1}{4} \le a \le \frac{1}{2}, \\ \frac{a}{2} + \frac{1}{2} & \text{if} \quad \frac{1}{2} \le a \le 1, \end{cases} \qquad ax_1 = \begin{cases} a & \text{if} \quad 0 \le a \le \frac{1}{2}, \\ 2a - \frac{1}{2} & \text{if} \quad \frac{1}{2} \le a \le \frac{5}{8}, \\ a + \frac{1}{8} & \text{if} \quad \frac{5}{8} \le a \le \frac{3}{4}, \\ \frac{a}{2} + \frac{1}{2} & \text{if} \quad \frac{3}{4} \le a \le 1. \end{cases}$$

Note that our generators x_0 and x_1 are the inverses of the generators in [1].

The *support* of an element f in F is the subset $\mathrm{Supp}(f) = \{x \in [0,1] \,|\, xf \ne x\}$. We can easily see that $\mathrm{Supp}(f)$ is a finite union of disjoint open intervals. Each of these open intervals will be called an *orbital* of f. The complement set of the support of f in F is denoted by $\mathrm{Fix}(f)$.

We say that $f \in F$ has *closure of support in an interval* J if the closure in $[0,1]$ of $\mathrm{Supp}(f)$ is contained in J. Let F_J be the set of all functions from F with closure of support in J. Then F_J is a subgroup of F. We note that $F_{(0,1)}$ is the set of all functions from F with slope 1 both at 0^+ and at 1^-. It is proved [1] that the commutator subgroup $[F,F]$ of F is exactly the subgroup $F_{(0,1)}$. It is also known [2] that for any $a,b \in [0,1]$ with $a < b$, $F_{(a,b)}$ is isomorphic to $F_{(0,1)}$.

For any finite subset X of $(0,1)$, let H_X be the stabilizer of X in F. That is, $H_X = \{f \in F \,|\, xf = x \text{ for each } x \in X\}$. Any finite subset Y of $[0,1]$ is subdivided into three subsets: $Y_1 = Y \cap \mathbb{Z}[1/2]$, $Y_2 = Y \cap (\mathbb{Q} \setminus \mathbb{Z}[1/2])$, and $Y_3 = Y \cap (\mathbb{R} \setminus \mathbb{Q})$.

Let $Y = \{r_1, \ldots, r_n\} \subset [0,1]$ with $r_j < r_{j+1}$, $r_1, r_n \notin Y_2$ and

$$B_Y = F_{[r_1, r_n]} \cap H_{Y \setminus \{r_1, r_n\}}.$$

The following proposition is a slight generalization of [2, Theorem 3.2] and can be proved in much the same way as it.

PROPOSITION 1. *The group B_Y is isomorphic to a semidirect product*

$$B_Y \cong [F,F]^{n-1} \rtimes \mathbb{Z}^{2|(Y \setminus \{r_1, r_n\})_1| + |Y_2| + |\{r_1, r_n\}_1|}.$$

COROLLARY 1. *Let $U = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_j < \alpha_{j+1}$ and $\alpha_1, \alpha_n \notin U_2$, and $V = \{\beta_1, \ldots, \beta_m\}$, where $\beta_j < \beta_{j+1}$ and $\beta_1, \beta_m \notin V_2$. If the subgroups B_U*

and B_V are isomorphic, then $n = m$ and $2|(U \setminus \{\alpha_1, \alpha_n\})_1| + |U_2| + |\{\alpha_1, \alpha_n\}_1| = 2|(V \setminus \{\beta_1, \beta_m\})_1| + |V_2| + |\{\beta_1, \beta_m\}_1|$.

Let $U = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_j < \alpha_{j+1}$. We define a type $\tau(U)$ as the word of length n in the alphabet $\{1, 2, 3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a dyadic rational, 2 if α_i is rational but not a dyadic rational, and 3 if α_i is irrational.

3. Isomorphism between stabilizers of finite sets

Recall that every finite subset U of $(0, 1)$ is subdivided into three subsets $U = U_1 \sqcup U_2 \sqcup U_3$, where $U_1 = U \cap \mathbb{Z}[1/2]$, $U_2 = U \cap (\mathbb{Q} \setminus \mathbb{Z}[1/2])$, and $U_3 = U \cap (\mathbb{R} \setminus \mathbb{Q})$. Write $U_1 \cup U_3 = \{r_1, \ldots, r_n\}$, where $r_j < r_{j+1}$ and $|U_1 \cup U_3| = n$. Let $r_0 = 0$, $r_{n+1} = 1$, and $U_{2,k} = \{q \in U_2 \mid r_k < q < r_{k+1}\}$. Then $U_2 = \bigsqcup_{k=0}^{n} U_{2,k}$. Recall (see subsection 4.2 in [2] for details), that

$$H_U = B_{\{r_0, r_1\} \cup U_{2,0}} \times \cdots \times B_{\{r_n, r_{n+1}\} \cup U_{2,n}}.$$

For any word $w_1 w_2 \in \{11, 13, 33\}$ and $j \in \{0, \ldots, |U_2|\}$, let $\Lambda_{U, w_1 w_2, j} = \{i \in \{0, \ldots, n\} \mid \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_1 2^j w_2 \text{ or } \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_2 2^j w_1\}$.

THEOREM 1. *Let U and V be finite sets of numbers in $(0, 1)$. Then the following statements are equivalent.*

(1) H_U *and* H_V *are isomorphic.*
(2) $|U_2| = |V_2|$, *and* $|\Lambda_{U, w_1 w_2, j}| = |\Lambda_{V, w_1 w_2, j}|$ *for each* $w_1 w_2 \in \{11, 13, 33\}$ *and each* $j \in \{0, \ldots, |U_2|\}$.

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