Isometries on a Banach space of analytic functions on the open unit disk

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1 Introduction

Let $(N, \|\cdot\|_N)$ be a normed linear space over $\mathbb{R}$ or $\mathbb{C}$. A mapping T on $(N, \|\cdot\|_N)$ is an *isometry* if

$$\|T(f) - T(g)\|_N = \|f - g\|_N \qquad (\forall f, g \in N).$$

Here, we don't assume linearity of T. Let $\mathbb{D}$ be the open unit disc and $\mathbb{T}$ the unit circle in $\mathbb{C}$. We denote by $H(\mathbb{D})$ the complex linear space of all analytic functions on $\mathbb{D}$. Let H^p be the Hardy space defined by

$$H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \sup_{0<r<1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right]^{1/p} < \infty \right\} \qquad (1 \leq p < \infty),$$

$$H^\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Complex linear isometries on the Hardy spaces were characterized in 1960's.

Theorem (deLeeuw, Rudin and Wermer [1]). *1. Let T be a surjective, complex linear isometry on $(H^\infty, \|\cdot\|_\infty)$. Then there exist a constant $\alpha \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and a conformal map $\phi \colon \mathbb{D} \to \mathbb{D}$ such that*

$$T(f)(z) = \alpha f(\phi(z)) \qquad (\forall f \in H^\infty, z \in \mathbb{D}).$$

2. Let T be a surjective, complex linear isometry on $(H^1, \|\cdot\|_1)$. Then there exist a constant $\alpha \in \mathbb{T}$ and a conformal map $\phi \colon \mathbb{D} \to \mathbb{D}$ such that

$$T(f)(z) = \alpha \phi'(z) f(\phi(z)) \qquad (\forall f \in H^1, z \in \mathbb{D}).$$

In 1959, Nagasawa [8] gave the characterization of surjective complex linear isometry on uniform algebras. The characterization of isometries on H^∞ by deLeeuw, Rudin and Wermer is a special case of the result by Nagasawa.

Forelli [3] investigated complex linear, not necessarily surjective, isometries on H^p. Here, I will introduce the result of surjective case.

Theorem (Forelli, [3]). *Let p be a real number with $1 \leq p < \infty$ and $p \neq 2$, and let T be a surjective complex linear isometry on $(H^p, \|\cdot\|_p)$. There exist a constant $\alpha \in \mathbb{T}$ and a conformal map $\phi\colon \mathbb{D} \to \mathbb{D}$ such that*

$$T(f)(z) = \alpha(\phi'(z))^{1/p} f(\phi(z)) \qquad (\forall f \in H^p, z \in \mathbb{D}).$$

Novinger and Oberlin [9] considered Banach spaces of analytic functions

$$\mathcal{S}^p = \{f \in H(\mathbb{D}) : f' \in H^p\} \qquad (1 \leq p < \infty)$$

with the following norms:

$$\|f\|_\sigma = |f(0)| + \|f'\|_p, \quad \|f\|_\Sigma = \|f\|_\infty + \|f'\|_p \qquad (f \in \mathcal{S}^p).$$

Here, it should be mentioned that $\|f\|_\infty$ is well-defined; in fact, if a function $f \in H(\mathbb{D})$ satisfies $f' \in H^p$ for some $p, 1 \leq p$ then f is extended to a continuous function on the closed unit ball $\bar{\mathbb{D}}$ (see, for example [2, Theorem 3.11]). Novinger and Oberlin [9] characterized complex linear isometries on $\mathcal{S}^p$ without assuming surjectivity. For the sake of simplicity, I will show you a surjective case of their results.

Theorem (Novinger and Oberlin [9]). *Let p be a real number with $1 \leq p < \infty$ and $p \neq 2$.*

1. *If T is a surjective complex linear isometry on $(\mathcal{S}^p, \|\cdot\|_\sigma)$, then there exist a constant $c \in \mathbb{T}$ and a conformal map $\phi\colon \mathbb{D} \to \mathbb{D}$ such that*

$$T(f)(z) = cf(0) + \int_{[0,z]} (\phi'(\zeta))^{1/p} f'(\phi(\zeta))\, d\zeta \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}).$$

2. *If T is a surjective complex linear isometry on $(\mathcal{S}^p, \|\cdot\|_\Sigma)$, then there exist a constant $c \in \mathbb{T}$ and a conformal map $\phi\colon \mathbb{D} \to \mathbb{D}$ such that*

$$T(f)(z) = cf(\phi(z)) \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}).$$

Novinger and Oberlin excluded the case when $p = \infty$ in the above result. But $\mathcal{S}^\infty$ is well-defined, and I believe the characterization of isometries on $\mathcal{S}^\infty$ is important to the theory of analytic functions. The purpose of this note is to give an answer to the above problem.

2 Main results

We define $\mathcal{S}^\infty = \{f \in H(\mathbb{D}) : f' \in H^\infty\}$. As is mentioned above, if $f \in \mathcal{S}^\infty$, then it can be extended to a continuous function on $\bar{\mathbb{D}}$. Thus, $\|f'\|_\infty$ is well-defined. We consider the following two norms on $\mathcal{S}^\infty$:

$$\|f\|_\sigma = |f(0)| + \|f'\|_\infty, \qquad \|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty \qquad (f \in \mathcal{S}^\infty).$$

We see that $(\mathcal{S}^\infty, \|\cdot\|_\sigma)$ and $(\mathcal{S}^\infty, \|\cdot\|_\Sigma)$ are both Banach spaces. Noviger and Oberlin characterized complex linear isometries on $\mathcal{S}^p$ $(1 \le p < \infty)$ without assuming surjectivity. Here we investigate *surjective*, not necessarily linear, isometries on $\mathcal{S}^\infty$. The main results of this note is as follows.

Theorem 1. *A map T is a surjective isometry on $(\mathcal{S}^\infty, \|\cdot\|_\Sigma)$ if and only if there exist constants $c, \lambda \in \mathbb{T}$ such that*

$$T(f)(z) = T(0)(z) + cf(\lambda z) \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad \textit{or}$$
$$T(f)(z) = T(0)(z) + \overline{cf(\overline{\lambda z})} \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}).$$

Outline of proof. By the Mazur-Ulam theorem [5], the map $T_0 = T - T(0)$, which sends $f \in \mathcal{S}^\infty$ to $T(f) - T(0)$, is real linear. In addition, we see that T_0 is a surjective isometry. Let $\widehat{f'}$ be the Gelfand transform of $f' \in H^\infty$ and let ∂_{H^∞} be the Shilov boundary for H^∞. Then $\sup_{\zeta \in \mathbb{D}} |f'(\zeta)| = \sup_{z \in \partial_{H^\infty}} |\widehat{f'}(\zeta)|$ for $f \in \mathcal{S}^\infty$. We denote by $\hat{f}$ the unique continuous extension of $f \in \mathcal{S}^\infty$ to $\bar{\mathbb{D}}$. By the maximal modulus principle, $\sup_{z \in \mathbb{D}} |f(z)| = \sup_{z \in \mathbb{T}} |\hat{f}(z)|$ for $f \in \mathcal{S}^\infty$. Therefore

$$\|f\|_\Sigma = \sup_{z \in \mathbb{D}} |f(z)| + \sup_{\zeta \in \mathbb{D}} |f'(\zeta)| = \sup_{z \in \mathbb{T}} |\hat{f}(z)| + \sup_{\zeta \in \partial_{H^\infty}} |\widehat{f'}(\zeta)| = \sup_{(z,w,\zeta) \in \mathbb{T}^2 \times \partial_{H^\infty}} |\hat{f}(z) + w\widehat{f'}(\zeta)|.$$

We now define a map $U\colon \mathcal{S}^\infty \to C(\mathbb{T}^2 \times \partial_{H^\infty})$ by

$$U(f)(z, w, \zeta) = \hat{f}(z) + w\widehat{f'}(\zeta) \qquad (\forall f \in \mathcal{S}^\infty, (z, w, \zeta) \in \mathbb{T}^2 \times \partial_{H^\infty}).$$

Set $B = U(\mathcal{S}^\infty)$, and then U is a surjective complex linear isometry from $(\mathcal{S}^\infty, \|\cdot\|_\Sigma)$ onto $(B, \|\cdot\|_\infty)$.

$$\begin{array}{ccc} \mathcal{S}^\infty & \xrightarrow{T_0} & \mathcal{S}^\infty \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ B & \xrightarrow[V]{} & B \end{array}$$

We set $V = UT_0U^{-1}$. Then V is a surjective *real linear* isometry on $(B, \|\cdot\|_\infty)$.

By a modified arguments of [10, Proof of Lemma 3.1], we can prove that

$$V_*(\{\lambda\delta_x : \lambda \in \mathbb{T}, x \in \mathbb{T}^2 \times \partial_{H^\infty}\}) = \{\lambda\delta_x : \lambda \in \mathbb{T}, x \in \mathbb{T}^2 \times \partial_{H^\infty}\},$$

where $V_*\colon B^* \to B^*$ is a map defined by

$$V_*(\eta)(a) = \mathrm{Re}\,\eta(V(a)) - i\mathrm{Re}\,\eta(V(ia)) \qquad (\forall \eta \in B^*, a \in B),$$

and $\delta_x\colon B \to \mathbb{C}$ is a point evaluation functional with $\delta_x(a) = a(x)$ for $a \in B$. Using the form of V, we can describe T_0 with extra variables, say $w \in \mathbb{T}$ and $\zeta \in \partial_{H^\infty}$. By straightforward, but complicated arguments, we obtain the desired form of T. The reader may refer to [7] for the detail. $\square$

Theorem 2. *Let T be a surjective isometry on $(\mathcal{S}^\infty, \|\cdot\|_\sigma)$. Then there exist constants $c_0, c_1, \lambda \in \mathbb{T}$ and $a \in \mathbb{D}$ such that*

$$T(f)(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 f'\left(\lambda \frac{z-a}{1-\overline{a}\zeta}\right) d\zeta \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad \text{or}$$

$$T(f)(z) = T(0)(z) + c_0 \overline{f(0)} + \int_{[0,z]} c_1 f'\left(\lambda \frac{z-a}{1-\overline{a}\zeta}\right) d\zeta \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad \text{or}$$

$$T(f)(z) = T(0)(z) + c_0 f(0) + \int_{[0,z]} c_1 \overline{f'\left(\overline{\lambda \frac{z-a}{1-\overline{a}\zeta}}\right)} d\zeta \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}) \quad \text{or}$$

$$T(f)(z) = T(0)(z) + c_0 \overline{f(0)} + \int_{[0,z]} c_1 \overline{f'\left(\overline{\lambda \frac{z-a}{1-\overline{a}\zeta}}\right)} d\zeta \qquad (\forall f \in \mathcal{S}^p, z \in \mathbb{D}).$$

Conversely, if T is one of the above four, then it is a surjective isometry on $(\mathcal{S}^\infty, \|\cdot\|_\sigma)$.

Outline of proof. The idea of this proof is quite similar to that of Theorem 1. We need the characterization of surjective, real linear isometries on uniform algebras (see [4, 6]). □

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