An introduction to large-scale geometry of asymmetric spaces

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1 Introduction

Coarse geometry, also known as large-scale geometry, is the study of large-scale properties of spaces, ignoring their local, small-scale ones. The origin of large-scale geometry goes back to Milnor's problems and Gromov's ideas from geometric group theory and Mostow's rigidity theorem ([13]).

Intuitively, two spaces are considered equivalent in coarse geometry if they look alike for an observer whose point of view is getting further and further away. For example, every bounded space is indistinguishable from a one-point space. Another possible example is the pair given by the integer numbers $\mathbb{Z}$ and the real numbers $\mathbb{R}$. From a topological perspective, these equivalences seem to loose too much information of the spaces. In fact, "small holes" and "small discontinuities" of the spaces are ignored, and, for example, we can identify a discrete space, $\mathbb{Z}$, with connected one, $\mathbb{R}$. However, and somehow unexpectedly, this theory found applications in several branches of mathematics, for example in geometric group theory (following the work of Gromov on finitely generated groups endowed with their word metrics), in Novikov conjecture, and in coarse Baum-Connes conjecture. We refer to [14] for a comprehensive introduction to large-scale geometry of metric spaces, and to [9] for applications to geometric group theory.

We have said that coarse geometry is the study of those properties of spaces that are preserved for an observer whose point of view is getting further and further away. Let us now more precisely describe the equivalences involved.

A map $f\colon (X,d_X)\to(Y,d_Y)$ between metric spaces is said to be *bornologous* if, for every $R\geq 0$, there exists $S_R\geq 0$ such that $d_Y(f(x),f(y))\leq S_R$ if $d_X(x,y)\leq R$.

Definition 1.1. Let $f\colon (X,d_X)\to(Y,d_Y)$ be a map two metric spaces. Then f is called a *coarse equivalence* if there exists another map $g\colon Y\to X$ and a value $R\geq 0$ such that both f and g are bornologous and

$$\max\{\sup_{x\in X} d_X(g(f(x)),x), \sup_{y\in Y} d_Y(f(g(y)),y)\}\leq R.$$

If f is a coarse equivalence, then the spaces X and Y are called *coarsely equivalent.*

Let us consider some easy examples of coarse equivalences. Every bounded metric space (X,d) (i.e., there exists $R\in\mathbb{R}_{\geq 0}=\{x\in\mathbb{R}\mid x\geq 0\}$ such that $X\subseteq B_d(x,R)=$

$\{y \in X \mid d(x,y) \leq R\}$, for every $x \in X$) is coarsely equivalent to a one point space $\{*\}$ (just take any inclusion map $f\colon \{*\} \to X$ and the constant map $g\colon X \to \{*\}$). The metric spaces $\mathbb{Z}$ and $\mathbb{R}$, endowed with their canonical euclidean metrics are coarsely equivalent (we can take the inclusion map $i\colon \mathbb{Z} \to \mathbb{R}$ and the *floor map* $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{Z}$ such that, for every $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$).

In order to give more interesting examples of coarse equivalences, we need to give two important classes of metric spaces.

Example 1.2. Let $\Gamma = (V, E)$ be a non-directed connected graph. Then the set of vertices V can be endowed with the *path metric* d_Γ defined as follows: for every $x, y \in X$,

$$d_\Gamma(x,y) = \min\{n \in \mathbb{N} \mid \exists x_0 = x, x_1, \dots, x_n = y \in V : \forall i = 1, \dots, n, \{x_{i-1}, x_i\} \in E\}.$$

Since Γ is connected, $d_\Gamma\colon V \times V \to \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$. If we consider also non-connected non-directed graphs, we can extend the path metric by putting $d_\Gamma(x,y) = \infty$ if and only if the vertices x and y are in different connected components.

Example 1.3. Let G be a group. We say that G is *finitely generated* if there exists a finite subset Σ of G such that, for every $g \in G$, there exist $n \in \mathbb{N}$ and $\sigma_1, \dots, \sigma_n \in \Sigma$ which satisfy $g = \sigma_1 \cdots \sigma_n$ (i.e., $G = \langle \Sigma \rangle$). If a group G is finitely generated by a finite subset Σ, we assume without loss of generality that $\Sigma = \Sigma^{-1}$ and the identity e_G of G belongs to Σ. In fact, we can replace Σ with $\Sigma \cup \Sigma^{-1} \cup \{e_G\}$.

Let G be a group which is generated by the finite subset $e \in \Sigma = \Sigma^{-1}$. Let us define the *(left) word metric* d_Σ as follows: for every pair of elements $x, y \in G$,

$$d_\Sigma(x,y) = \min\{n \mid \exists \sigma_1, \dots, \sigma_n \in \Sigma : y = x\sigma_1 \cdots \sigma_n\}.$$

Note that d_Σ^λ is *left-invariant*, i.e., for every $x, y, z \in G$, $d_\Sigma(zx, zy) = d_\Sigma(x,y)$.

To every finitely generated group G and every finite generating set Σ, we can associate a non-directed graph $\mathrm{Cay}(G, \Sigma) = (G, E)$, called *Cayley graph of* G *associated to* Σ, where a pair $\{g, h\} \in G \times G$ belongs to E if and only if there exists $\sigma \in \Sigma$ such that $h = g\sigma$. Note that the map $id_G\colon (G, d_\Sigma) \to (G, d_{\mathrm{Cay}(G,\Sigma)})$ is an isometry.

A finitely generated group G can be endowed with several word metrics, in fact, they strongly depend on the finite generating set associated. However, from the large-scale point of view, they coincide as the following result shows.

Proposition 1.4. *Let G be a finitely generated group, and Σ and Δ be two symmetric finite generating subsets of G. Then the identity map $id_G\colon (G, d_\Sigma) \to (G, d_\Delta)$ is a coarse equivalence.*

Proposition 1.4 can be interpreted as follows: every finitely generated group has precisely one large-scale geometry. Finitely generated groups are a very important object in geometric group theory where the large-scale approach turned out to be very fruitful (see, for example, [8] and [9] for a wide discussion of the subject).

A classical generalisation of the notion of metric space is the one of uniform spaces. Uniform spaces have been widely studied since their introduction by the work of Weil and Tukey in the first half of the last century, and successfully applied in different areas. If

X is a set, every subset $U \subseteq X \times X$ is called an *entourage*. For every pair of entourages U, V, we define the *composite of U and V* as the entourage

$$U \circ V = \{(x,z) \mid \exists y \in X : (x,y) \in U, (y,z) \in V\},$$

and the *inverse of U* as $U^{-1} = \{(y,x) \mid (x,y) \in U\}$.

Definition 1.5 ([11]). A *uniform space* is a pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a *uniformity* over it, i.e., a family of subsets of $X \times X$ that satisfies the following properties:
(U1) $\mathcal{U}$ is a *filter* (i.e., a family closed under taking finite intersubsections and supersets);
(U2) for every $U \in \mathcal{U}$, $\Delta_X = \{(x,x) \mid x \in X\} \subseteq U$;
(U3) for every $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$;
(U4) for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

For instance, if (X,d) is a metric space, then, the family

$$\mathcal{U}_d = \{V \supseteq U_R \mid R \geq 0\}, \text{ where, for every } R > 0,\ U_R = \bigcup_{x \in X} \{x\} \times B_d(x,R), \tag{1}$$

is a uniformity over X, called *metric uniformity*. The metric uniformity captures the small-scale properties (e.g., the topological properties) of a metric space.

In order to generalise the large-scale properties of metric spaces, Roe introduced coarse spaces ([19]), as a counterpart of Weil's definition of uniform spaces via entourages, and Protasov and Banakh ([16]) defined balleans, generalising the ball structure of metric spaces. Furthermore, Dydak and Hoffland with large-scale structures ([5]) and Protasov with asymptotic proximities ([15]) independently developed the approach via coverings, as Tukey did for uniform spaces. As for the definition of coarse structures and coarse spaces, we refer to Definition 2.1. Coarse structures are also very useful to encode the large-scale properties of groups.

A very important *coarse invariant* (i.e., a cardinal associated to every metric space in such a way that two coarsely equivalent coarse spaces have the same cardinal associated to them) is the *asymptotic dimension* which was introduced by Gromov ([8]) as the large-scale counterpart of the classical Čech-Lebesgue covering dimension (see [6]). We refer to [1] for a comprehensive introduction of this notion. In [20], the authors introduce and study some notions of asymptotic dimension in the realm of quasi-coarse spaces, generalising the usual concept.

In mathematics some weakened version of metrics appeared. Let X be a set and $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a map such that $d(x,x) = 0$, for every $x \in X$. The map d is a *semi-positive-definite map*. Moreover d is a

- *pseudo-semi-metric* if, for every $x, y \in X$, $d(x,y) = d(y,x)$;
- *pseudo-quasi-metric* if, for every $x, y, z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$ (with the convention that $\infty + a = a + \infty = \infty$, for every $a \in \mathbb{R}$).

In particular, a pseudo-metric is both a pseudo-semi-metric and a pseudo-quasi-metric. Note that we allow that the distance between two points is infinite. Usually, the prefix "pseudo" is dropped if, for every $x, y \in X$, $d(x,y) = 0$ if and only if $x = y$. However, for the sake of simplicity, we call a pseudo-semi-metric a *semi-metric* and a pseudo-quasi-metric a *quasi-metric*. The pair (X,d) is a *semi-metric space* if d is a semi-metric, and a *quasi-metric space* if d is a quasi-metric.

Let us now give some examples of quasi-metric spaces in order to motivate our interest in those structures. The first example (Example 1.6) is due to Hausdorff himself, while Examples 1.8 and 1.9 are the asymmetric counterparts of Examples 1.2 and 1.3, respectively.

Example 1.6 ([10])**.** Let (X, d) be a metric space. On the power set $\mathcal{P}(X) = \{A \subseteq X \mid A \subseteq X\}$ of X we define a map $d_H^q \colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as follows: for every $Y, Z \subseteq X$,

$$d_H^q(Y, Z) = \inf\{R \geq 0 \mid Z \subseteq B_d(Y, R)\},$$

where $\inf \emptyset = \infty$. The map d_H^q is actually a quasi-metric, called *Hausdorff quasi-metric*.

Example 1.7. Let $(X, \geq)$ be a preordered set. Then the preorder $\geq$ induces a quasi-metric $d_{\geq}$ on X, called *preorder quasi-metric*, is defined as follows: for every $x, y \in X$,

$$d_{\geq}(x, y) = \begin{cases} 0 & \text{if } x \geq y, \\ \infty & \text{otherwise.} \end{cases}$$

Example 1.8. Let $\Gamma = (V, E)$ be a directed graph. Then the set of vertices V can be endowed with the *path quasi-metric* d_Γ defined as follows: for every $x, y \in X$,

$$d_\Gamma(x, y) = \min\{n \in \mathbb{N} \mid \exists x_0 = x, x_1, \ldots, x_n = y \in V : \forall i = 1, \ldots, n, (x_{i-1}, x_i) \in E\}.$$

Again $\min \emptyset = \infty$, and thus $d_\Gamma(x, y) = \infty$ if and only if there is no directed path from x to Y. It is easy to check that d_Γ is actually a quasi-metric.

Before introducing the next example, let us recall some algebraic definitions. A *magma* is a pair $(M, \cdot)$, where M is a set and $\cdot \colon M \times M \to M$ is a map. A magma $(M, \cdot)$ is called *unitary* if there exists a *neutral element* $e \in M$ such that $g \cdot e = e \cdot g = g$, for every $g \in M$. A unitary magma is a *monoid*, if $\cdot$ is associative.

Example 1.9. Let M be a monoid. We say that M is *finitely generated* if there exists a finite subset Σ of M such that, for every $g \in M$ there exist $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_n \in \Sigma$ which satisfy $g = \sigma_1 \cdots \sigma_n$.

Let M be a monoid which is finitely generated by Σ. Let us define the *left word quasi-metric* d_Σ^λ as follows: for every pair of elements $x, y \in M$,

$$d_\Sigma^\lambda(x, y) = \min\{n \mid \exists \sigma_1, \ldots, \sigma_n \in \Sigma : y = x\sigma_1 \cdots \sigma_n\}$$

(we denote $\min \emptyset = \infty$). The map $d_\Sigma^\lambda \colon M \times M \to \mathbb{N} \cup \{\infty\}$ is actually a quasi-metric. Similarly, we define the *right word quasi-metric* d_Σ^ρ on M, as follows: for every $x, y \in M$,

$$d_\Sigma^\rho(x, y) = \min\{n \mid \exists \sigma_1, \ldots, \sigma_n \in \Sigma : y = \sigma_1 \cdots \sigma_n x\}.$$

Moreover, note that d_Σ^λ (d_Σ^ρ) is *left-non-expanding*, i.e., for every $x, y, z \in M$, $d_\Sigma^\lambda(zx, zy) \leq d_\Sigma^\lambda(x, y)$ (*right-non-expanding*, respectively, i.e., for every $x, y, z \in M$, $d_\Sigma^\rho(xz, yz) \leq d_\Sigma^\rho(x, y)$).

It is possible to extend the notion of Cayley graph, which is a useful tool to represent a finitely generated group, in the framework of finitely generated monoids. Let M be a monoid and $\Sigma \subseteq M$ a finite subset which generates M. Then the (*left*) *Cayley graph of M associated to* Σ is the directed graph $\mathrm{Cay}^\lambda(M,\Sigma) = (M,E)$, where $(x,y) \in E$ if and only if there exists $\sigma \in \Sigma$ such that $y = x\sigma$ or, equivalently, $d^\lambda_\Sigma(x,y) = 1$. Similarly $\mathrm{Cay}^\rho(M,\Sigma)$, the (*right*) *Cayley graph*, can be constructed. Also in this case, the maps $id_M\colon (M,d^\lambda_\Sigma) \to (M, d_{\mathrm{Cay}^\lambda(M,\Sigma)})$ and $id_M\colon (M,d^\rho_\Sigma) \to (M, d_{\mathrm{Cay}^\rho(M,\Sigma)})$ are isometries.

We refer to [21] for a general introduction to the subject of quasi-metric spaces. Quasi-metrics are innerly non symmetric, so, if we consider the family $\mathcal{U}_d$ as in (1), then (U3) may not be satisfied.

In order to fill the gap, quasi-uniform spaces were introduced: a *quasi-uniform space* is a pair $(X,\mathcal{U})$, where $\mathcal{U}$ is a *quasi-uniformity* over the set X, i.e. a family of entourages that satisfies (U1), (U2) and (U4). There is a wide literature investigating those structures and also important applications to computer science were discovered (see the monograph [7] and the survey [12] for a wide-range introduction and a broad bibliography). Similarly, a *semi-uniform space* is a pair $(X,\mathcal{U})$, where $\mathcal{U}$ is a *semi-uniformity* over the set X, i.e., a family of entourages that satisfies (U1)–(U3) (see, for example, [2]).

Following the paper [22], we introduce large-scale counterparts of quasi-uniform spaces and semi-uniform spaces, respectively, in order to generalise coarse spaces. In particular, we define quasi-coarse spaces and semi-coarse spaces (Definition 2.1). Moreover, in order to provide a more comprehensive introduction to these new objects, we consider also entourage spaces, which are structures that generalise both quasi-coarse spaces and semi-coarse spaces. First of all, scratching the surface of this topic, we focus on adapting basic notions of coarse geometry (e.g., morphisms, as bornologous maps, connectedness, boundedness) to this more general setting. Moreover, we present a different characterisation of those structures by using ball structures ([16]). We motivate our interest in quasi-coarse spaces and semi-coarse spaces by providing a wide list of examples in which those structures naturally appear. Most of them are extensions of some classical examples of coarse spaces. In particular, we prove that also every finitely generated monoid can be endowed with precisely just two word quasi-metrics up to asymorphism (Proposition 4.5), which coincide if the monoid is abelian. This result is a generalisation of the classical situation involving finitely generated groups and their word metrics (Proposition 1.4).

Furthermore, we provide a generalisation of the notion of coarse equivalence between spaces in the realm of asymmetric objects, namely, the Sym-coarse equivalence. Using this equivalence, we could provide important characterisations of classes of quasi-coarse spaces: metric entourage spaces induced by quasi-metrics and graphic quasi-coarse spaces, giving an answer to a problem posed by Protasov and Banakh ([16, Problem 9.4]).

Let us now describe the structure of the paper. In §2 we introduce the basic notions and properties of entourage, semi-coarse, quasi-coarse and coarse spaces, and the morphisms between them (§2.1). Then we alternatively describe them in terms of ball structures in §3. Examples of those structures are provided and studied in §4: relation entourage structures (§4.1), graphic quasi-coarse structures (§4.2), entourage hyperstructures and semi-coarse hyperstructures (§4.3), and quasi-coarse structures on finitely generated monoids (§4.4). The notion of Sym-coarse equivalence is introduced in §5, and then applied in §6 in order to characterise some special classes of quasi-coarse spaces.

2 Coarse spaces and their generalisations

Definition 2.1. Let X be a set. A family $\mathcal{E}\subseteq\mathcal{P}(X\times X)$ is an *entourage structure over* X if it is an ideal on $X\times X$ that contains the diagonal Δ_X. Moreover, an entourage structure $\mathcal{E}$ over X is

- a *semi-coarse structure* if $E^{-1}\in\mathcal{E}$, for every $E\in\mathcal{E}$;
- a *quasi-coarse structure* if $E\circ F\in\mathcal{E}$, for every $E,F\in\mathcal{E}$;
- a *coarse structure* if it is both a semi-coarse and a quasi-coarse structure.

The pair $(X,\mathcal{E})$ is an *entourage space* (a *semi-coarse space*, a *quasi-coarse space*, a *coarse space*) if $\mathcal{E}$ is an entourage structure (a semi-coarse structure, a quasi-coarse structure, a coarse structure, respectively) over X.

If $\mathcal{E}$ is an entourage structure on a set X, then also $\mathcal{E}^{-1}=\{E^{-1}\mid E\in\mathcal{E}\}$ is an entourage structure. Of course, $\mathcal{E}=\mathcal{E}^{-1}$ if and only if $\mathcal{E}$ is a semi-coarse structure. Moreover, if $\mathcal{E}$ is a quasi-coarse structure, then $\mathcal{E}^{-1}$ is a quasi-coarse structure.

Let $(X,\mathcal{E})$ be an entourage space and Y be a subset of X. Then Y can be endowed with the *subspace entourage structure* $\mathcal{E}|_Y=\{E\cap(Y\times Y)\mid E\in\mathcal{E}\}$, and $(Y,\mathcal{E}|_Y)$ is called an *entourage subspace of* $(X,\mathcal{E})$. If $\mathcal{E}$ is a quasi-coarse structure (semi-coarse structure), then $\mathcal{E}|_Y$ is a quasi-coarse structure (semi-coarse structure, respectively).

If X is a set, a family $\mathcal{B}$ of subsets of $X\times X$ such that $\mathcal{E}=\mathfrak{cl}(\mathcal{B})$ is an entourage structure (semi-coarse structure, quasi-coarse structure, coarse structure, respectively) is a *base of the entourage structure* (*base of the semi-coarse structure*, *base of the quasi-coarse structure*, *base of the coarse structure*, respectively) $\mathcal{E}$.

Let us now give some example of these structures.

Example 2.2. (a) Every set X can be endowed with two entourage structures which are actually coarse structures: the *discrete coarse structure* $\mathcal{E}_{dis}=\mathfrak{cl}(\{\{\Delta_X\}\})$, and the *trivial* (or *indiscrete*) *coarse structure* $\mathcal{E}_{triv}=\mathcal{P}(X\times X)$. Moreover, the discrete and the trivial coarse structures coincide if the set is a singleton.

(b) A leading example of entourage structures is the metric entourage structure. Let (X,d) be a set endowed with an extended semi-positive-definite map d. We define the following entourage structure:

$$\mathcal{E}_d=\mathfrak{cl}(\{E_R\subseteq X\times X\mid R\geq 0\}),\ \text{where, for every } R\geq 0,\ E_R=\bigcup_{x\in X}(\{x\}\times B_d(x,R)). \tag{2}$$

Even though it is not precise, for the sake of simplicity, we call $\mathcal{E}_d$ a *metric entourage structure*. If d is a semi-metric, then $\mathcal{E}_d$ is a semi-coarse structure, while, if d is a quasi-metric, then $\mathcal{E}_d$ is a quasi-coarse structure. There are non-symmetric quasi-metrics and semi-metrics that do not satisfy the triangular inequality which induce coarse structures. For example, consider the quasi-metric d_1 and the semi-metric d_2 on $\mathbb{N}$ defined as follows: for every two points $m,n\in\mathbb{N}$,

$$d_1(m,n)=\max\{|m-n|-1,0\},\quad\text{and}\quad d_2(m,n)=\begin{cases}n-m & \text{if } m\leq n,\\ 2(m-n) & \text{otherwise.}\end{cases} \tag{3}$$

Although d_1 does not satisfy the triangular inequality, and d_2 is not symmetric, both $\mathcal{E}_{d_1}$ and $\mathcal{E}_{d_2}$ coincide with the metric coarse structure induced by the usual metric, and so they are coarse structures.
In the sequel, for the sake of simplicity and for consistency with the previous literature, if d is a metric, we call $\mathcal{E}_d$ a *metric coarse structure*.

More examples of entourage spaces will be given in §4.

Remark 2.3. While uniformities capture the small-scale properties of spaces, coarse structures encode their large-scale behaviour. In order to clarify this idea, let us consider the following constructions. Let (X, d) be a metric space, and let us derived two more metrics from d: for every $x, y \in X$,

$$d_1(x,y) = \min\{d(x,y), 1\}, \quad \text{and} \quad d_2(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{d(x,y), 1\} & \text{otherwise.} \end{cases}$$

From the large-scale point of view, d_1 loses a lot of information, in fact $\mathcal{E}_{d_1} = \mathcal{E}_{triv}$, while it keeps all the important features from the small-scale point of view, and, in fact, $\mathcal{U}_d = \mathcal{U}_{d_1}$. Conversely, the metric space (X, d_2) is discrete and $\mathcal{U}_d \neq \mathcal{U}_{d_2}$, but $\mathcal{E}_d = \mathcal{E}_{d_2}$.

An entourage space $(X, \mathcal{E})$ satisfies the property C_4 (or X is *strongly connected*) if $\bigcup \mathcal{E} = X \times X$.

Example 2.4. One may ask whether there are quasi-coarse spaces that satisfy C_4, but they are not semi-coarse spaces.

Let (X, d) be a metric space and let $h \colon X \to \mathbb{R}$ be an arbitrary function. Then the function $d_h \colon X \to \mathbb{R}_{\geq 0}$, defined by the law

$$d_h(x,y) = \begin{cases} d(x,y) + h(y) - h(x) & \text{if } h(y) - h(x) \geq 0, \\ d(x,y) & \text{otherwise,} \end{cases}$$

for every $x, y \in X$, is a quasi-metric.

Let now $X = \mathbb{Z}$, d be the usual euclidean metric, and $h(x) = x^3$. Then $(\mathbb{Z}, \mathcal{E}_{d_h})$ is a quasi-coarse space, since d_h is a quasi-metric, and it is C_4. However, it is not a coarse space. In fact, for every $R \geq 0$ and every $z \in \mathbb{R}$, $d_h(z + R, z) = R$, while $d_h(z, z + R) = R(1 + 3z^2 + 3zR + R^2)$, and the latter strongly depends on the point z. Hence, even though $\{(z + R, z) \mid z \in \mathbb{Z}\} \subseteq E_R \in \mathcal{E}_d$, there exists no $S \geq 0$ such that $\{(z, z + R) \mid z \in \mathbb{R}\} \subseteq E_S$.

In Example 2.2 we introduced metric entourage structures. We now want to characterise those structures. If $(X, \mathcal{E})$ is an entourage structure, its *cofinality* is $\operatorname{cf} \mathcal{E} = \inf\{|\mathcal{B}| \mid \mathfrak{cl}(\mathcal{B}) = \mathcal{E}\}$.

Proposition 2.5. *Let $(X, \mathcal{E})$ be an entourage space.*

(a) There exists an extended semi-positive-definite map d on X such that $\mathcal{E} = \mathcal{E}_d$ if and only if $\operatorname{cf} \mathcal{E} \leq \omega$.

(b) Suppose that $\mathcal{E}$ is a semi-coarse structure. Then there exists a semi-metric d on X such that $\mathcal{E} = \mathcal{E}_d$ if and only if $\operatorname{cf} \mathcal{E} \leq \omega$.

Proof. First of all, the "only if" implications in both items (a) and (b) are trivial since he family $\{E_n \mid n \in \mathbb{N}\}$, in the notation of (2), is a base of $\mathcal{E}_d$.

(a,$\leftarrow$) Let $\{F_n \mid n \in \mathbb{N}\}$ be a countable base of $\mathcal{E}$, and, without loss of generality, we can ask that $F_0 = \Delta_X$ and $F_n \subseteq F_{n+1}$, for every $n \in \mathbb{N}$. Then define a map $d\colon X \times X \to \mathbb{N}$ as follows: for every $x, y \in X$,

$$d(x,y) = \begin{cases} \min\{n \mid y \in F_n[x]\} & \text{if it exists,} \\ \infty & \text{otherwise.} \end{cases} \tag{4}$$

It is easy to check that d satisfies the required properties.

(b,$\leftarrow$) Suppose that $\mathcal{E}$ is a semi-coarse structure with $\operatorname{cf}\mathcal{E} \leq \omega$. Then we can choose a base $\{F_n \mid n \in \mathbb{N}\}$ as in item (a) with the further property that $F_n = F_n^{-1}$, for every $n \in \mathbb{N}$. Then the map d defined as in (4) satisfies the desired properties. $\square$

The maps d in Proposition 2.5 do not assume value ∞ if and only if $(X, \mathcal{E})$ is C_4.

The case where the entourage space is a quasi-coarse space (or a coarse space, in particular, which is a classical result) will be discussed in §6.

2.1 Morphisms between entourage spaces

Let $f\colon X \to Y$ is a map between sets. If $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets of X and Y, respectively, we denote by $f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\}$ and $f^{-1}(\mathcal{B}) = \{f^{-1}(B) \mid B \in \mathcal{B}\}$. Moreover, denote by $f \times f\colon X \times X \to Y \times Y$ the map defined by the law $(f \times f)(x, y) = (f(x), f(y))$, for every $(x, y) \in X \times X$.

Definition 2.6. A map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between entourage spaces is said to be

- *bornologous* (*uniformly boundedness preserving*, *coarsely uniform*) if $(f \times f)(\mathcal{E}_X) \subseteq \mathcal{E}_Y$;
- *uniformly weakly boundedness copreserving* if, for every $E \in \mathcal{E}_Y$, there exists $F \in \mathcal{E}_X$ such that $(f \times f)(F) = E \cap (f(X) \times f(X))$;
- *uniformly boundedness copreserving* if, for every $E \in \mathcal{E}_Y$, there exists $F \in \mathcal{E}_X$ such that, for every $x \in X$, $E[f(x)] \cap f(X) \subseteq f(F[x])$;
- *effectively proper* (or *uniformly proper*) if $(f \times f)^{-1}(\mathcal{E}_Y) \subseteq \mathcal{E}_X$;
- a *coarse embedding* if it is both bornologous and effectively proper;
- an *asymorphism* if it is bijective and both f and f^{-1} are bornologous.

Note that all the properties introduced in Definition 2.6 can be checked just for all the entourages that belong to some base of the entourage structures.

We can provide first trivial examples of the properties enlisted in Definition 2.6.

Example 2.7. Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between two entourage spaces. Then the following properties trivially hold:

(a) if $\mathcal{E}_X$ is the discrete coarse structure, then f is bornologous;
(b) if $\mathcal{E}_Y$ is the discrete coarse structure, then f is uniformly boundedness copreserving;
(c) if $\mathcal{E}_X$ is the trivial coarse structure, then f is effectively proper;
(d) if $\mathcal{E}_Y$ is the trivial coarse structure, then f is bornologous.

It is easy to check that composites of bornologous maps are bornologous. Moreover, we have the following result.

Proposition 2.8. *Let* $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ *be a map between entourage spaces. Then:*
(a) if f *is effectively proper, then* f *is uniformly boundedness copreserving;*
(b) if f *is uniformly boundedness copreserving, then* f *is uniformly weakly boundedness copreserving.*

Proof. (a) Suppose that f is effectively proper and let $E\in\mathcal{E}_Y$. Then, for every $x\in X$, $E[f(x)]\cap f(X)\subseteq f((f\times f)^{-1}(E)[x])$. In fact, for every $y\in X$ such that $(f(x),f(y))\in E$, $(x,y)\in(f\times f)^{-1}(E)$ and so $f(y)\in f((f\times f)^{-1}(E)[x])$.

(b) Suppose now that f is uniformly boundedness copreserving and let $E\in\mathcal{E}_Y$. Let $F\in\mathcal{E}_X$ be an entourage such that, for every $x\in X$, $E[f(x)]\cap f(X)\subseteq f(F[x])$. We claim that $E\cap(f(X)\times f(X))\subseteq(f\times f)(F)$. Let $(u,v)\in E\cap(f(X)\times f(X))$. There exists $z\in f^{-1}(u)$, and so $v\in E[f(z)]\cap f(X)$, which implies that there exists $w\in F[z]\cap f^{-1}(v)$. Finally, note that $(z,w)\in F$ and $(u,v)=(f(z),f(w))\in(f\times f)(F)$. ◻

If f is injective, then both implications of Proposition 2.8 can be easily reverted. Proposition 2.9 gives another condition that implies their reversibility.

Note that a map $f\colon (X,\mathcal{E}_X)\to Y$ from an entourage space to a set has uniformly bounded fibres if and only if $R_f=\{(x,y)\in X\times X\mid f(x)=f(y)\}\in\mathcal{E}_X$. We call such a map *large-scale injective*.

Proposition 2.9. *Let* $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ *be a map between entourage spaces. If* f *is effectively proper, then* f *is large-scale injective. Moreover, if* $\mathcal{E}_X$ *is a quasi-coarse structure, then the following properties are equivalent:*
(a) f *is large-scale injective and it is uniformly weakly boundedness copreserving;*
(b) f *is large-scale injective and it is uniformly boundedness copreserving;*
(c) f *is effectively proper.*

Proof. The first statement can be easily proved: since $\Delta_Y\in\mathcal{E}_Y$, then $R_f=(f\times f)^{-1}(\Delta_Y)\in\mathcal{E}_X$.

In view of Proposition 2.8, we just need to show the implication (a)→(c). Suppose now that f is uniformly weakly boundedness copreserving and $R_f\in\mathcal{E}_X$. Let $E\in\mathcal{E}_Y$ and (x,y) be an arbitrary pair in $(f\times f)^{-1}(E)$. Let $F\in\mathcal{E}_X$ such that $(f\times f)(F)=E\cap(f(X)\times f(X))$. Then there exists $(z,w)\in F$ such that $(f(x),f(y))=(f(z),f(w))$ and thus

$$(x,y)=(x,z)\circ(z,w)\circ(w,y)\in R_f\circ F\circ R_f\in\mathcal{E}_X. \qquad \square$$

Trivially, for a bijective map $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ between entourage spaces the following properties are equivalent:
- f is an asymorphism;
- f is bornologous and uniformly weakly boundedness copreserving;
- f is bornologous and uniformly boundedness copreserving;
- f is bornologous and effectively proper.

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two asymorphic entourage spaces. Then $\mathcal{E}_X$ is a semi-coarse structure (quasi-coarse structure) if and only if $\mathcal{E}_Y$ is a semi-coarse structure (quasi-coarse structure, respectively). For the proof of this fact, we address to [16], where the authors used the equivalent approach through ball structures (see §3 for the introduction of these structures).

Furthermore, if X and Y are two asymorphic entourage spaces, then X satisfies C_4 if and only if Y satisfies C_4.

3 Approach via ball structures

Let $(X, \mathcal{E})$ be an entourage space. Then we can associate to $\mathcal{E}$ a triple $\mathfrak{B}_\mathcal{E} = (X, P_\mathcal{E}, B_\mathcal{E})$, where $P_\mathcal{E} = \{E \in \mathcal{E} \mid \Delta_X \subseteq E\}$ and $B_\mathcal{E}(x, E) = E[x]$, for every $x \in X$ and every $E \in P$. It is an example of ball structure.

Definition 3.1. ([16, 18]) A *ball structure* is a triple $\mathfrak{B} = (X, P, B)$ where X and P are sets, $P \neq \emptyset$, and $B\colon X \times P \to \mathcal{P}(X)$ is a map, such that $x \in B(x, r)$ for every $x \in X$ and every $r \in P$. The set X is called *support of the ball structure*, P – *set of radii of a ball structure*, and $B(x, r)$ – *ball of center x and radius r*. In case $X = \emptyset$, the map B is the empty map.

The terminology and the intuition come from the metric setting: if (X, d) is a metric space, then $\mathfrak{B}_d = (X, \mathbb{R}_{\geq 0}, B_d)$ is a ball structure, called *metric ballean*.

For a ball structure (X, P, B), $x \in X$, $r \in P$ and a subset A of X, one puts

$$B^*(x, r) = \{y \in X \mid x \in B(y, r)\} \quad \text{and} \quad B(A, r) = \bigcup\{B(x, r) \mid x \in A\}.$$

A ball structure $\mathfrak{B} = (X, P, B)$ is said to be:

- *weakly upper multiplicative* if, for every pair of radii $r, s \in P$ there exists $t \in P$ such that $B(x, r) \cup B(x, s) \subseteq B(x, t)$, for every $x \in X$;
- *upper multiplicative* if, for every pair of radii $r, s \in P$ there exists $t \in P$ such that $B(B(x, r), s) \subseteq B(x, t)$, for every $x \in X$;
- *upper symmetric* if, for every pair of radii $r, s \in P$ there exist $r', s' \in P$ such that $B^*(x, r) \subseteq B(x, r')$ and $B(x, s) \subseteq B^*(x, s')$, for every $x \in X$.

It is trivial that upper multiplicativity implies weak upper multiplicativity since every ball contains its center.

Definition 3.2. A ball structure is

- a *semi-ballean* if it is weakly upper multiplicative and upper symmetric;
- a *quasi-ballean* if it is upper multiplicative;
- a *ballean* ([16]) if it is both a semi-ballean and a quasi-ballean.

For every entourage space $(X, \mathcal{E})$, $\mathfrak{B}_\mathcal{E}$ is indeed a weakly upper multiplicative ball structure. Moreover, if $\mathcal{E}$ is a semi-coarse structure, then $\mathfrak{B}_\mathcal{E}$ is a semi-ballean, while, if $\mathcal{E}$ is a quasi-coarse structure, then $\mathfrak{B}_\mathcal{E}$ is a quasi-ballean.

We have seen how we construct ball structures from entourage structures. Let us now discuss the opposite construction. Let $\mathfrak{B} = (X, P, B)$ be a weakly upper multiplicative ball structure. Then we can define an associated entourage structure $\mathcal{E}_{\mathfrak{B}}$ of X as follows: for every $r \in P$,

$$E_r = \bigcup_{x \in X} (\{x\} \times B(x, r)),$$

and the family $\{E_r \mid r \in P\}$ is a base for the entourage structure $\mathcal{E}_{\mathfrak{B}}$. Moreover,

- if $\mathfrak{B}$ is a semi-ballean, then $\mathcal{E}_{\mathfrak{B}}$ is a semi-coarse structure;
- if $\mathfrak{B}$ is a quasi-ballean, then $\mathcal{E}_{\mathfrak{B}}$ is a quasi-coarse structure;
- if $\mathfrak{B}$ is a ballean, then $\mathcal{E}_{\mathfrak{B}}$ is a coarse structure.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two weakly upper multiplicative ball structures on the same support X. Then we identify those two ball structures and we write $\mathfrak{B} = \mathfrak{B}'$, if $\mathcal{E}_{\mathfrak{B}} = \mathcal{E}_{\mathfrak{B}'}$. We soon give a characterization of the equality between ball structures. Hence, for every entourage space $(X, \mathcal{E})$ and every weakly upper multiplicative ball structure $\mathfrak{B}$ on X,

$$\mathcal{E}_{\mathfrak{B}_{\mathcal{E}}} = \mathcal{E} \quad \text{and} \quad \mathfrak{B}_{\mathcal{E}_{\mathfrak{B}}} = \mathfrak{B}.$$

The equivalence between coarse structures and balleans have been widely discussed for example in [18, 4].

If $\mathfrak{B}$ and $\mathfrak{B}'$ are two ball structure on a set X, then $\mathfrak{B}$ is *finer than* $\mathfrak{B}'$ and we write $\mathfrak{B} \prec \mathfrak{B}'$ if $id_X \colon (\mathcal{E}_{\mathfrak{B}}) \to (\mathcal{E}_{\mathfrak{B}'})$ is bornologous. Moreover,

$$\mathfrak{B} = \mathfrak{B}' \text{ if and only if } \mathfrak{B} \prec \mathfrak{B}' \text{ and } \mathfrak{B}' \prec \mathfrak{B}. \tag{5}$$

4 Examples of entourage spaces

In this subsection we enlist some examples of entourage spaces.

4.1 Relation entourage structures and para-bornologies

Let $\mathscr{R}$ be a reflexive relation over a set X. In other words, $\mathscr{R} \subseteq X \times X$ is an entourage containing the diagonal Δ_X. Then we can canonically define an entourage structure $\mathcal{E}_{\mathscr{R}} = \mathfrak{cl}(\{\mathscr{R}\})$, which is called *relation entourage structure*. Moreover, $\mathscr{R}$ is symmetric if and only if $\mathcal{E}_{\mathscr{R}}$ is a semi-coarse structure, while $\mathscr{R}$ is transitive if and only if $\mathcal{E}_{\mathscr{R}}$ is a quasi-coarse structure. Furthermore, note that, $(\mathcal{E}_{\mathscr{R}})^{-1} = \mathcal{E}_{\mathscr{R}^{-1}}$, where $\mathscr{R}^{-1}$ denotes the inverse of $\mathscr{R}$ as an entourage.

Remark 4.1. Note that, if $(X, \geq)$ is a preordered set and $d_{\geq}$ is defined as in Example 1.7, then $\mathcal{E}_{d_{\geq}} = \mathcal{E}_{\geq}$.

Another entourage structure that can be defined from a reflexive relation $\mathscr{R}$ on a set X is the following: $\mathcal{E}_{\mathscr{R}}^{fin} = \mathfrak{cl}([\mathscr{R}]^{<\omega} \cup \{\Delta_X\})$.

It is easy to verify the following result.

Proposition 4.2. *Let $f \colon (X, \mathscr{R}_X) \to (Y, \mathscr{R}_Y)$ be a map between sets endowed with reflexive relations. Then the following properties are equivalent:*

(a) f preserves the relation *(i.e., for every $x, y \in X$, $f(x)\mathscr{R}_Y f(y)$ provided that $x\mathscr{R}_X y$);*
(b) $f\colon (X, \mathcal{E}_{\mathscr{R}_X}) \to (Y, \mathcal{E}_{\mathscr{R}_Y})$ *is bornologous;*
(c) $f\colon (X, (\mathcal{E}_{\mathscr{R}_X})^{-1}) \to (Y, (\mathcal{E}_{\mathscr{R}_Y})^{-1})$ *is bornologous;*
(d) $f\colon (X, \mathcal{E}^{fin}_{\mathscr{R}_X}) \to (Y, \mathcal{E}^{fin}_{\mathscr{R}_Y})$ *is bornologous;*
(e) $f\colon (X, (\mathcal{E}^{fin}_{\mathscr{R}_X})^{-1}) \to (Y, (\mathcal{E}^{fin}_{\mathscr{R}_Y})^{-1})$ *is bornologous.*

We have discussed how one can construct entourage structures from reflexive relations. Now, we focus on the opposite process. Let $(X, \mathcal{E})$ be an entourage space. Then we define $\mathscr{R}_{\mathcal{E}} = \bigcup \mathcal{E}$, which is a reflexive relation since $\Delta_X \in \mathcal{E}$. Moreover, if $\mathcal{E}$ is a semi-coarse structure, then $\mathscr{R}_{\mathcal{E}}$ is symmetric, and, if $\mathcal{E}$ is a quasi-coarse structure, then $\mathscr{R}_{\mathcal{E}}$ is transitive.

Note that, if $\mathscr{R}$ is a reflexive relation on X, then

$$\mathscr{R} = \mathscr{R}_{\mathcal{E}_{\mathscr{R}}} = \mathscr{R}_{\mathcal{E}^{fin}_{\mathscr{R}}}.$$

Meanwhile, if $(X, \mathcal{E})$ is an entourage space, then

$$\mathcal{E}^{fin}_{\mathscr{R}_{\mathcal{E}}} \subseteq \mathcal{E} \subseteq \mathcal{E}_{\mathscr{R}_{\mathcal{E}}}. \tag{6}$$

The inclusions in (6) can be strict. Consider, for example, $\mathbb{R}$ endowed with the usual metric d. Then $\mathcal{E}^{fin}_{\mathscr{R}_{\mathcal{E}_d}} \subsetneq \mathcal{E}_d \subsetneq \mathcal{E}_{\mathscr{R}_{\mathcal{E}_d}}$. Furthermore, note that $\mathcal{E} = \mathcal{E}_{\mathscr{R}_{\mathcal{E}}}$ if and only if $\bigcup \mathcal{E} \in \mathcal{E}$ and, thus, every entourage structure $\mathcal{E}$ on a finite set X is a relation entourage structure. This observation will be used also in Remark 5.9.

4.2 Graphic quasi-coarse structures

In Example 1.8, we described how the family of vertices of a directed graph can be endowed with a quasi-metric, namely, the path quasi-metric. The induced metric entourage structure $\mathcal{E}_d$, which is a quasi-coarse structure, is called *graphic quasi-coarse structure*.

The graphic quasi-coarse space can be extended to the points on the graph edges, by identifying every edge with the interval $[0,1]$ endowed with the relation quasi-coarse structure associated to the usual order $\leq$ on $[0,1]$. More precisely, if $\Gamma = (V, E)$ is a directed graph and $(v, w) \in E$, then we identify 0 with v and 1 with w, respectively. This new quasi-coarse structure is called *extended graphic quasi-coarse structure*.

Let $f\colon \Gamma(V, E) \to \Gamma'(V', E')$ be a map between oriented graphs. Then f is said to be a *graph homomorphism* if, for every $(x, y) \in E$, either $f(x) = f(y)$ or $(f(x), f(y)) \in E'$. If $f\colon \Gamma(V, E) \to \Gamma'(V', E')$ is a graph homomorphism, then f sends directed paths into non-longer directed paths. Hence $f\colon (V, d) \to (V', d)$ is *non-expanding* (i.e., $d(f(x), f(y)) \leq d(x, y)$, for every $x, y \in V$), and thus $f\colon (V, \mathcal{E}_d) \to (V', \mathcal{E}_d)$ is bornologous.

4.3 Entourage hyperstructures

Let $(X, \mathcal{E})$ be an entourage structure. We define the following two entourage structures on $\mathcal{P}(X)$:

$$\mathcal{H}(\mathcal{E}) = \mathfrak{cl}(\{\mathcal{H}(E) \mid \Delta_X \subseteq E \in \mathcal{E}\}) \text{ and } \exp \mathcal{E} = \mathfrak{cl}(\{\exp E \mid \Delta_X \subseteq E \in \mathcal{E}\}) = \mathcal{H}(\mathcal{E}) \cap \mathcal{H}(\mathcal{E})^{-1},$$

where, for every $E \in \mathcal{E}$,

$$\mathcal{H}(E) = \{(A,B) \mid B \subseteq E[A]\} \quad \text{and} \quad \exp(E) = \mathcal{H}(E) \cap \mathcal{H}(E)^{-1},$$

named *entourage hyperstructure* and *semi-coarse hyperstructure*, respectively.

Remark 4.3. Let (X,d) be a metric space. In Example 1.6, we described the Hausdorff quasi-metric d_H^q on $\mathcal{P}(X)$. It is easy to prove that actually $\mathcal{E}_{d_H^q} = \mathcal{H}(\mathcal{E}_d)$.

First of all, note that, if $\mathcal{E}$ is an entourage structure, then both $\mathcal{H}(\mathcal{E})$ and $\exp\mathcal{E}$ are entourage structures since $\mathcal{H}(E) \cup \mathcal{H}(F) \subseteq \mathcal{H}(E \cup F)$, for every $E, F \in \mathcal{E}$. More precisely, $\exp\mathcal{E}$ is actually a semi-coarse structure. Furthermore, if $\mathcal{E}$ is quasi-coarse structure, then $\mathcal{H}(\mathcal{E})$ is a quasi-coarse structure, while $\exp\mathcal{E}$ is a coarse structure. In fact, for every $E, F \in \mathcal{E}$, if $(A,C) \in \mathcal{H}(E) \circ \mathcal{H}(F)$, there exists $B \subseteq X$ such that $(A,B) \in \mathcal{H}(E)$ and $(B,C) \in \mathcal{H}(F)$. Then $B \subseteq E[A]$ and $C \subseteq F[B]$, which implies that $C \subseteq F[E[A]] = (F \circ E)[A]$ and so $(A,C) \in \mathcal{H}(F \circ E)$. Note that $\mathcal{H}(\mathcal{E})$ is not a semi-coarse structure, unless the support X of $\mathcal{E}$ is empty: in fact, $(X,\emptyset) \in \mathcal{H}(\Delta_X)$, although, for every $E \in \mathcal{E}$, $E[\emptyset] = \emptyset$. Moreover, even if we consider the subspace $(\mathcal{P}(X) \setminus \{\emptyset\}, \mathcal{H}(\mathcal{E})|_{\mathcal{P}(X)\setminus\{\emptyset\}})$, it is a semi-coarse structure if and only if X satisfies (B_3). In fact, $(X,\{x\}) \in \mathcal{H}(\Delta_X)$, for every $x \in X$.

Every map $f \colon X \to Y$ between sets can be extended to a map $\overline{f} \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ such that, for every $A \in \mathcal{P}(X)$, $\overline{f}(A) = f(A) \in \mathcal{P}(Y)$.

Proposition 4.4. *Let $f \colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ be a map between entourage spaces. The following properties are equivalent:*
(a) $f \colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ is bornologous;
(b) $\overline{f} \colon (\mathcal{P}(X),\mathcal{H}(\mathcal{E}_X)) \to (\mathcal{P}(Y),\mathcal{H}(\mathcal{E}_Y))$ is bornologous.

Proof. As for the implication (a)$\to$(b), if f is bornologous, then the inclusion $(\overline{f} \times \overline{f})(\mathcal{H}(E)) \subseteq \mathcal{H}((f \times f)(E))$, for every $E \in \mathcal{E}_X$, holds, and the thesis follows. Conversely, (b)$\to$(a) is a consequence of the fact that, for every entourage space $(Z,\mathcal{E}_Z)$, if $E \in \mathcal{E}_Z$ and $x, y \in Z$, then $(x,y) \in E$ if and only if $(\{x\},\{y\}) \in \mathcal{H}(E)$. □

The study of the hyperspace of a coarse space was started in [17] and push forward in [3], using the language of ball structures (see §3 for their introduction).

4.4 Finitely generated monoids

In this subsubsection we want to briefly discuss the existence of precisely two inner quasi-coarse structures on a finitely generated monoid (see Proposition 4.5). The proof we give is similar to the proof of Proposition 1.4, which is the case of finitely generated groups (see, for example, [9]).

Let M be a monoid which is finitely generated by Σ. In Example 1.9 we defined the left, d_Σ^λ, and the right, d_Σ^ρ, word quasi-metrics. These quasi-metrics induce quasi-coarse structures on the monoid.

Proposition 4.5. *Let M be a monoid and Σ and Δ be two finite subsets of M which generate the whole monoid. Then $\mathcal{E}_{d_\Sigma^\lambda} = \mathcal{E}_{d_\Delta^\lambda}$ and $\mathcal{E}_{d_\Sigma^\rho} = \mathcal{E}_{d_\Delta^\rho}$.*

Proof. Define $k = \max\{d_\Delta^\lambda(e,\sigma) \mid \sigma \in \Sigma\}$ and $l = \max\{d_\Sigma^\lambda(e,\delta) \mid \delta \in \Delta\}$. Let $x, y \in M$, suppose that $d_\Sigma^\lambda(x,y) = n$ and let $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $y = x\sigma_1 \cdots \sigma_n$. Suppose that $\sigma_i = \delta_{i,1} \cdots \delta_{i,k_i}$, for every $i = 1, \ldots, n$, where $k_i = d_\Delta^\lambda(e, \sigma_i)$ and $\delta_{i,j} \in \Delta$, for every $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$. Then

$$y = x\sigma_1 \cdots \sigma_n = x\delta_{1,1} \cdots \delta_{1,k_1}\delta_{2,1} \cdots \delta_{n,k_n}$$

and so $d_\Delta^\lambda(x,y) \le \sum_{i=1}^n k_i \le nk = kd_\Sigma^\lambda(x,y)$. Hence, $\mathcal{E}_{d_\Sigma^\lambda} \subseteq \mathcal{E}_{d_\Delta^\lambda}$. Similarly, $d_\Sigma^\lambda(x,y) \le ld_\Delta^\lambda(x,y)$ and then $\mathcal{E}_{d_\Delta^\lambda} \subseteq \mathcal{E}_{d_\Sigma^\lambda}$. A similar proof shows that $\mathcal{E}_{d_\Sigma^\rho} = \mathcal{E}_{d_\Delta^\rho}$. □

5 The Sym-coarse equivalence

In this subsection we focus on quasi-coarse spaces. We want to introduce another equivalence notion, which will be more flexible than the one of asymorphism. In order to do that, we need to fix some terminology and notation. Two maps $f, g\colon S \to (X, \mathcal{E})$ from a set to a quasi-coarse space are Sym-*close*, and we denote it by $f \sim_{\mathrm{Sym}} g$, if $\{(f(x), g(x)), (g(x), f(x)) \mid x \in X\} \in \mathcal{E}$. Note that the Sym-closeness relation just defined is an equivalence relation.

Remark 5.1. Let $f, g\colon S \to (X, \mathcal{E})$ be two maps from a set to a quasi-coarse space. If $f = g$, then $f \sim_{\mathrm{Sym}} g$. The converse implication is not always true. However, if $\mathcal{E}_Y$ is the discrete coarse structure over Y, then $f \sim_{\mathrm{Sym}} g$ if and only if $f = g$.

Remark 5.2. It will be useful to check that some large-scale properties of a map are shared by all the maps in its equivalent class under Sym-closeness. Let us fix a pair of Sym-close maps $f, g\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between quasi-coarse spaces. Since they are Sym-close, $M = \{(f(x), g(x)), (g(x), f(x)) \mid x \in X\} \in \mathcal{E}_Y$.

(a) We claim that f is bornologous if and only if g is bornologous. In fact, let us assume that f is bornologous, and let $E \in \mathcal{E}_X$ be an arbitrary entourage. Then, for every $(x, y) \in E$

$$(g \times g)(x, y) = (g(x), f(x)) \circ (f(x), f(y)) \circ (f(y), g(y)) \in M \circ (f \times f)(E) \circ M,$$

which shows that $(g \times g)(E) \subseteq M \circ (f \times f)(E) \circ M \in \mathcal{E}_Y$.

(b) Similarly to what we have done for the item (a), we can prove that f is effectively proper if and only if g is effectively proper. In fact, if f is effectively proper, for every $E \in \mathcal{E}_Y$, $(g \times g)^{-1}(E) \subseteq (f \times f)^{-1}(M \circ E \circ M)$.

(c) The map f is large-scale surjective if and only if g is large-scale surjective. Let $E \in \mathcal{E}_Y$ be an entourage such that $E[f(X)] = Y$. Then

$$(M \circ E)[g(X)] = E[M[g(X)]] \subseteq E[f(X)] = Y.$$

Let $f\colon X \to Y$ be a map between quasi-coarse spaces. Then a map $g\colon Y \to X$ is a Sym-*coarse inverse* of f if $g \circ f \sim_{\mathrm{Sym}} id_X$ and $f \circ g \sim_{\mathrm{Sym}} id_Y$.

Definition 5.3. Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between quasi-coarse spaces. Then f is a Sym-*coarse equivalence* if it is bornologous and has a Sym-coarse inverse $g\colon Y \to X$ which is bornologous.

Two quasi-coarse spaces are Sym-*coarsely equivalent* if there exists a Sym-coarse equivalence between them.

In Theorem 5.6 we give other characterisations of Sym-coarse equivalences.

A subset L of a quasi-coarse space $(X,\mathcal{E})$ is Sym-*large* if there exists a symmetric entourage $E=E^{-1}\in\mathcal{E}$ such that $E[L]=X$. A map $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ between quasi-coarse spaces is *large-scale surjective* if $f(X)$ is Sym-large in Y. If f is also large-scale injective, then it is *large-scale bijective*. The following result characterises large-scale bijective maps between quasi-coarse spaces.

Proposition 5.4. *Let $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a map between quasi-coarse spaces. Then f is large-scale bijective if and only if it has a* Sym-*coarse inverse. In particular, every* Sym-*coarse inverse is large-scale bijective.*

Proof. ($\to$) Let $M=M^{-1}\in\mathcal{E}_Y$ be an entourage such that $M[f(X)]=Y$. For every $y\in Y$, there exists $x_y\in X$ such that $(y,f(x_y))\in M$. If $y\in f(X)$, suppose that $x_y\in f^{-1}(y)$. Define $g\colon Y\to X$ with the following law: $g(y)=x_y$, for every $y\in Y$. Then $(f(g(y)),y)\in M$ for every $y\in Y$, which witnesses that $f\circ g\sim_{\text{Sym}} id_Y$. The fact that f is large-scale injective proves that $g\circ f\sim_{\text{Sym}} id_X$.

($\leftarrow$) Let now $g\colon Y\to X$ be a Sym-coarse inverse of f. Let $M=M^{-1}\in\mathcal{E}_X$ and $N=N^{-1}\in\mathcal{E}_Y$ be two entourages showing that $g\circ f\sim_{\text{Sym}} id_X$ and $f\circ g\sim_{\text{Sym}} id_Y$, respectively. Note that, for every $y\in Y$, $f(g(y))\in f(X)$ and $(y,f(g(y))),(f(g(y)),y)\in N$. Hence f is large-scale surjective. Moreover, since $R_f=\{(x,y)\in X\times X\mid f(x)=f(y)\}\subseteq M\circ M$, f is large-scale injective.

The last assertion is trivial since, if g is a Sym-coarse inverse of f, then f is a Sym-coarse inverse of g. □

Proposition 5.5. *Let $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a large-scale bijective map between quasi-coarse spaces and let g be a* Sym-*coarse inverse of f. Then, the following properties are equivalent:*

(a) f is bornologous;

(b) g is uniformly weakly boundedness copreserving;

(c) g is uniformly boundedness copreserving;

(d) g is effectively proper.

Moreover, every other Sym-*coarse inverse h of g satisfies $h\sim_{\text{Sym}} f$.*

Proof. Since g is large-scale injective, the equivalences (b)$\leftrightarrow$(c)$\leftrightarrow$(d) descend from Proposition 2.9. Suppose now that f is bornologous. Let $E\in\mathcal{E}_X$ and consider $(g\times g)^{-1}(E)$. Denote by $M=M^{-1}$ the entourage of $\mathcal{E}_Y$ such that $(f(g(z)),z)\in M$, for every $z\in Y$. Then, for every $(x,y)\in(g\times g)^{-1}(E)$,

$$(x,y)=(x,f(g(x)))\circ(f(g(x)),f(g(y)))\circ(f(g(y)),y)\in M\circ(f\times f)(E)\circ M\in\mathcal{E}_Y.$$

Conversely, suppose that g is effectively proper. Denote by $N=N^{-1}\in\mathcal{E}_X$ the entourage showing that $g\circ f\sim_{\text{Sym}} id_X$. Let $E\in\mathcal{E}_X$ and $(x,y)\in E$. Then

$$(g(f(x)),g(f(y)))=(g(f(x)),x)\circ(x,y)\circ(y,g(f(y)))\in N\circ E\circ N\in\mathcal{E}_X$$

and thus $(f(x), f(y)) \in (g \times g)^{-1}(N \circ E \circ N) \in \mathcal{E}_Y$.

Finally, if h is another Sym-coarse inverse of g, then, for every $x \in X$,

$$(g(f(x)), g(h(x))) = (g(f(x)), x) \circ (x, g(h(x))) \in N \circ K,$$

where $K = K^{-1} \in \mathcal{E}_X$ is an entourage that shows that $g \circ h \sim_{\mathrm{Sym}} id_X$. Hence $(f(x), h(x)) \in (g \times g)^{-1}(N \circ K)$ and so $f \sim_{\mathrm{Sym}} h$ since $(g \times g)^{-1}(N \circ K) = ((g \times g)^{-1}(N \circ K))^{-1} \in \mathcal{E}_Y$. □

Note that, with an easy variation of the proof of Proposition 5.5, one can prove that every large-scale injective map $f\colon (X, \mathcal{E}_X) \to Y$ from a quasi-coarse space to a set has a *partial* Sym-*coarse inverse*, i.e., a map $g\colon Y' \to (X, \mathcal{E}_X)$, where $Y' \subseteq Y$, such that $g \circ f \sim_{\mathrm{Sym}} id_X$.

Theorem 5.6. *Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between quasi-coarse spaces. Then the following are equivalent:*

(a) f is a Sym-*coarse equivalence;*

(b) $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is large-scale bijective, bornologous and uniformly weakly boundedness copreserving;

(c) $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is large-scale bijective, bornologous and uniformly boundedness copreserving;

(d) $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is large-scale surjective, bornologous and effectively proper.

Proof. The equivalences (b)↔(c)↔(d) follow from Proposition 2.9.

(a)→(b) Since f has a Sym-coarse inverse g, it is large-scale bijective, thanks to Proposition 5.4. Moreover, g is bornologous and thus Proposition 5.5 implies that f is uniformly weakly boundedness copreserving.

(d)→(a) Let us construct a map $g\colon Y \to X$ with the desired properties. Since f is large-scale surjective, there exists $M = M^{-1} \in \mathcal{E}_Y$ such that $Y = M[f(X)]$. Hence, for every point $y \in Y$, we can fix another point $x \in X$ with the property that $(f(x_y), y) \in M$. Define the map g by putting $g(y) = x_y$, for every $y \in Y$. Let now $x \in X$. Then $(f(x), f(g(f(x)))) = (f(x), f(x_{f(x)})) \in M$, and so $(x, g(f(x))) \in (f \times f)^{-1}(M) \in \mathcal{E}_X$ since f is effectively proper. Thus $g \circ f \sim_{\mathrm{Sym}} id_X$. If now $y \in Y$, the pair $(y, f(g(y))) \in M$ because of the definition of g, which implies that $f \circ g \sim_{\mathrm{Sym}} id_Y$, and so g is a Sym-coarse inverse of f. The conclusion then follows from Proposition 5.5. □

Theorem 5.7. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two quasi-coarse spaces. Then X and Y are* Sym-*coarsely equivalent if and only if there exist two subspaces $X' \subseteq X$ and $Y' \subseteq Y$, which are* Sym-*large in X and in Y, respectively, and an asymorphism $f'\colon X' \to Y'$.*

Proof. (→) Let assume that there exists a Sym-coarse equivalence $f\colon X \to Y$. According to Theorem 5.6, f is large-scale surjective, bornologous and effectively proper. Let $X' \subseteq X$ be a subset with the following property: for every $x \in X$, $|X' \cap f^{-1}(f(x))| = 1$. Then $f' = f|_{X'}\colon X' \to Y'$, where $Y' = f(X) = f(X')$, is bijective. Moreover, $f'\colon (X', \mathcal{E}_X|_{X'}) \to (Y', \mathcal{E}_Y|_{Y'})$ is bornologous and effectively proper, since it is a restriction of f. Finally, since $f\colon X \to Y$ is large-scale injective (Proposition 2.9), X' is Sym-large in X.

(←) Let $M = M^{-1} \in \mathcal{E}_X$ be an entourage such that $M[X'] = X$. Then define a map $h\colon X \to X'$ as follows: if $x \in X'$, then $h(x) = x$, and, if otherwise $x \in X \setminus X'$, then $h(x)$

is a point such that $(h(x), x) \in M$. Similarly we can define a map $k\colon Y \to Y'$. We claim that h and k are bornologous. Let $E \in \mathcal{E}_X$. Then note that $(h \times h)(E) \subseteq M \circ E \circ M \in \mathcal{E}_X$ and thus h is bornologous. The same property can be similarly proved for k. Then the maps $f = f' \circ h$ and $g = (f')^{-1} \circ k$ are bornologous. We claim that g is a Sym-coarse inverse of f. For every $x \in X$, since $k|_{Y'} = id_{Y'}$,

$$(x, g(f(x))) = (x, (f')^{-1}(k(f'(h(x))))) = (x, (f')^{-1}(f'(h(x)))) = (x, h(x)) \in M,$$

and thus $g \circ f \sim_{\mathrm{Sym}} id_X$. The other request can be similarly proved. □

Proposition 5.8. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two* Sym*-coarsely equivalent quasi-coarse spaces. If $(X, \mathcal{E}_X)$ is a coarse space, then so it is $(Y, \mathcal{E}_Y)$.*

Proof. Let $f\colon X \to Y$ be a Sym-coarse equivalence and let $g\colon Y \to X$ be a Sym-coarse inverse of f. Moreover, let $E = E^{-1} \in \mathcal{E}_X$ and $F^{-1} = F \in \mathcal{E}_Y$ be two symmetric entourages which witness that $g \circ f \sim_{\mathrm{Sym}} id_X$ and $f \circ g \sim_{\mathrm{Sym}} id_Y$, respectively. Then, for every $K \in \mathcal{E}_Y$ and $(x, y) \in K$,

$$(y, x) = (y, f(g(y))) \circ (f(g(y)), f(g(x))) \circ (f(g(x)), x) \in F \circ (f \times f)(((g \times g)(K))^{-1}) \circ F \in \mathcal{E}_Y,$$

and then $K^{-1} \in \mathcal{E}_Y$. □

Remark 5.9. Let $(X, \mathcal{E})$ be a finite quasi-coarse space. According to the discussion contained in §4.1, there exists a pre-order $\geq$ on X such that $\mathcal{E} = \mathcal{E}_{\geq}$ (actually, $\geq = \bigcup \mathcal{E}$). Moreover, $\geq$ induces an equivalence relation $\cong$ on X in the usual way: for every $x, y \in X$, $x \cong y$ if and only if $x \geq y$ and $y \geq x$. Let $q\colon X \to X/_{\cong}$ be the quotient map. Then $\geq$ induces a partial order $\overline{\geq} = (q \times q)(\geq)$ on $\overline{X} = X/_{\cong}$. Moreover, the map $q\colon (X, \mathcal{E}_{\geq}) \to (\overline{X}, \mathcal{E}_{\overline{\geq}})$ is a Sym-coarse equivalence. Hence, finite quasi-coarse spaces from the large-scale point of view are just partial ordered sets.

6 Characterisation of some special classes of quasi-coarse spaces

Let $(X, \mathcal{E})$ be a quasi-coarse space. Then $(X, \mathcal{E})$ is *monogenic* if there exists an entourage $E \in \mathcal{E}$ such that the family $\{E^n \mid n \in \mathbb{N}\}$ forms a base of $\mathcal{E}$, where E^n is the composite of n copies of E. In the realm of coarse spaces, monogenicity is a classical notion (see, for example [19]). In particular, every monogenic quasi-coarse space has a countable base. Note that, if $(X, \mathcal{E})$ is an entourage space such that there exists $E \in \mathcal{E}$ with the property that $\mathfrak{cl}(\{E^n \mid n \in \mathbb{N}\}) = \mathcal{E}$, then $\mathcal{E}$ is a quasi-coarse structure. An example of a monogenic quasi-coarse space is a directed graph endowed with its graphic quasi-coarse structure.

Proposition 6.1. *If X and Y are* Sym*-coarsely equivalent quasi-coarse spaces, then:*
(a) X satisfies C_4 if and only if Y satisfies C_4;
(b) X is monogenic if and only if Y is monogenic.

Proof. First of all, note that all those properties are invariant under asymorphism. Thanks to Theorem 5.7, it is enough to prove the claim when Y is a Sym-large subspace of X, and, in this case, item (a) is not hard to shown. Let us now prove item (b).

Suppose that X is monogenic and $E \in \mathcal{E}_X$ is an entourage such that $\{E^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_X$. Let $F \in \mathcal{E}_X|_Y$. Then there exists $n_F \in \mathbb{N}$ such that $F \subseteq E^{n_F}$. Let $(x, y) \in F$. Thus there exist $z_0 = x, z_1, \dots, z_n = y \in X$ such that $(z_i, z_{i+1}) \in E$, for every $i = 0, \dots, n-1$. Moreover, for every $i = 1, \dots, n-1$, there exists $z_i' \in Y$ such that $(z_i, z_i') \in M$. Then, if we define $z_0' = x$ and $z_n' = y$, for every $i = 0, \dots, n-1$, $(z_i', z_{i+1}') \in (M \circ E \circ M) \cap (Y \times Y)$. Hence $\{((M \circ E \circ M) \cap (Y \times Y))^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_Y$.

Conversely, suppose that Y is monogenic and $\{E^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_Y$, for some $E \in \mathcal{E}_Y$. By using a similar argument, it is easy to show that $\{(M \circ E \circ M)^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_X$. □

Lemma 6.2. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two* Sym*-coarsely equivalent quasi-coarse spaces. Then* $\operatorname{cf} \mathcal{E}_X = \operatorname{cf} \mathcal{E}_Y$.

Proof. By applying Theorem 5.7, we can assume that Y is an entourage subspace of X and the inclusion map $i\colon Y \to X$ is large-scale surjective. It is trivial that $\operatorname{cf} \mathcal{E}_Y \leq \operatorname{cf} \mathcal{E}_X$. Let $f\colon X \to Y$ be a Sym-coarse inverse of i and $M = M^{-1} \in \mathcal{E}_X$ be an entourage such that $(x, f(x)) \in M$, for every $x \in X$. Then, for every base $\{E_i\}_{i \in I}$ of $\mathcal{E}_Y$, we claim that $\{M \circ E_i \circ M\}_i$ is a base of $\mathcal{E}_X$, and thus $\operatorname{cf} \mathcal{E}_X \leq \operatorname{cf} \mathcal{E}_Y$. In fact, let $F \in \mathcal{E}_X$ and $i \in I$ be an index such that $(M \circ F \circ M)|_{Y \times Y} \subseteq E_i$. Then $F \subseteq M \circ E_i \circ M$. □

We are now ready to prove the generalisations of some classical classification results in the framework of quasi-coarse spaces ([16, Theorems 9.1, 9.2], [18, Theorem 2.11]). The following results, together with Proposition 2.5, give a complete characterisation of metric entourage structures.

Theorem 6.3. *Let $(X, \mathcal{E})$ be a quasi-coarse space. The following properties are equivalent:*
(a) there exists a quasi-metric $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ on X such that $\mathcal{E} = \mathcal{E}_d$;
(b) there exists a quasi-metric space (Y, d) which is Sym*-coarsely equivalent to $(X, \mathcal{E})$;*
(c) $\operatorname{cf} \mathcal{E} \leq \omega$.

Proof. The implications (a)→(b)→(c) are trivial: in particular, (b)→(c) is implied by Lemma 6.2.

(c)→(a) Let $\{F_n\}_n$ be a base of $\mathcal{E}$ as in the proof of Proposition 2.5(a) with the following further property: for every $m, n \in \mathbb{N}$, $F_m \circ F_n \subseteq F_{m+n}$. We claim that the map $d\colon X \times X \to [0, \infty]$ defined as in (4) is a quasi-metric and, in order to show it, proving that d satisfies the triangle inequality is enough. Let $x, y, z \in X$ be three arbitrary points. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ trivially holds if $d(x, y) = \infty$ or $d(y, z) = \infty$. Suppose now that $d(x, y) \leq m$ and $d(y, z) \leq n$. Then $(x, z) = (x, y) \circ (y, z) \in F_m \circ F_n \subseteq F_{m+n}$ and thus $d(x, z) \leq m + n$. Finally, the equality $\mathcal{E} = \mathcal{E}_d$ can be easily proved. □

A quasi-coarse space satisfying the hypothesis of the previous theorem is called *quasi-metrisable*. Since the extended quasi-metric defined in the proof of Theorem 6.3 does not assume the value ∞ if and only if the quasi-coarse space is strongly connected, in view of Proposition 6.1, Theorem 6.3 can be specialised as follows.

Corollary 6.4. *Let $(X,\mathcal{E})$ be a quasi-coarse space. The following properties are equivalent:*

(a) there exists a quasi-metric d on X which does not assume the value ∞ and satisfies $\mathcal{E}=\mathcal{E}_d$;

(b) there exists a quasi-metric space (Y,d) which does not assume the value ∞ and is Sym-*coarsely equivalent to $(X,\mathcal{E})$;*

(c) $(X,\mathcal{E})$ satisfies C_4 and $\operatorname{cf}\mathcal{E}\leq\omega$.

[18, Proposition 2.1.1] implies that the quasi-metrics in Theorem 6.3 and in Corollary 6.4 can be taken as metrics if and only if the initial space is a coarse space.

Finally we can answer to a problem posed by Protasov and Banakh [16, Problem 9.4].

Theorem 6.5. *Let $(X,\mathcal{E})$ be a connected quasi-coarse space. Then the following properties are equivalent:*

(a) $(X,\mathcal{E})$ is a graphic quasi-coarse space;

(b) $(X,\mathcal{E})$ is Sym-*coarsely equivalent to a graphic quasi-coarse space;*

(c) $(X,\mathcal{E})$ is monogenic.

Proof. The implication (a)→(b) is trivial. As for the implication (b)→(c), since graphic quasi-coarse spaces are monogenic, Proposition 6.1(b) implies that also $(X,\mathcal{E})$ has the same property.

(c)→(a) Let $\Delta_X\subseteq E\in\mathcal{E}$ be an entourage such that $\mathfrak{cl}(\{E^n\mid n\in\mathbb{N}\})=\mathcal{E}$. Consider the directed graph $\Gamma=(X,E)$ whose set of edges is the entourage E (i.e., a pair of points (x,y) of X is an edge of Γ if and only if $(x,y)\in E$). Then the graphic quasi-coarse space associated to the graph Γ is asymorphic to $(X,\mathcal{E})$. □

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