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Noether's problem and rationality problem for multiplicative invariant fields: a survey

By

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Abstract

In this paper, we give a brief survey of recent developments on Noether's problem and rationality problem for multiplicative invariant fields including author's recent papers Hoshi [Hos15] about Noether's problem over $\mathbb{Q}$, Hoshi, Kang and Kunyavskii [HKK13], Chu, Hoshi, Hu and Kang [CHHK15], Hoshi [Hos16] and Hoshi, Kang and Yamasaki [HKY16] about Noether's problem over $\mathbb{C}$, and Hoshi, Kang and Kitayama [HKK14] and Hoshi, Kang and Yamasaki [HKY] about rationality problem for multiplicative invariant fields.

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§ 1. Introduction

Let k be a field and G be a finite group acting on the rational function field $k(x_g \mid g \in G)$ by k-automorphisms $h(x_g) = x_{hg}$ for any $g, h \in G$. We denote the fixed field $k(x_g \mid g \in G)^G$ by $k(G)$. Emmy Noether [Noe13, Noe17] asked whether $k(G)$

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is rational (= purely transcendental) over k. This is called Noether's problem for G over k, and is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions (see Swan [Swa81, Swa83], Saltman [Sal82], Manin and Tsfasman [MT86], Garibaldi, Merkurjev and Serre [GMS03, Section 33.1, page 86], Colliot-Thélène and Sansuc [CTS07]).

Theorem 1.1 (Fischer [Fis15], see also Swan [Swa83, Theorem 6.1]). *Let G be a finite abelian group with exponent e. Assume that* (i) *either* $\mathrm{char}\ k = 0$ *or* $\mathrm{char}\ k > 0$ *with* $\mathrm{char}\ k \nmid e$, *and* (ii) *k contains a primitive e-th root of unity. Then $k(G)$ is rational over k. In particular, $\mathbb{C}(G)$ is rational over $\mathbb{C}$.*

Theorem 1.2 (Kuniyoshi [Kun54, Kun55, Kun56], see also Gaschütz [Gas59]). *Let G be a p-group and k be a field with* $\mathrm{char}\ k = p > 0$. *Then $k(G)$ is rational over k.*

Definition 1.3. Let K/k and L/k be finitely generated extensions of fields.
(1) K is said to be *rational* over k (for short, *k-rational*) if K is purely transcendental over k, i.e. $K \simeq k(x_1, \ldots, x_n)$ for some algebraically independent elements $x_1, \ldots, x_n$ over k;
(2) K is said to be *stably k-rational* if $K(y_1, \ldots, y_m)$ is k-rational for some algebraically independent elements $y_1, \ldots, y_m$ over K;
(3) K and L are said to be *stably k-isomorphic* if $K(y_1, \ldots, y_m) \simeq L(z_1, \ldots, z_n)$ for some algebraically independent elements $y_1, \ldots, y_m$ over K and $z_1, \ldots, z_n$ over L;
(4) (Saltman, [Sal84b, Definition 3.1]) when k is an infinite field, K is said to be *retract k-rational* if there exists a k-algebra A contained in K such that (i) K is the quotient field of A, (ii) there exist a non-zero polynomial $f \in k[x_1, \ldots, x_n]$ and k-algebra homomorphisms $\varphi\colon A \to k[x_1, \ldots, x_n][1/f]$ and $\psi\colon k[x_1, \ldots, x_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$;
(5) K is said to be *k-unirational* if $k \subset K \subset k(x_1, \ldots, x_n)$ for some integer n.

We see that if K and L are stably k-isomorphic and K is retract k-rational, then L is also retract k-rational (see [Sal84b, Proposition 3.6]), and hence it is not difficult to verify the following implications:

$$k\text{-rational} \ \Rightarrow\ \text{stably } k\text{-rational} \ \Rightarrow\ \text{retract } k\text{-rational} \ \Rightarrow\ k\text{-unirational}.$$

Note that $k(G)$ is retract k-rational if and only if there exists a generic G-Galois extension over k (see [Sal82, Theorem 5.3], [Sal84b, Theorem 3.12]). In particular, if k is a Hilbertian field, e.g. number field, and $k(G)$ is retract k-rational, then inverse Galois problem for G over k has a positive answer, i.e. there exists a Galois extension K/k with $\mathrm{Gal}(K/k) \simeq G$.

§ 2. Noether's problem over $\mathbb{Q}$

Masuda [Mas55, Mas68] gave an idea to use a technique of Galois descent to Noether's problem for cyclic groups C_p of order p. Let ζ_p be a primitive p-th root of unity, $L = \mathbb{Q}(\zeta_p)$ and $\pi = \mathrm{Gal}(L/\mathbb{Q})$. Then, by Theorem 1.1, we have $\mathbb{Q}(C_p) = \mathbb{Q}(x_1, \ldots, x_p)^{C_p} = (L(x_1, \ldots, x_p)^{C_p})^{\pi} = L(y_0, \ldots, y_{p-1})^{\pi} = L(M)^{\pi}(y_0)$ where $y_0 = \sum_{i=1}^{p} x_i$ is π-invariant, M is free $\mathbb{Z}[\pi]$-module and π acts on $y_1, \ldots, y_{p-1}$ by $\sigma(y_i) = \prod_{j=1}^{p-1} y_j^{a_{ij}}$, $[a_{ij}] \in GL_n(\mathbb{Z})$ for any $\sigma \in \pi$. Thus the field $L(M)^{\pi}$ may be regarded as the function field of some algebraic torus of dimension $p-1$ (see e.g. [Vos98, Chapter 3], [HY17, Chapter 1]).

Theorem 2.1 (Masuda [Mas55, Mas68], see also [Swa83, Lemma 7.1])**.**
(1) *M is projective $\mathbb{Z}[\pi]$-module of rank one;*
(2) *If M is a permutation $\mathbb{Z}[\pi]$-module, i.e. M has a $\mathbb{Z}$-basis which is permuted by π, then $L(M)^{\pi}$ is $\mathbb{Q}$-rational. In particular, $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational for $p \leq 11$.*[1]

Swan [Swa69] gave the first negative solution to Noether's problem by investigating a partial converse to Masuda's result.

Theorem 2.2 (Swan [Swa69], Voskresenskii [Vos70])**.**
(1) *If $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational, then there exists $\alpha \in \mathbb{Z}[\zeta_{p-1}]$ such that $N_{\mathbb{Q}(\zeta_{p-1})/\mathbb{Q}}(\alpha) = \pm p$;*
(2) (Swan [Swa69, Theorem 1]) *$\mathbb{Q}(C_{47})$, $\mathbb{Q}(C_{113})$ and $\mathbb{Q}(C_{233})$ are not $\mathbb{Q}$-rational;*
(3) (Voskresenskii [Vos70, Theorem 2]) *$\mathbb{Q}(C_{47})$, $\mathbb{Q}(C_{167})$, $\mathbb{Q}(C_{359})$, $\mathbb{Q}(C_{383})$, $\mathbb{Q}(C_{479})$, $\mathbb{Q}(C_{503})$ and $\mathbb{Q}(C_{719})$ are not $\mathbb{Q}$-rational.*

Theorem 2.3 (Voskresenskii [Vos71, Theorem 1])**.** *$\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational if and only if there exists $\alpha \in \mathbb{Z}[\zeta_{p-1}]$ such that $N_{\mathbb{Q}(\zeta_{p-1})/\mathbb{Q}}(\alpha) = \pm p$.*

Hence if the cyclotomic field $\mathbb{Q}(\zeta_{p-1})$ has class number one, then $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational. However, it is known that such primes are exactly $p \leq 43$ and $p = 61, 67, 71$ (see Masley and Montgomery [MM76, Main theorem] or Washington's book [Was97, Chapter 11]).

Endo and Miyata [EM73] refined Masuda-Swan's method and gave some further consequences on Noether's problem when G is abelian (see also [Vos73]).

Theorem 2.4 (Endo and Miyata [EM73, Theorem 2.3])**.** *Let G_1 and G_2 be finite groups and k be a field with $\operatorname{char} k = 0$. If $k(G_1)$ and $k(G_2)$ are k-rational (resp. stably k-rational), then $k(G_1 \times G_2)$ is k-rational (resp. stably k-rational).*[2]

[1] The author [Hos05, Chapter 5] generalized Theorem 2.1 (2) to Frobenius groups F_{pl} of order pl with $l \mid p-1$ ($p \leq 11$).

[2] Kang and Plans [KP09, Theorem 1.3] showed that Theorem 2.4 is also valid for any field k.

Theorem 2.5 (Endo and Miyata [EM73, Theorem 3.1]). *Let p be an odd prime and l be a positive integer. Let k be a field with* char $k = 0$ *and* $[k(\zeta_{p^l}) : k] = p^{m_0} d_0$ *with* $0 \leq m_0 \leq l-1$ *and* $d_0 \mid p-1$. *Then the following conditions are equivalent:*
(1) *For any faithful $k[C_{p^l}]$-module V, $k(V)^{C_{p^l}}$ is k-rational;*
(2) *$k(C_{p^l})$ is k-rational;*
(3) *There exists $\alpha \in \mathbb{Z}[\zeta_{p^{m_0} d_0}]$ such that*

$$N_{\mathbb{Q}(\zeta_{p^{m_0} d_0})/\mathbb{Q}}(\alpha) = \begin{cases} \pm p & m_0 > 0 \\ \pm p^l & m_0 = 0. \end{cases}$$

Further suppose that $m_0 > 0$. Then the above conditions are equivalent to each of the following conditions:
(1′) *For any $k[C_{p^l}]$-module V, $k(V)^{C_{p^l}}$ is k-rational;*
(2′) *For any $1 \leq l' \leq l$, $k(C_{p^{l'}})$ is k-rational.*

Theorem 2.6 (Endo and Miyata [EM73, Proposition 3.2]). *Let p be an odd prime and k be a field with* char $k = 0$. *If k contains $\zeta_p + \zeta_p^{-1}$, then $k(C_{p^l})$ is k-rational for any l. In particular, $\mathbb{Q}(C_{3^l})$ is $\mathbb{Q}$-rational for any l.*

Theorem 2.7 (Endo and Miyata [EM73, Proposition 3.4, Corollary 3.10]).
(1) *For primes $p \leq 43$ and $p = 61, 67, 71$, $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational;*
(2) *For $p = 5, 7$, $\mathbb{Q}(C_{p^2})$ is $\mathbb{Q}$-rational;*
(3) *For $l \geq 3$, $\mathbb{Q}(C_{2^l})$ is not stably $\mathbb{Q}$-rational.*

Theorem 2.8 (Endo and Miyata [EM73, Theorem 4.4]). *Let G be a finite abelian group of odd order and k be a field with* char $k = 0$. *Then there exists an integer $m > 0$ such that $k(G^m)$ is k-rational.*

Theorem 2.9 (Endo and Miyata [EM73, Theorem 4.6]). *Let G be a finite abelian group. Then $\mathbb{Q}(G)$ is $\mathbb{Q}$-rational if and only if $\mathbb{Q}(G)$ is stably $\mathbb{Q}$-rational.*

Ultimately, Lenstra [Len74] gave a necessary and sufficient condition of Noether's problem for abelian groups.

Theorem 2.10 (Lenstra [Len74, Main Theorem, Remark 5.7]). *Let k be a field and G be a finite abelian group. Let $k_{\rm cyc}$ be the maximal cyclotomic extension of k in an algebraic closure. For $k \subset K \subset k_{\rm cyc}$, we assume that $\rho_K = \mathrm{Gal}(K/k) = \langle \tau_k \rangle$ is finite cyclic. Let p be an odd prime with $p \neq$ char k and $s \geq 1$ be an integer. Let $\mathfrak{a}_K(p^s)$ be a $\mathbb{Z}[\rho_K]$-ideal defined by*

$$\mathfrak{a}_K(p^s) = \begin{cases} \mathbb{Z}[\rho_K] & \text{if } K \neq k(\zeta_{p^s}) \\ (\tau_K - t, p) & \text{if } K = k(\zeta_{p^s}) \text{ where } t \in \mathbb{Z} \text{ satisfies } \tau_K(\zeta_p) = \zeta_p^t \end{cases}$$

and put $\mathfrak{a}_K(G) = \prod_{p,s} \mathfrak{a}_K(p^s)^{m(G,p,s)}$ *where* $m(G,p,s) = \dim_{\mathbb{Z}/p\mathbb{Z}}(p^{s-1}G/p^sG)$*. Then the following conditions are equivalent:*
(1) $k(G)$ *is* k*-rational;*
(2) $k(G)$ *is stably* k*-rational;*
(3) *for* $k \subset K \subset k_{\rm cyc}$*, the* $\mathbb{Z}[\rho_K]$*-ideal* $\mathfrak{a}_K(G)$ *is principal and if* char $k \neq 2$*, then* $k(\zeta_{r(G)})/k$ *is cyclic extension where* $r(G)$ *is the highest power of* 2 *dividing the exponent of* G*.*

Theorem 2.11 (Lenstra [Len74, Corollary 7.2], [Len80, Proposition 2, Corollary 3])**.** *Let* n *be a positive integer. Then the following conditions are equivalent:*
(1) $\mathbb{Q}(C_n)$ *is* $\mathbb{Q}$*-rational;*
(2) $k(C_n)$ *is* k*-rational for any field* k*;*
(3) $\mathbb{Q}(C_{p^s})$ *is* $\mathbb{Q}$*-rational for any* $p^s \mid\mid n$*;*
(4) $8 \nmid n$ *and for any* $p^s \mid\mid n$*, there exists* $\alpha \in \mathbb{Z}[\zeta_{\varphi(p^s)}]$ *such that* $N_{\mathbb{Q}(\zeta_{\varphi(p^s)})/\mathbb{Q}}(\alpha) = \pm p$*.*

Theorem 2.12 (Lenstra [Len74, Corollary 7.6], [Len80, Proposition 6])**.** *Let* k *be a field which is finitely generated over its prime field. Let* P_k *be the set of primes* p *for which* $k(C_p)$ *is* k*-rational. Then* P_k *has Dirichlet density* 0 *inside the set of all primes. In particular,*

$$\lim_{x\to\infty} \frac{\pi^*(x)}{\pi(x)} = 0$$

where $\pi(x)$ *is the number of primes* $p \leq x$*, and* $\pi^*(x)$ *is the number of primes* $p \leq x$ *for which* $\mathbb{Q}(C_p)$ *is* $\mathbb{Q}$*-rational.*

Theorem 2.13 (Lenstra [Len80, Proposition 4])**.** *Let* p *be a prime and* $s \geq 2$ *be an integer. Then* $\mathbb{Q}(C_{p^s})$ *is* $\mathbb{Q}$*-rational if and only if* $p^s \in \{2^2, 3^m, 5^2, 7^2 \mid m \geq 2\}$*.*

By using Theorem 2.4, Endo and Miyata [EM73, Appendix] checked whether $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational for some primes $p < 2000$. By using PARI/GP [PARI2], Hoshi [Hos15] confirmed that for primes $p < 20000$, $\mathbb{Q}(C_p)$ is not $\mathbb{Q}$-rational except for 17 rational cases with $p \leq 43$ and $p = 61, 67, 71$ and undetermined 46 cases. Eventually, Plans [Pla17] determined the complete set of primes for which $\mathbb{Q}(C_p)$ is $\mathbb{Q}$-rational:

Theorem 2.14 (Plans [Pla17, Theorem 1.1])**.** *Let* p *be a prime. Then* $\mathbb{Q}(C_p)$ *is* $\mathbb{Q}$*-rational if and only if* $p \leq 43$*,* $p = 61, 67$ *or* 71*.*

Combining Theorem 2.11, Theorem 2.13 and Theorem 2.14, we have:

Corollary 2.15 (Plans [Pla17, Corollary 1.2])**.** *Let* n *be a positive integer. Then* $\mathbb{Q}(C_n)$ *is* $\mathbb{Q}$*-rational if and only if* n *divides*

$$2^2 \cdot 3^m \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 61 \cdot 67 \cdot 71$$

for some integer $m \geq 0$*.*

On the other hand, just a handful of results about Noether's problem are obtained when the groups are non-abelian.

Theorem 2.16 (Maeda [Mae89, Theorem, page 418]). *Let k be a field and A_5 be the alternating group of degree 5. Then $k(A_5)$ is k-rational.*

Theorem 2.17 (Rikuna [Rik], Plans [Pla07], see also [HKY11, Example 13.7]). *Let k be a field with* char $k \neq 2$*. Then $k(SL_2(\mathbb{F}_3))$ and $k(GL_2(\mathbb{F}_3))$ are k-rational.*

Theorem 2.18 (Serre [GMS03, Chapter IX], see also Kang [Kan05]). *Let G be a finite group with a 2-Sylow subgroup which is cyclic of order ≥ 8 or the generalized quaternion Q_{16} of order 16. Then $\mathbb{Q}(G)$ is not stably $\mathbb{Q}$-rational.*

Theorem 2.19 (Plans [Pla09, Theorem 2]). *Let A_n be the alternating group of degree n. If $n \geq 3$ is odd integer, then $\mathbb{Q}(A_n)$ is rational over $\mathbb{Q}(A_{n-1})$. In particular, if $\mathbb{Q}(A_{n-1})$ is $\mathbb{Q}$-rational, then so is $\mathbb{Q}(A_n)$.*

However, it is an open problem whether $k(A_n)$ is k-rational for $n \geq 6$.

§ 3. Noether's problem over $\mathbb{C}$ and unramified Brauer groups

We consider Noether's problem for G over $\mathbb{C}$, i.e. the rationality problem for $\mathbb{C}(G)$ over $\mathbb{C}$. Let G be a p-group. Then, by Theorem 1.1 and Theorem 1.2, we may focus on the case where G is a non-abelian p-group and k is a field with char $k \neq p$. For p-groups of small order, the following results are known.

Theorem 3.1 (Chu and Kang [CK01]). *Let p be any prime and G be a p-group of order $\leq p^4$ and of exponent e. If k is a field containing a primitive e-th root of unity, then $k(G)$ is k-rational. In particular, $\mathbb{C}(G)$ is $\mathbb{C}$-rational.*

Theorem 3.2 (Chu, Hu, Kang and Prokhorov [CHKP08]). *Let G be a group of order 32 and of exponent e. If k is a field containing a primitive e-th root of unity, then $k(G)$ is k-rational. In particular, $\mathbb{C}(G)$ is $\mathbb{C}$-rational.*

Saltman introduced a notion of retract k-rationality (see Definition 1.3) and the unramified Brauer group:

Definition 3.3 (Saltman [Sal84a, Definition 3.1], [Sal85, page 56]). Let K/k be an extension of fields. The *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(K/k)$ of K over k is defined to be

$$\mathrm{Br}_{\mathrm{nr}}(K/k) = \bigcap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$$

where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R. We write just $\mathrm{Br}_{\mathrm{nr}}(K)$ when the base field k is clear from the context.

Proposition 3.4 (Saltman [Sal84a], [Sal85, Proposition 1.8], [Sal87]). *If K is retract k-rational, then $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(K)$. In particular, if k is an algebraically closed field and K is retract k-rational, then $\mathrm{Br}_{\mathrm{nr}}(K) = 0$.*

Theorem 3.5 (Bogomolov [Bog88, Theorem 3.1], Saltman [Sal90, Theorem 12]). *Let G be a finite group and k be an algebraically closed field with $\mathrm{char}\ k = 0$ or $\mathrm{char}\ k = p \nmid |G|$. Then $\mathrm{Br}_{\mathrm{nr}}(k(G)/k)$ is isomorphic to the group $B_0(G)$ defined by*

$$B_0(G) = \bigcap_{A} \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

Remark 3.6. For a smooth projective variety X over $\mathbb{C}$ with function field K, $\mathrm{Br}_{\mathrm{nr}}(K/\mathbb{C})$ is isomorphic to the birational invariant $H^3(X, \mathbb{Z})_{\mathrm{tors}}$ which was used by Artin and Mumford [AM72] to provide some elementary examples of k-unirational varieties which are not k-rational (see also [Bog88, Theorem 1.1 and Corollary]).

Note that $B_0(G)$ is a subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})$ which is isomorphic to the Schur multiplier $H_2(G, \mathbb{Z})$ of G (see Karpilovsky [Kar87]). We call $B_0(G)$ the *Bogomolov multiplier of G* (cf. Kunyavskii [Kun10]). Because of Theorem 3.5, we will not distinguish $B_0(G)$ and $\mathrm{Br}_{\mathrm{nr}}(k(G)/k)$ when k is an algebraically closed field, and $\mathrm{char}\ k = 0$ or $\mathrm{char}\ k = p \nmid |G|$. Using $B_0(G)$, Saltman and Bogomolov gave counter-examples to Noether's problem for non-abelian p-groups over algebraically closed field.

Theorem 3.7 (Saltman [Sal84a], Bogomolov [Bog88]). *Let p be any prime and k be any algebraically closed field with $\mathrm{char}\ k \neq p$.*
(1) (Saltman [Sal84a, Theorem 3.6]) *There exists a meta-abelian group G of order p^9 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not (retract, stably) k-rational;*
(2) (Bogomolov [Bog88, Lemma 5.6]) *There exists a group G of order p^6 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not (retract, stably) k-rational.*

Colliot-Thélène and Ojanguren [CTO89] generalized the notion of the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(K/k)$ to the unramified cohomology $H^i_{\mathrm{nr}}(K/k, \mu_n^{\otimes j})$ of degree $i \geq 1$, that is $F_n^{i,j}(K/k)$ in [CTO89, Definition 1.1].

Definition 3.8 (Colliot-Thélène and Ojanguren [CTO89], [CT95, Sections 2–4]). Let n be a positive integer and k be a field with $\mathrm{char}\ k = 0$ or $\mathrm{char}\ k = p$ with $p \nmid n$. Let

K/k be a function field, that is finitely generated field extension as a field over k. For any positive integer $i \geq 2$, any integer j, the *unramified cohomology group* $H^i_{\rm nr}(K/k, \mu_n^{\otimes j})$ of K over k of degree i is defined to be

$$H^i_{\rm nr}(K/k, \mu_n^{\otimes j}) := \bigcap_R {\rm Ker}\{r_R : H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(\Bbbk_R, \mu_n^{\otimes (j-1)})\}$$

where R runs over all the discrete valuation rings R of rank one such that $k \subset R \subset K$ and K is the quotient field of R, $\Bbbk_R$ is the residue field of R and r_R is the residue map of K at R.

By [CT95, Theorem 4.1.1, page 30], if it is assumed furthermore that K is the function field of a complete smooth variety over k, the unramified cohomology group $H^i_{\rm nr}(K/k, \mu_n^{\otimes j})$ may be defined as well by

$$H^i_{\rm nr}(K/k, \mu_n^{\otimes j}) = \bigcap_R {\rm Image}\{H^i_{\rm \acute{e}t}(R, \mu_n^{\otimes j}) \to H^i_{\rm \acute{e}t}(K, \mu_n^{\otimes j})\}$$

where R runs over all the discrete valuation rings R of rank one such that $k \subset R \subset K$ and K is the quotient field of R.

Note that the unramified cohomology groups of degree two are isomorphic to the n-torsion part of the unramified Brauer group: ${}_n{\rm Br}_{\rm nr}(K/k) \simeq H^2_{\rm nr}(K/k, \mu_n)$.

Theorem 3.9. *Let n be a positive integer and k be an algebraically closed field with* char $k = 0$ *or* char $k = p \nmid n$.
(1) (Colliot-Thélène and Ojanguren [CTO89, Proposition 1.2]) *If K and L are stably k-isomorphic, then $H^i_{\rm nr}(K/k, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\rm nr}(L/k, \mu_n^{\otimes j})$. In particular, K is stably k-rational, then $H^i_{\rm nr}(K/k, \mu_n^{\otimes j}) = 0$;*
(2) ([Mer08, Proposition 2.15], see also [CTO89, Remarque 1.2.2], [CT95, Sections 2–4], [GS10, Example 5.9]) *If K is retract k-rational, then $H^i_{\rm nr}(K/k, \mu_n^{\otimes j}) = 0$.*

Colliot-Thélène and Ojanguren [CTO89, Section 3] produced the first example of not stably $\mathbb{C}$-rational but $\mathbb{C}$-unirational field K with $H^3_{\rm nr}(K, \mu_2^{\otimes 3}) \neq 0$, where K is the function field of a quadric of the type $\langle\langle f_1, f_2\rangle\rangle = \langle g_1 g_2\rangle$ over the rational function field $\mathbb{C}(x, y, z)$ with three variables x, y, z for a 2-fold Pfister form $\langle\langle f_1, f_2\rangle\rangle$, as a generalization of Artin and Mumford [AM72]. Peyre [Pey93, Corollary 3] gave a sufficient condition for $H^i_{\rm nr}(K/k, \mu_p^{\otimes i}) \neq 0$ and produced an example of the function field K with $H^3_{\rm nr}(K/k, \mu_p^{\otimes 3}) \neq 0$ and ${\rm Br}_{\rm nr}(K/k) = 0$ using a result of Suslin [Sus91] where K is the function field of a product of some norm varieties associated to cyclic central simple algebras of degree p (see [Pey93, Proposition 7]). Using a result of Jacob and Rost [JR89], Peyre [Pey93, Proposition 9] also gave an example of $H^4_{\rm nr}(K/k, \mu_2^{\otimes 4}) \neq 0$ and ${\rm Br}_{\rm nr}(K/k) = 0$ where K is the function field of a product of quadrics associated to a 4-fold Pfister form $\langle\langle a_1, a_2, a_3, a_4\rangle\rangle$ (see also [CT95, Section 4.2]).

In case char $k = 0$, take the direct limit with respect to n:

$$H^i(K/k, \mathbb{Q}/\mathbb{Z}(j)) = \varinjlim_{n} H^i(K/k, \mu_n^{\otimes j})$$

and we may define the unramified cohomology group

$$H^i_{\rm nr}(K/k, \mathbb{Q}/\mathbb{Z}(j)) = \bigcap_R \operatorname{Ker}\{r_R : H^i(K/k, \mathbb{Q}/\mathbb{Z}(j)) \to H^{i-1}(\Bbbk_R, \mathbb{Q}/\mathbb{Z}(j-1))\}.$$

We write simply $H^i_{\rm nr}(K, \mu_n^{\otimes j})$ and $H^i_{\rm nr}(K, \mathbb{Q}/\mathbb{Z}(j))$ when the base field k is understood. When k is an algebraically closed field with char $k = 0$, we will write $H^i_{\rm nr}(K/k, \mathbb{Q}/\mathbb{Z})$ for $H^i_{\rm nr}(K/k, \mathbb{Q}/\mathbb{Z}(j))$. Then we have $\mathrm{Br}_{\rm nr}(K/k) \simeq H^2_{\rm nr}(K/k, \mathbb{Q}/\mathbb{Z})$.

Peyre [Pey08] constructed an example of a field K, as $K = \mathbb{C}(G)$, whose unramified Brauer group vanishes, but unramified cohomology of degree three does not vanish:

Theorem 3.10 (Peyre [Pey08, Theorem 3]). *Let p be any odd prime. Then there exists a p-group G of order p^{12} such that $B_0(G) = 0$ and $H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.*

The idea of Peyre's proof is to find a subgroup $K^3_{\rm max}/K^3$ of $H^3_{\rm nr}(\mathbb{C}(G), \mathbb{Q}/\mathbb{Z})$ and to show that $K^3_{\rm max}/K^3 \neq 0$ (see [Pey08, page 210]).

Asok [Aso13] generalized Peyre's argument [Pey93] and established the following theorem for a smooth proper model X (resp. a smooth projective model Y) of the function field of a product of quadrics of the type $\langle\langle s_1, \ldots, s_{n-1}\rangle\rangle = \langle s_n\rangle$ (resp. Rost varieties) over some rational function field over $\mathbb{C}$ with many variables.

Theorem 3.11 (Asok [Aso13], see [AM11, Theorem 3] for retract rationality).
(1) ([Aso13, Theorem 1]) *For any $n > 0$, there exists a smooth projective complex variety X that is $\mathbb{C}$-unirational, for which $H^i_{\rm nr}(\mathbb{C}(X), \mu_2^{\otimes i}) = 0$ for each $i < n$, yet $H^n_{\rm nr}(\mathbb{C}(X), \mu_2^{\otimes n}) \neq 0$, and so X is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational;*
(2) ([Aso13, Theorem 3]) *For any prime l and any $n \geq 2$, there exists a smooth projective rationally connected complex variety Y such that $H^n_{\rm nr}(\mathbb{C}(Y), \mu_l^{\otimes n}) \neq 0$. In particular, Y is not $\mathbb{A}^1$-connected, nor (retract, stably) $\mathbb{C}$-rational.*

Namely, the triviality of the unramified Brauer group or the unramified cohomology of higher degree is just a necessary condition of $\mathbb{C}$-rationality of fields. It is unknown whether the vanishing of all the unramified cohomologies is a sufficient condition for $\mathbb{C}$-rationality. It is interesting to consider an analog of Theorem 3.11 for quotient varieties V/G, e.g. the case of Noether's problem $\mathbb{C}(V_{\rm reg}/G) = \mathbb{C}(G)$.

Colliot-Thélène and Voisin [CTV12] established:

Theorem 3.12 (Colliot-Thélène and Voisin [CTV12], [Voi14, Theorem 6.18]). *For any smooth projective complex variety X, there is an exact sequence*

$$0 \to H^3_{\rm nr}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{\rm nr}(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tors}(Z^4(X)) \to 0$$

where

$$Z^4(X) = \mathrm{Hdg}^4(X,\mathbb{Z})/\mathrm{Hdg}^4(X,\mathbb{Z})_{\mathrm{alg}}$$

and the lower index "alg" means that we consider the group of integral Hodge classes which are algebraic. In particular, if X is rationally connected, then we have

$$H^3_{\mathrm{nr}}(X,\mathbb{Q}/\mathbb{Z}) \simeq Z^4(X).$$

Using Peyre's method [Pey08], we obtain the following theorem which is an improvement of Theorem 3.10 and gives an explicit counter-example to integral Hodge conjecture with the aid of Theorem 3.12.

Theorem 3.13 (Hoshi, Kang and Yamasaki [HKY16, Theorem 1.4])**.** *Let p be any odd prime. Then there exists a p-group G of order p^9 such that $B_0(G) = 0$ and $H^3_{\mathrm{nr}}(\mathbb{C}(G),\mathbb{Q}/\mathbb{Z}) \neq 0$. In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.*

The case where G is a group of order p^5 ($p \geq 3$).

From Theorem 3.7 (2), Bogomolov [Bog88, Remark 1] raised a question to classify the groups of order p^6 with $B_0(G) \neq 0$. He also claimed that if G is a p-group of order $\leq p^5$, then $B_0(G) = 0$ ([Bog88, Lemma 5.6]). However, this claim was disproved by Moravec:

Theorem 3.14 (Moravec [Mor12, Section 8])**.** *Let G be a group of order 243. Then $B_0(G) \neq 0$ if and only if $G = G(3^5, i)$ with $28 \leq i \leq 30$, where $G(3^5, i)$ is the i-th group of order 243 in the GAP database* [GAP]. *Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq \mathbb{Z}/3\mathbb{Z}$.*

Moravec [Mor12] gave a formula for $B_0(G)$ by using a nonabelian exterior square $G \wedge G$ of G and an implemented algorithm **b0g.g** in computer algebra system GAP [GAP], which is available from his website `www.fmf.uni-lj.si/~moravec/Papers/b0g.g`. The number of all solvable groups G of order ≤ 729 apart from the orders 512, 576 and 640 with $B_0(G) \neq 0$ was given as in [Mor12, Table 1].

Hoshi, Kang and Kunyavskii [HKK13] determined p-groups G of order p^5 with $B_0(G) \neq 0$ for any $p \geq 3$. It turns out that they belong to the same isoclinism family.

Definition 3.15 (Hall [Hal40, page 133])**.** Let G be a finite group. Let $Z(G)$ be the center of G and $[G,G]$ be the commutator subgroup of G. Two p-groups G_1 and G_2 are called *isoclinic* if there exist group isomorphisms $\theta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi\colon [G_1,G_1] \to [G_2,G_2]$ such that $\phi([g,h]) = [g',h']$ for any $g,h \in G_1$ with $g' \in$

$\theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$:

$$\begin{array}{ccc} G_1/Z_1 \times G_1/Z_1 & \xrightarrow{(\theta,\theta)} & G_2/Z_2 \times G_2/Z_2 \\ {\scriptstyle [\cdot,\cdot]}\downarrow & \circlearrowleft & \downarrow{\scriptstyle [\cdot,\cdot]} \\ [G_1, G_1] & \xrightarrow{\quad\phi\quad} & [G_2, G_2]. \end{array}$$

For a prime p and an integer n, we denote by $G_n(p)$ the set of all non-isomorphic groups of order p^n. In $G_n(p)$, consider an equivalence relation: two groups G_1 and G_2 are equivalent if and only if they are isoclinic. Each equivalence class of $G_n(p)$ is called an *isoclinism family*, and the j-th isoclinism family is denoted by Φ_j.

For $p \geq 5$ (resp. $p = 3$), there exist $2p + 61 + \gcd\{4, p-1\} + 2\gcd\{3, p-1\}$ (resp. 67) groups G of order p^5 which are classified into ten isoclinism families $\Phi_1, \ldots, \Phi_{10}$ (see [Jam80, Section 4]). The main theorem of [HKK13] can be stated as follows:

Theorem 3.16 (Hoshi, Kang and Kunyavskii [HKK13, Theorem 1.12]). *Let p be any odd prime and G be a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10}. Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq \mathbb{Z}/p\mathbb{Z}$.*

For the last statement, see [Kan14, Remark, page 424]. The proof of Theorem 3.16 was given by purely algebraic way. There exist exactly 3 groups which belong to Φ_{10} if $p = 3$, i.e. $G = G(243, i)$ with $28 \leq i \leq 30$. This agrees with Moravec's computational result (Theorem 3.14). For $p \geq 5$, there exist exactly $1 + \gcd\{4, p-1\} + \gcd\{3, p-1\}$ groups which belong to Φ_{10} (see [Jam80, page 621]).

The following result for the k-rationality of $k(G)$ supplements Theorem 3.14 although it is unknown whether $k(G)$ is k-rational for groups G which belong to Φ_7:

Theorem 3.17 (Chu, Hoshi, Hu and Kang [CHHK15, Theorem 1.13]). *Let G be a group of order 243 with exponent e. If $B_0(G) = 0$ and k be a field containing a primitive e-th root of unity, then $k(G)$ is k-rational except possibly for the five groups G which belong to Φ_7, i.e. $G = G(243, i)$ with $56 \leq i \leq 60$.*

In [HKK13] and [CHHK15], not only the evaluation of the Bogomolov multiplier $B_0(G)$ and the k-rationality of $k(G)$ but also the k-isomorphisms between $k(G_1)$ and $k(G_2)$ for some groups G_1 and G_2 belonging to the same isoclinism family were given.

Bogomolov and Böhning [BB13] gave an answer to the question raised as [HKK13, Question 1.11] in the affirmative as follows.

Theorem 3.18 (Bogomolov and Böhning [BB13, Theorem 6]). *If G_1 and G_2 are isoclinic, then $\mathbb{C}(G_1)$ and $\mathbb{C}(G_2)$ are stably $\mathbb{C}$-isomorphic. In particular, $H^i_{\rm nr}(\mathbb{C}(G_1), \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\rm nr}(\mathbb{C}(G_2), \mu_n^{\otimes j})$.*

A partial result of Theorem 3.18 was already given by Moravec. Indeed, Moravec [Mor14, Theorem 1.2] proved that if G_1 and G_2 are isoclinic, then $B_0(G_1) \simeq B_0(G_2)$.

The case where G is a group of order 64.

The classification of the groups G of order $64 = 2^6$ with $B_0(G) \neq 0$ was obtained by Chu, Hu, Kang and Kunyavskii [CHKK10]. Moreover, they investigated Noether's problem for groups G with $B_0(G) = 0$. There exist 267 groups G of order 64 which are classified into 27 isoclinism families $\Phi_1, \ldots, \Phi_{27}$ by Hall and Senior [HS64] (see also [JNO90, Table I]). The main result of [CHKK10] can be stated in terms of the isoclinism families as follows.

Theorem 3.19 (Chu, Hu, Kang and Kunyavskii [CHKK10]). *Let $G = G(2^6, i)$, $1 \leq i \leq 267$, be the i-th group of order 64 in the GAP database* [GAP].
(1) ([CHKK10, Theorem 1.8]) *$B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16}, i.e. $G = G(2^6, i)$ with $149 \leq i \leq 151$, $170 \leq i \leq 172$, $177 \leq i \leq 178$ or $i = 182$. Moreover, if $B_0(G) \neq 0$, then $B_0(G) \simeq \mathbb{Z}/2\mathbb{Z}$ (see* [Kan14, Remark, page 424] *for this statement);*
(2) ([CHKK10, Theorem 1.10]) *If $B_0(G) = 0$ and k is an quadratically closed field, then $k(G)$ is k-rational except possibly for five groups which belong to Φ_{13}, i.e. $G = G(2^6, i)$ with $241 \leq i \leq 245$.*

For groups G which belong to Φ_{13}, k-rationality of $k(G)$ is unknown. The following two propositions supplement the cases Φ_{13} and Φ_{16} of Theorem 3.19. For the proof, the case of $G = G(2^6, 149)$ is given in [HKK14, Proof of Theorem 6.3], see also [CHKK10, Example 5.11, page 2355] and the proof for other cases can be obtained by the similar manner.

Definition 3.20. Let k be a field with char $k \neq 2$ and $k(X_1, X_2, X_3, X_4, X_5, X_6)$ be the rational function field over k with variables $X_1, X_2, X_3, X_4, X_5, X_6$.
(1) *The field $L_k^{(0)}$* is defined to be $k(X_1, X_2, X_3, X_4, X_5, X_6)^H$ where $H = \langle \sigma_1, \sigma_2 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ acts on $k(X_1, X_2, X_3, X_4, X_5, X_6)$ by k-automorphisms

$$\sigma_1 : X_1 \mapsto X_3,\ X_2 \mapsto \frac{1}{X_1X_2X_3},\ X_3 \mapsto X_1,\ X_4 \mapsto X_6,\ X_5 \mapsto \frac{1}{X_4X_5X_6},\ X_6 \mapsto X_4,$$
$$\sigma_2 : X_1 \mapsto X_2,\ X_2 \mapsto X_1,\ X_3 \mapsto \frac{1}{X_1X_2X_3},\ X_4 \mapsto X_5,\ X_5 \mapsto X_4,\ X_6 \mapsto \frac{1}{X_4X_5X_6}.$$

(2) *The field $L_k^{(1)}$* is defined to be $k(X_1, X_2, X_3, X_4)^{\langle \tau \rangle}$ where $\langle \tau \rangle \simeq C_2$ acts on $k(X_1, X_2, X_3, X_4)$ by k-automorphisms

$$\tau : X_1 \mapsto -X_1,\ X_2 \mapsto \frac{X_4}{X_2},\ X_3 \mapsto \frac{(X_4 - 1)(X_4 - X_1^2)}{X_3},\ X_4 \mapsto X_4.$$

Proposition 3.21 ([CHKK10, Proposition 6.3], see also [HY17, Proposition 12.5]). *Let G be a group of order 64 which belongs to Φ_{13}, i.e. $G = G(2^6, i)$ with $241 \le i \le 245$. There exists a $\mathbb{C}$-injective homomorphism $\varphi : L_{\mathbb{C}}^{(0)} \to \mathbb{C}(G)$ such that $\mathbb{C}(G)$ is rational over $\varphi(L_{\mathbb{C}}^{(0)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(0)}$ are stably $\mathbb{C}$-isomorphic and $B_0(G) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(0)}) = 0$.*

Proposition 3.22 ([CHKK10, Example 5.11], [HKK14, Proof of Theorem 6.3]). *Let G be a group of order 64 which belongs to Φ_{16}, i.e. $G = G(2^6, i)$ with $149 \le i \le 151$, $170 \le i \le 172$, $177 \le i \le 178$ or $i = 182$. There exists a $\mathbb{C}$-injective homomorphism $\varphi : L_{\mathbb{C}}^{(1)} \to \mathbb{C}(G)$ such that $\mathbb{C}(G)$ is rational over $\varphi(L_{\mathbb{C}}^{(1)})$. In particular, $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are stably $\mathbb{C}$-isomorphic, $B_0(G) \simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(1)}) \simeq \mathbb{Z}/2\mathbb{Z}$ and hence $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(1)}$ are not (retract, stably) $\mathbb{C}$-rational.*

Question 3.23 ([CHKK10, Section 6], [HY17, Section 12]). Is $L_k^{(0)}$ k-rational?

The case where G is a group of order 128.

There exist 2328 groups of order 128 which are classified into 115 isoclinism families $\Phi_1, \ldots, \Phi_{115}$ ([JNO90, Tables I, II, III]).

Theorem 3.24 (Moravec [Mor12, Section 8, Table 1]). *Let G be a group of order 128. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{16}, Φ_{30}, Φ_{31}, Φ_{37}, Φ_{39}, Φ_{43}, Φ_{58}, Φ_{60}, Φ_{80}, Φ_{106} or Φ_{114}. Moreover, we have*

$$B_0(G) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } G \text{ belongs to } \Phi_{16}, \Phi_{31}, \Phi_{37}, \Phi_{39}, \Phi_{43}, \Phi_{58}, \Phi_{60}, \Phi_{80}, \Phi_{106} \text{ or } \Phi_{114}, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & \text{if } G \text{ belongs to } \Phi_{30}. \end{cases}$$

In particular, $\mathbb{C}(G)$ is not (retract, stably) $\mathbb{C}$-rational.

It turns out that there exist 220 groups G of order 128 with $B_0(G) \neq 0$:

Family	Φ_{16}	Φ_{31}	Φ_{37}	Φ_{39}	Φ_{43}	Φ_{58}	Φ_{60}	Φ_{80}	Φ_{106}	Φ_{114}	Φ_{30}
$\exp(G)$	8	4	8	4 or 8	8	8	8	16	8	8	4
$B_0(G)$	$\mathbb{Z}/2\mathbb{Z}$										$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
# of G's	48	55	18	6	26	20	10	9	2	2	34

It is natural to ask the (stably) birational classification of $\mathbb{C}(G)$ for groups G of order 128. In particular, what happens to $\mathbb{C}(G)$ with $B_0(G) \neq 0$? The following theorem (Theorem 3.26) gives a partial answer to this question.

Definition 3.25. Let k be a field with char $k \neq 2$ and $k(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ be the rational function field over k with variables $X_1, X_2, X_3, X_4, X_5, X_6, X_7$.

(1) *The field* $L_k^{(2)}$ is defined to be $k(X_1,X_2,X_3,X_4,X_5,X_6)^{\langle\rho\rangle}$ where $\langle\rho\rangle\simeq C_4$ acts on $k(X_1,X_2,X_3,X_4,X_5,X_6)$ by k-automorphisms

$$\rho : X_1\mapsto X_2, X_2\mapsto -X_1, X_3\mapsto X_4, X_4\mapsto X_3,$$
$$X_5\mapsto X_6, X_6\mapsto \frac{(X_1^2X_2^2-1)(X_1^2X_3^2+X_2^2-X_3^2-1)}{X_5}.$$

(2) *The field* $L_k^{(3)}$ is defined to be $k(X_1,X_2,X_3,X_4,X_5,X_6,X_7)^{\langle\lambda_1,\lambda_2\rangle}$ where $\langle\lambda_1,\lambda_2\rangle\simeq C_2\times C_2$ acts on $k(X_1,X_2,X_3,X_4,X_5,X_6,X_7)$ by k-automorphisms

$$\lambda_1 : X_1\mapsto X_1, X_2\mapsto \frac{X_1}{X_2}, X_3\mapsto \frac{1}{X_1X_3}, X_4\mapsto \frac{X_2X_4}{X_1X_3},$$
$$X_5\mapsto -\frac{X_1X_6^2-1}{X_5}, X_6\mapsto -X_6, X_7\mapsto X_7,$$
$$\lambda_2 : X_1\mapsto \frac{1}{X_1}, X_2\mapsto X_3, X_3\mapsto X_2, X_4\mapsto \frac{(X_1X_6^2-1)(X_1X_7^2-1)}{X_4},$$
$$X_5\mapsto -X_5, X_6\mapsto -X_1X_6, X_7\mapsto -X_1X_7.$$

Theorem 3.26 (Hoshi [Hos16, Theorem 1.31])**.** *Let G be a group of order* 128*. Assume that $B_0(G)\neq 0$. Then $\mathbb{C}(G)$ and $L_{\mathbb{C}}^{(m)}$ are stably $\mathbb{C}$-isomorphic where*

$$m=\begin{cases}1 & \text{if } G \text{ belongs to } \Phi_{16},\Phi_{31},\Phi_{37},\Phi_{39},\Phi_{43},\Phi_{58},\Phi_{60} \text{ or } \Phi_{80},\\ 2 & \text{if } G \text{ belongs to } \Phi_{106} \text{ or } \Phi_{114},\\ 3 & \text{if } G \text{ belongs to } \Phi_{30}.\end{cases}$$

In particular, $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(1)})\simeq \mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(2)})\simeq \mathbb{Z}/2\mathbb{Z}$ *and* $\mathrm{Br}_{\mathrm{nr}}(L_{\mathbb{C}}^{(3)})\simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ *and hence the fields $L_{\mathbb{C}}^{(1)}$, $L_{\mathbb{C}}^{(2)}$ and $L_{\mathbb{C}}^{(3)}$ are not (retract, stably) $\mathbb{C}$-rational.*

For $m=1,2$, the fields $L_{\mathbb{C}}^{(m)}$ and $L_{\mathbb{C}}^{(3)}$ are not stably $\mathbb{C}$-isomorphic because their unramified Brauer groups are not isomorphic. However, we do not know whether the fields $L_{\mathbb{C}}^{(1)}$ and $L_{\mathbb{C}}^{(2)}$ are stably $\mathbb{C}$-isomorphic. If not, it is interesting to evaluate the higher unramified cohomologies.

§ 4. Rationality problem for multiplicative invariant fields

Let k be a field, G be a finite group and $\rho : G\to GL(V)$ be a faithful representation of G where V is a finite-dimensional vector space over k. Then G acts on the rational function field $k(V)$.

We consider the rationality problem for $k(V)^G$. By No-name Lemma (cf. Miyata [Miy71, Remark 3]), it is known that $k(G)$ is stably k-rational if and only if so is $k(V)^G$

where $\rho : G \to GL(V)$ is any faithful representation of G over k. Thus the rationality problem of $k(V)^G$ over k is also called Noether's problem.

In order to solve the rationality problem of $k(V)^G$, it is natural and almost inevitable that we reduce the problem to that of the multiplicative invariant field $k(M)^G$ defined in Definition 4.2; an illustration of reducing Noether's problem to the multiplicative invariant field can be found in, e.g. [CHKK10], [HKY11, Example 13.7].

When M is a G-lattice with $\mathrm{rank}_{\mathbb{Z}} M = n$, the multiplicative invariant field $k(M)^G$ is nothing but $k(x_1, \ldots, x_n)^G$, the fixed field of the rational function field $k(x_1, \ldots, x_n)$ on which G acts by multiplicative actions.

Definition 4.1. Let G be a finite group and $\mathbb{Z}[G]$ be the group ring. A finitely generated $\mathbb{Z}[G]$-module M is called a *G-lattice* if, as an abelian group, M is a free abelian group of finite rank. We will write $\mathrm{rank}_{\mathbb{Z}} M$ for the rank of M as a free abelian group. A G-lattice M is called *faithful* if, for any $\sigma \in G \setminus \{1\}$, $\sigma \cdot x \neq x$ for some $x \in M$.

Suppose that G is any finite group and $\Phi : G \to GL_n(\mathbb{Z})$ is a group homomorphism, i.e. an integral representation of G. Then the group $\Phi(G)$ acts naturally on the free abelian group $M := \mathbb{Z}^{\oplus n}$; thus M becomes a $\mathbb{Z}[G]$-module. We call M the *G-lattice associated to Φ (or $\Phi(G)$)*. Conversely, if M is a G-lattice with $\mathrm{rank}_{\mathbb{Z}} M = n$, write $M = \oplus_{1 \le i \le n} \mathbb{Z} \cdot x_i$. Then there is a group homomorphism $\Phi : G \to GL_n(\mathbb{Z})$ defined as follows: If $\sigma \cdot x_i = \sum_{1 \le j \le n} a_{ij}\, x_j$ where $\sigma \in G$ and $a_{ij} \in \mathbb{Z}$, define $\Phi(\sigma) = (a_{ij})_{1 \le i,j \le n} \in GL_n(\mathbb{Z})$.

When the group homomorphism $\Phi : G \to GL_n(\mathbb{Z})$ is injective, the corresponding G-lattice is a faithful G-lattice. For examples, any finite subgroup G of $GL_n(\mathbb{Z})$ gives rise to a faithful G-lattice of rank n.

The list of all the finite subgroups of $GL_n(\mathbb{Z})$ (with $n \le 4$), up to conjugation, can be found in the book [BBNWZ78] and in GAP [GAP]. As to the situations of $GL_n(\mathbb{Z})$ (with $n \ge 5$), Plesken etc. found the lists of all the finite subgroups of $GL_n(\mathbb{Z})$ (with $n = 5$ and 6); see [PS00] and the references therein. These lists may be found in the GAP package CARAT [CARAT] and also in [HY17, Chapter 3].

Here is a list of the total number of lattices, up to isomorphism, of a given rank:

rank	1	2	3	4	5	6
# of G-lattices	2	13	73	710	6079	85308

Definition 4.2. Let M be a G-lattice of rank n and write $M = \oplus_{1 \le i \le n} \mathbb{Z} \cdot x_i$. For any field k, define $k(M) = k(x_1, \ldots, x_n)$ the rational function field over k with n variables $x_1, \ldots, x_n$. Define a *multiplicative action* of G on $k(M)$: For any $\sigma \in G$, if $\sigma \cdot x_i = \sum_{1 \le j \le n} a_{ij}\, x_j$ in the G-lattice M, then we define $\sigma \cdot x_i = \prod_{1 \le j \le n} x_j^{a_{ij}}$ in the field $k(M)$. Note that G acts trivially on k. The above multiplicative action is called a

purely monomial action of G on $k(M)$ in [HK92] and $k(M)^G$ is called a *multiplicative invariant field* in [Sal87].

When M is the G-lattice $\mathbb{Z}[G]$ where $M = \oplus_{g\in G}\mathbb{Z}\cdot x_g$ and $h\cdot x_g = x_{hg}$ for $h,g\in G$, we have $k(M) = k(x_g \mid g\in G)$ and $k(M)^G = k(G)$ (see Section 1). Note that $k(G) = k(V_{\rm reg})^G$ where $G\to GL(V_{\rm reg})$ is the regular representation of G over k.

Theorem 4.3 (Hajja [Haj87]). *Let k be a field and G be a finite group acting on $k(x_1,x_2)$ by monomial k-automorphisms. Then $k(x_1,x_2)^G$ is k-rational.*

Theorem 4.4 (Hajja and Kang [HK92, HK94], Hoshi and Rikuna [HR08]). *Let k be a field and G be a finite group acting on $k(x_1,x_2,x_3)$ by purely monomial k-automorphisms. Then $k(x_1,x_2,x_3)^G$ is k-rational.*

Theorem 4.5 (Hoshi, Kang and Kitayama [HKK14, Theorem 1.16]). *Let k be a field, G be a finite group and M be a G-lattice with* $\mathrm{rank}_{\mathbb{Z}}M = 4$ *such that G acts on $k(M)$ by purely monomial k-automorphisms. If M is decomposable, i.e. $M = M_1\oplus M_2$ as $\mathbb{Z}[G]$-modules where* $1\le \mathrm{rank}_{\mathbb{Z}}M_1\le 3$*, then $k(M)^G$ is k-rational.*

Theorem 4.6 (Hoshi, Kang and Kitayama [HKK14, Theorem 6.2]). *Let k be a field, G be a finite group and M be a G-lattice such that G acts on $k(M)$ by purely monomial k-automorphisms. Assume that* (i) *$M = M_1\oplus M_2$ as $\mathbb{Z}[G]$-modules where* $\mathrm{rank}_{\mathbb{Z}}M_1 = 3$ *and* $\mathrm{rank}_{\mathbb{Z}}M_2 = 2$, (ii) *either M_1 or M_2 is a faithful G-lattice. Then $k(M)^G$ is k-rational except the following situation:* char $k\neq 2$, *$G = \langle\sigma,\tau\rangle\simeq D_4$ and $M_1 = \bigoplus_{1\le i\le 3}\mathbb{Z}x_i$, $M_2 = \bigoplus_{1\le j\le 2}\mathbb{Z}y_j$ such that $\sigma: x_1\leftrightarrow x_2,\ x_3\mapsto -x_1-x_2-x_3,\ y_1\mapsto y_2\mapsto -y_1$, $\tau: x_1\leftrightarrow x_3,\ x_2\mapsto -x_1-x_2-x_3,\ y_1\leftrightarrow y_2$ where the $\mathbb{Z}[G]$-module structure of M is written additively. For the exceptional case, $k(M)^G$ is not retract k-rational.*

Definition 4.7. Let k be a field and μ be a multiplicative subgroup of $k\setminus\{0\}$ containing all the roots of unity in k. If M is a G-lattice, a *μ-extension* is an exact sequence of $\mathbb{Z}[G]$-modules given by $(\alpha): 1\to\mu\to M_\alpha\to M\to 0$ where G acts trivially on μ. Be aware that $M_\alpha = \mu\oplus M$ as abelian groups, but not as $\mathbb{Z}[G]$-modules except when the extension (α) splits.

As in Definition 4.2, if $M = \oplus_{1\le i\le n}\mathbb{Z}\cdot x_i$ and M_α is a μ-extension, we define the field $k_\alpha(M) = k(x_1,\ldots,x_n)$ the rational function field over k with n variables $x_1,\ldots,x_n$; the action of G on $k_\alpha(M)$ will be described in the next paragraph. Note that M_α is embedded into the multiplicative group $k_\alpha(M)\setminus\{0\}$ by sending $(\epsilon,\sum_{1\le i\le n}b_ix_i)\in\mu\oplus M$ to the element $\epsilon\prod_{1\le i\le n}x_i^{b_i}$ in the field $k_\alpha(M) = k(x_1,\ldots,x_n)$.

The group G acts on $k_\alpha(M)$ by a twisted multiplicative action: Suppose that, in M we have $\sigma\cdot x_i = \sum_{1\le j\le n}a_{ij}\,x_j$, and in M_α we have $\sigma\cdot x_i = \varepsilon_i(\sigma)+\sum_{1\le j\le n}a_{ij}\,x_j$ where $\varepsilon_i(\sigma)\in\mu$. Then we define $\sigma\cdot x_i = \varepsilon_i(\sigma)\prod_{1\le j\le n}x_j^{a_{ij}}$ in $k_\alpha(M)$. Again G acts trivially on the coefficient field k. The above group action is called *monomial group action* in [HK92] and $k_\alpha(M)^G$ is called *twisted multiplicative invariant field* in [Sal90].

Note that, if the extension $(\alpha) : 1 \to \mu \to M_\alpha \to M \to 0$ is a split extension, then $k_\alpha(M) = k(M)$ and the twisted multiplicative action is reduced to the multiplicative action in Definition 4.2.

For any faithful linear representation $G \to GL(V)$ of G, we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G) \simeq B_0(G)$ by No-name Lemma (see [Sal90]).

The formula in [Sal90, Theorem 12] (Theorem 3.5) can be used to compute not only $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G)$, but also $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}_\alpha(M)^G)$ where $\mathbb{C}_\alpha(M)$ is the rational function field associated to the μ-extension M_α:

Theorem 4.8 (Saltman [Sal90, Theorem 12]). *Let k be an algebraically closed field with char $k = 0$, and G be a finite group. If M is a G-lattice and* $(\alpha) : 1 \to \mu \to M_\alpha \to M \to 0$ *is a μ-extension such that* (i) *M is a faithful G-lattice, and* (ii) $H^2(G, \mu) \to H^2(G, M_\alpha)$ *is injective, then*

$$\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G) = \bigcap_A \mathrm{Ker}\{\mathrm{res} : H^2(G, M_\alpha) \to H^2(A, M_\alpha)\}$$

where A runs over all the bicyclic subgroups of G.

In particular, if the μ-extension $(\alpha) : 1 \to \mu \to M_\alpha \to M \to 0$ *splits, then* $\mathrm{Br}_{\mathrm{nr}}(k(M)^G) \simeq B_0(G) \oplus \bigcap_A \mathrm{Ker}\{\mathrm{res} : H^2(G, M) \to H^2(A, M)\}$ *where A runs over bicyclic subgroups of G.*

Definition 4.9. By Definition 3.3, $\mathrm{Br}_{\mathrm{nr}}(K)$ is a subgroup of the Brauer group $\mathrm{Br}(K)$. On the other hand, the map of the Brauer groups $\mathrm{Br}(k_\alpha(M)^G) \to \mathrm{Br}(k_\alpha(M))$ sends $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ to $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M))$ [Sal87, Theorem 2.1]. Since $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)) = 0$ by [Sal87, Proposition, 2.2], it follows that the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ is a subgroup of the relative Brauer group $\mathrm{Br}(k_\alpha(M)/k_\alpha(M)^G)$. As $\mathrm{Br}(k_\alpha(M)/k_\alpha(M)^G)$ is isomorphic to the cohomology group $H^2(G, k_\alpha(M)^\times)$, we may regard $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ as a subgroup of $H^2(G, k_\alpha(M)^\times)$.

Through the embedding $M_\alpha \hookrightarrow k_\alpha(M)^\times$, there is a canonical injection $H^2(G, M_\alpha) \hookrightarrow \mathrm{Br}(k_\alpha(M)^G)$ [Sal90, page 536]. Identifying $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ and $H^2(G, M_\alpha)$ as subgroups of $H^2(G, k_\alpha(M)^\times)$, we see that $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ is a subgroup of $H^2(G, M_\alpha)$ [Sal90, page 536]. Thus we write $H^2_{\mathrm{nr}}(G, M_\alpha)$ for $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ (see [Sal90]).

Note that there is a natural map $H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, M_\alpha)$. Clearly this map is injective if the μ-extension $(\alpha) : 1 \to \mu \to M_\alpha \to M \to 0$ splits. In this case, regarding $H^2(G, \mathbb{Q}/\mathbb{Z})$ and $H^2(G, M)$ as subgroups of $H^2(G, M_\alpha)$, we define $H^2_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}) = H^2(G, \mathbb{Q}/\mathbb{Z}) \cap \mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$ and $H^2_{\mathrm{nr}}(G, M) = H^2(G, M) \cap \mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G)$. It follows that $\mathrm{Br}_{\mathrm{nr}}(k_\alpha(M)^G) = H^2_{\mathrm{nr}}(G, \mathbb{Q}/\mathbb{Z}) \oplus H^2_{\mathrm{nr}}(G, M)$. By Theorems 3.5 and 4.8, we have

$H^2_{\rm nr}(G,\mathbb{Q}/\mathbb{Z})\simeq B_0(G)$ and $H^2_{\rm nr}(G,M)\simeq\bigcap_A {\rm Ker}\{{\rm res}: H^2(G,M)\to H^2(A,M)\}$ where A runs over bicyclic subgroups of G.

Theorem 4.10 (Barge [Bar89, Theorem II.7]). *Let G be a finite group. Then the following conditions are equivalent:*
(1) *All the Sylow subgroups of G are bicyclic;*
(2) ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)=0$ *for all G-lattices M.*

Theorem 4.11 (Barge [Bar97, Theorem IV-1]). *Let G be a finite group. Then the following conditions are equivalent:*
(1) *All the Sylow subgroups of G are cyclic;*
(2) ${\rm Br}_{\rm nr}(\mathbb{C}_\alpha(M)^G)=0$ *for all G-lattices M, for all short exact sequences of $\mathbb{Z}[G]$-modules $\alpha: 0\to\mathbb{C}^\times\to M_\alpha\to M\to 0$.*

As in Definition 4.9, we have ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\simeq B_0(G)\oplus H^2_{\rm nr}(G,M)$ where $B_0(G)$ is the Bogomolov multiplier and $H^2_{\rm nr}(G,M)\le H^2(G,M)$. We remark that $B_0(G)$ is related to the rationality of $\mathbb{C}(V)^G$ where $G\to GL(V)$ is any faithful linear representation of G over $\mathbb{C}$; on the other hand, $H^2_{\rm nr}(G,M)$ arises from the multiplicative nature of the field $\mathbb{C}(M)^G$.

In case ${\rm rank}_\mathbb{Z} M\le 3$, ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)=0$ for all G-lattices M because $\mathbb{C}(M)^G$ are always $\mathbb{C}$-rational (see Theorem 4.3 and Theorem 4.4). The following theorem [HKY, Theorem 1.10] gives the classification of all the lattices M with ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$ when ${\rm rank}_\mathbb{Z} M\le 6$. Thus $\mathbb{C}(M)^G$ are not retract $\mathbb{C}$-rational for these lattices (and thus are not $\mathbb{C}$-rational).

Let C_n (resp. D_n, QD_{8n}, Q_{8n}) be the cyclic group of order n (resp. the dihedral group of order $2n$, the quasi-dihedral group of order $16n$, the generalized quaternion group of order $8n$).

Theorem 4.12 (Hoshi, Kang and Yamasaki [HKY, Theorem 1.10]). *Let G be a finite group and M be a faithful G-lattice.*
(1) *If* ${\rm rank}_\mathbb{Z} M\le 3$, *then* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)=0$.
(2) *If* ${\rm rank}_\mathbb{Z} M=4$, *then* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$ *if and only if M is one of the* 5 *cases in* Table 1. *Moreover, if M is one of the* 5 *G-lattices with* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$, *then* $B_0(G)=0$ *and* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)=H^2_{\rm nr}(G,M)$.
(3) *If* ${\rm rank}_\mathbb{Z} M=5$, *then* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$ *if and only if M is one of the* 46 *cases in* [HKY, Table 2]. *Moreover, if M is one of the* 46 *G-lattices with* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$, *then* $B_0(G)=0$ *and* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)=H^2_{\rm nr}(G,M)$.
(4) *If* ${\rm rank}_\mathbb{Z} M=6$, *then* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$ *if and only if M is one of the* 1073 *cases as in* [HKY, Table 3]. *Moreover, if M is one of the* 1073 *G-lattices with* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$, *then* $B_0(G)=0$ *and* ${\rm Br}_{\rm nr}(\mathbb{C}(M)^G)=H^2_{\rm nr}(G,M)$, *except for* 24 *cases with* $B_0(G)=$

$\mathbb{Z}/2\mathbb{Z}$ *where the* CARAT ID *of* G *are* $(6,6458,i)$, $(6,6459,i)$, $(6,6464,i)$ $(1 \le i \le 8)$. *Note that* 22 *cases out of the exceptional* 24 *cases satisfy* $H^2_{\rm nr}(G,M)=0$.

Table 1: 5 G-lattices M of rank 4 with $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G) \neq 0$

$G(n,i)$	G	GAP ID	$B_0(G)$	$H^2_{\rm nr}(G,M)$
$(8,3)$	D_4	$(4,12,4,12)$	0	$\mathbb{Z}/2\mathbb{Z}$
$(8,4)$	Q_8	$(4,32,1,2)$	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
$(16,8)$	QD_8	$(4,32,3,2)$	0	$\mathbb{Z}/2\mathbb{Z}$
$(24,3)$	$SL_2(\mathbb{F}_3)$	$(4,33,3,1)$	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
$(48,29)$	$GL_2(\mathbb{F}_3)$	$(4,33,6,1)$	0	$\mathbb{Z}/2\mathbb{Z}$

Remark 4.13. (1) The above theorem remains valid if we replace the coefficient field $\mathbb{C}$ by any algebraically closed field k with char $k=0$.
(2) If M is of rank ≤ 6 and $\mathrm{Br}_{\rm nr}(\mathbb{C}(M^G)) \neq 0$, then G is solvable and non-abelian, and $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G) \simeq \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The case where $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G) \simeq \mathbb{Z}/3\mathbb{Z}$ occurs only for 4 groups G of order 27, 27, 54, 54 with the CARAT ID $(6,2865,1)$, $(6,2865,3)$, $(6,2899,3)$, $(6,2899,5)$ which are isomorphic to $C_9 \rtimes C_3$, $C_9 \rtimes C_3$, $(C_9 \rtimes C_3) \rtimes C_2$, $(C_9 \rtimes C_3) \rtimes C_2$ respectively. For CARAT ID, see Hoshi and Yamasaki [HY17, Chapter 3].
(3) The group G ($\simeq D_4$) which appears as the exceptional case in Theorem 4.6 (i.e. [HKK14, Theorem 6.2]) satisfies the property that $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G) = H^2_{\rm nr}(G,M) \neq 0$ where M is the associated lattice. It follows that $\mathbb{C}(M)^G$ is not retract rational.

In Theorem 4.6, note that both $\mathbb{C}(M_1)^G$ and $\mathbb{C}(M_2)^G$ are rational by Theorem 4.4 and Theorem 4.3. Thus $\mathrm{Br}_{\rm nr}(\mathbb{C}(M_2)^G)=0$ and $H^2_{\rm nr}(G,M_2)=0$. But M_1 is not a faithful G-lattice and we cannot apply Theorem 4.8 to $\mathbb{C}(M_1)^G$. Hence it is possible that $H^2_{\rm nr}(G,M_1)$ is non-trivial. Because $H^2_{\rm nr}(G,M) \simeq H^2_{\rm nr}(G,M_1) \oplus H^2_{\rm nr}(G,M_2)$, this allows for the possibility that $H^2_{\rm nr}(G,M)$ is non-trivial. Indeed, it can be shown that $H^2_{\rm nr}(G,M_1) \simeq \mathbb{Z}/2\mathbb{Z}$ and therefore $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G) = H^2_{\rm nr}(G,M_1) \simeq \mathbb{Z}/2\mathbb{Z}$.
(4) Here is a summary of Theorem 4.12:

$\mathrm{rank}_{\mathbb{Z}} M$	1	2	3	4	5	6
# of G-lattices M	2	13	73	710	6079	85308
# of G-lattices M with $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G) \neq 0$	0	0	0	5	46	1073

Theorem 4.14 (Hoshi, Kang and Yamasaki [HKY, Theorem 4.4]). *The following fields K are stably equivalent each other:*
(1) $\mathbb{C}(G)$ *where* G *is a group of order* 64 *which belongs to the* 16*th isoclinism class* Φ_{16} (*see the* 9 *groups defined as in* Theorem 3.19 (1));

(2) $\mathbb{C}(x_1,x_2,x_3,x_4)^{D_4}$ *where* $D_4=\langle\sigma,\tau\rangle$ *acts on* $\mathbb{C}(x_1,x_2,x_3,x_4)$ *by*

$$\sigma: x_1\mapsto x_2x_3, x_2\mapsto x_1x_3, x_3\mapsto x_4, x_4\mapsto \tfrac{1}{x_3},$$
$$\tau: x_1\mapsto \tfrac{1}{x_2}, x_2\mapsto \tfrac{1}{x_1}, x_3\mapsto \tfrac{1}{x_4}, x_4\mapsto \tfrac{1}{x_3}$$

(*see* Theorem 4.12 (2) and Table 1)*;*
(3) $\mathbb{C}(y_1,y_2,y_3,y_4,y_5)^{D_4}$ *where* $D_4=\langle\sigma,\tau\rangle$ *acts on* $\mathbb{C}(y_1,y_2,y_3,y_4,y_5)$ *by*

$$\sigma: y_1\mapsto y_2, y_2\mapsto y_1, y_3\mapsto \tfrac{1}{y_1y_2y_3}, y_4\mapsto y_5, y_5\mapsto \tfrac{1}{y_4},$$
$$\tau: y_1\mapsto y_3, y_2\mapsto \tfrac{1}{y_1y_2y_3}, y_3\mapsto y_1, y_4\mapsto y_5, y_5\mapsto y_4$$

(*see* Theorem 4.6)*;*
(4) $\mathbb{C}(z_1,z_2,z_3,z_4)^{C_2\times C_2}$ *where* $C_2\times C_2=\langle\sigma,\tau\rangle$ *acts on* $\mathbb{C}(z_1,z_2,z_3,z_4)$ *by*

$$\sigma: z_1\mapsto z_2, z_2\mapsto z_1, z_3\mapsto \tfrac{1}{z_1z_2z_3}, z_4\mapsto \tfrac{-1}{z_4},$$
$$\tau: z_1\mapsto z_3, z_2\mapsto \tfrac{1}{z_1z_2z_3}, z_3\mapsto z_1, z_4\mapsto -z_4$$

(*see* [HKK14, Proof of Theorem 6.4])*;*
(5) $\mathbb{C}(w_1,w_2,w_3,w_4)^{C_2}$ *where* $C_2=\langle\sigma\rangle$ *acts on* $\mathbb{C}(w_1,w_2,w_3,w_4)$ *by*

$$\sigma: w_1\mapsto -w_1, w_2\mapsto \tfrac{w_4}{w_2}, w_3\mapsto \tfrac{(w_4-1)(w_4-w_1^2)}{w_3}, w_4\mapsto w_4$$

(*see* [HKK14, Theorem 6.3])*.*

In particular, the unramified cohomology groups $H^i_{\rm nr}(K,\mathbb{Q}/\mathbb{Z})$ *of the fields* K *in* (1)–(5) *coincide and* $\mathrm{Br}_{\rm nr}(K)\simeq\mathbb{Z}/2\mathbb{Z}$*.*

As in Remark 4.13 (2), all the G-lattices M with $\mathrm{rank}_{\mathbb{Z}}M\le 6$ and $H^2_{\rm nr}(G,M)\neq 0$ in Theorem 4.12 satisfy the condition that G is non-abelian and solvable. Examples of G-lattices M with $H^2_{\rm nr}(G,M)\neq 0$ where G is abelian (resp. non-solvable; in fact, simple) are given in [HKY] as follows:

Theorem 4.15 (Hoshi, Kang and Yamasaki [HKY, Theorem 6.1])**.** *Let G be an elementary abelian group of order 2^n in $GL_7(\mathbb{Z})$ and M be the associated G-lattice of rank 7. Then* $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^G)\neq 0$ *if and only if G is isomorphic up to conjugation to one of the nine groups $G_1,\ldots,G_9\le GL_7(\mathbb{Z})$ as in* [HKY, Theorem 6.1] *where each of G_i is isomorphic to $(C_2)^3$ as an abstract group. Moreover,* $\mathrm{Br}_{\rm nr}(\mathbb{C}(M)^{G_i})=H^2_{\rm nr}(G_i,M)\simeq$ $\mathbb{Z}/2\mathbb{Z}$ *(resp.* $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$*) for $1\le i\le 8$ (resp. $i=9$).*

Theorem 4.16 (Hoshi, Kang and Yamasaki [HKY, Theorem 6.2])**.** *Embed A_6 into S_{10} through the isomorphism $A_6\simeq PSL_2(\mathbb{F}_9)$, which acts on the projective line $\mathbb{P}^1_{\mathbb{F}_9}$ via fractional linear transformations. Thus we may regard A_6 as a transitive subgroup of*

S_{10}. Let $N=\oplus_{1\le i\le 10}\mathbb{Z}\cdot x_i$ be the S_{10}-lattice defined by $\sigma\cdot x_i=x_{\sigma(i)}$ for any $\sigma\in S_{10}$; it becomes an A_6-lattice by restricting the action of S_{10} to A_6. Define $M=N/(\mathbb{Z}\cdot\sum_{i=1}^{10}x_i)$ with $\mathrm{rank}_{\mathbb{Z}}M=9$. There exist exactly six A_6-lattices $M=M_1, M_2,\ldots,M_6$ which are $\mathbb{Q}$-conjugate but not $\mathbb{Z}$-conjugate to each other; in fact, all these M_i form a single $\mathbb{Q}$-class, but this $\mathbb{Q}$-class consists of six $\mathbb{Z}$-classes. Then we have

$$H^2_{\mathrm{nr}}(A_6,M_1)\simeq H^2_{\mathrm{nr}}(A_6,M_3)\simeq\mathbb{Z}/2\mathbb{Z},\quad H^2_{\mathrm{nr}}(A_6,M_i)=0\text{ for }i=2,4,5,6.$$

In particular, $\mathbb{C}(M_1)^{A_6}$ and $\mathbb{C}(M_3)^{A_6}$ are not retract $\mathbb{C}$-rational. Furthermore, the lattices M_1 and M_3 may be distinguished by the Tate cohomology groups:

$$\begin{aligned} &H^1(A_6,M_1)=0, && \widehat{H}^{-1}(A_6,M_1)=\mathbb{Z}/10\mathbb{Z},\\ &H^1(A_6,M_3)=\mathbb{Z}/5\mathbb{Z}, && \widehat{H}^{-1}(A_6,M_3)=\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

Motivated by the G-lattices in Theorem 4.12 (2) (see Table 1), the following G-lattices M of rank $2n+2$, $4n$ and $p(p-1)$ (n is any positive integer and p is any odd prime number) with $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G)\neq 0$ were constructed in [HKY]:

Theorem 4.17 (Hoshi, Kang and Yamasaki [HKY, Theorem 7.2]). *Let $G=\langle\sigma,\tau\mid\sigma^{4n}=\tau^2=1,\tau^{-1}\sigma\tau=\sigma^{-1}\rangle\simeq D_{4n}$, the dihedral group of order $8n$ where n is any positive integer. Let M be the G-lattice of rank $2n+2$ defined in* [HKY, Definition 7.1]. *Then $H^2_{\mathrm{nr}}(G,M)\simeq\mathbb{Z}/2\mathbb{Z}$. Consequently, $\mathbb{C}(M)^G$ is not retract $\mathbb{C}$-rational.*

Theorem 4.18 (Hoshi, Kang and Yamasaki [HKY, Theorem 7.5]).
(1) *Let n be any positive integer and $G=\langle\sigma,\tau\mid\sigma^{8n}=\tau^2=1,\tau^{-1}\sigma\tau=\sigma^{4n-1}\rangle\simeq QD_{8n}$ be the quasi-dihedral group of order $16n$. Let M be the G-lattice of rank $4n$ defined in* [HKY, Definition 7.4]. *Then $H^2_{\mathrm{nr}}(G,M)\simeq\mathbb{Z}/2\mathbb{Z}$. Consequently, $\mathbb{C}(M)^G$ is not retract $\mathbb{C}$-rational.*
(2) *Let $\widehat{G}=\langle\sigma^2,\sigma\tau\rangle\simeq Q_{8n}\le G$ be the generalized quaternion group of order $8n$. Let $\widehat{M}=\mathrm{Res}^G_{\widehat{G}}(M)$ be the $\widehat{G}$-lattice of rank $4n$ defined in* [HKY, Definition 7.4]. *Then $H^2_{\mathrm{nr}}(\widehat{G},\widehat{M})\simeq\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Consequently, $\mathbb{C}(\widehat{M})^{\widehat{G}}$ is not retract $\mathbb{C}$-rational.*

Theorem 4.19 (Hoshi, Kang and Yamasaki [HKY, Theorem 7.7]). *Let p be an odd prime and $G=\langle\sigma,\tau\mid\sigma^{p^2}=\tau^p=1,\tau^{-1}\sigma\tau=\sigma^{p+1}\rangle\simeq C_{p^2}\rtimes C_p$. Let M be the G-lattice of rank $p(p-1)$ defined in* [HKY, Definition 7.6]. *Then $H^2_{\mathrm{nr}}(G,M)\simeq\mathbb{Z}/p\mathbb{Z}$. Consequently, $\mathbb{C}(M)^G$ is not retract $\mathbb{C}$-rational.*

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