A combinatorial bijection on k-noncrossing partitions*

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Abstract

For any integer $k \geq 2$, we prove combinatorially the following Euler (binomial) transformation identity

$$\mathrm{NC}_{n+1}^{(k)}(t) = t\sum_{i=0}^{n}\binom{n}{i}\mathrm{NW}_i^{(k)}(t),$$

where $\mathrm{NC}_m^{(k)}(t)$ (resp. $\mathrm{NW}_m^{(k)}(t)$) is the enumerative polynomial on partitions of $\{1,\ldots,m\}$ avoiding k-crossings (resp. enhanced k-crossings) by number of blocks. The special $k=2$ and $t=1$ case, asserting the Euler transformation of Motzkin numbers are Catalan numbers, was discovered by Donaghey 1977. The result for $k=3$ and $t=1$, arising naturally in a recent study of pattern avoidance in ascent sequences and inversion sequences, was proved only analytically.

It is based on the preprint (arXiv:1905.10526) with Zhicong Lin.

§1 Introduction

We begin with the definition of set partition of $[n]=\{1,2,\ldots,n\}$. A family of nonempty subsets of $[n]$, $P=\{B_1,B_2,\ldots,B_k\}$, is a set partition of $[n]$ with k blocks, if B_i's are mutually disjoint, and $\cup_i B_i=[n]$. Let Π_n denote the set of all set partitions of $[n]$. The Stirling number of the second kind, $S(n,k)$, is the number of set partitions of $[n]$ with k blocks.

Example 1 *Elements of* Π_4 *are usually listed as follows:* 1234, 123−4, 124−3, 134−2, 1−234, 12−34, 13−24, 14−23, 12−3−4, 13−2−4, 14−2−3, 1−23−4, 1−24−3, 1−2−34, 1−2−3−4

There are many different ways of representing set partitions of $[n]$. One of them is a representation by arc diagram. Any $P \in \Pi_n$ can be identified with its *arc diagram* defined as follows:

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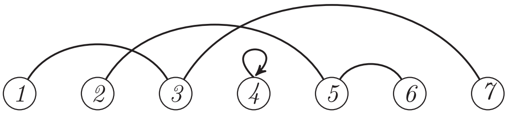
Definition 2 (Arc diagram of a partition) *Nodes are* $1, 2, \ldots, n$ *from left to right. There is an arc from* i *to* j, $i < j$, *whenever both* i *and* j *belong to the same block, say* $B \in P$, *and* B *contains no* l *with* $i < l < j$. *There is a loop from* i *to itself if* $\{i\}$ *is a block in* P.

Example 3 *The arc diagram of* $\{\{1,3,7\},\{2,5,6\},\{4\}\} \in \Pi_7$.



Arc diagram representation allows us to define 'crossing' in a set partiton. A partition has a *crossing* if there exist two arcs (i_1, j_1) and (i_2, j_2) in its arc diagram such that $i_1 < i_2 < j_1 < j_2$.

It is well known that the number of partitions in Π_n with no crossings is given by the n-th *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The crossings of partitions have a natural generalization called k-crossings for any fixed integer $k \geq 2$. For instance, the arc diagram of $\{\{1,3,7\},\{2,5,6\},\{4\}\} \in \Pi_7$ has two crossings:

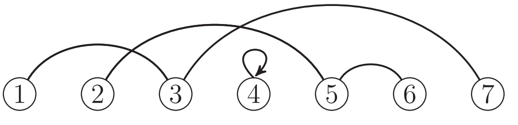
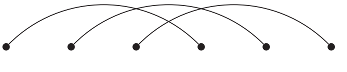


Figure 1: $\{(1,3),(2,5)\}$ and $\{(2,5),(3,7)\}$ are crossings.

Crossings in set partitions can be generalized into k-crossings. A *k-crossing* of $P \in \Pi_n$ is a k-subset $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ of arcs in the arc diagram of P such that

$$i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k.$$

A partition without any k-crossing is called a *k-noncrossing* partition. A 3-crossing is depicted below:



A *weak k-crossing* of $P \in \Pi_n$ is a k-subset $\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ of arcs in the arc diagram of P such that

$$i_1 < i_2 < \cdots < i_k = j_1 < j_2 < \cdots < j_k.$$

The k-crossings and weak k-crossings of P are collectively called the *enhanced k-crossings* of P. A partition without any enhanced k-crossing is an *enhanced k-noncrossing* partition. A 3-crossing and a weak 3-crossing are depicted below:

We can classify set partitions by the number of k-crossings.

Definition 4 *Let* $\mathrm{NC}_n^{(k)}$ *be the set of all* k*-noncrossing partitions in* Π_n*.*

Note that *k-noncrossing* means no k-crossing.

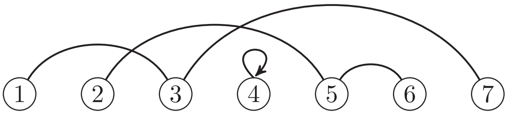
Definition 5 *Let* $\mathrm{NW}_n^{(k)}$ *be the set of all enhanced* k*-noncrossing partitions in* Π_n*.*

Note that *enhanced k-noncrossing* means no k-crossing and no weak k-crossing.

If k is sufficently large, i.e. $k > \frac{n+1}{2}$, then we have $\mathrm{NW}_n^{(k)} = \mathrm{NC}_n^{(k)} = \Pi_n$.

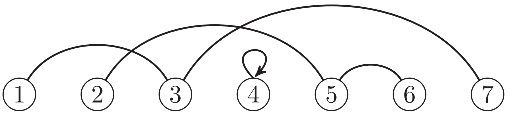
Definition 6 *Let* $\mathrm{NC}_m^{(k)}(t)$ *be the generating polynomial of* k*-noncrossing partitions of* $[m]$ *by number of blocks.*

The following contributes t^3 to $\mathrm{NC}_7^{(3)}(t)$.



Definition 7 *Let* $\mathrm{NW}_m^{(k)}(t)$ *be the generating polynomial of enhanced* k*-noncrossing partitions of* $[m]$ *by number of blocks.*

The following contributes t^3 to $\mathrm{NW}_7^{(4)}(t)$.



§2 Main Result

The following is the main result. A bijective proof of this theorem will be introduced later.

Theorem 8 *For* $n \geq 1$ *and* $k \geq 2$*,*

$$\mathrm{NC}_{n+1}^{(k)}(t) = t \sum_{i=0}^{n} \binom{n}{i} \mathrm{NW}_i^{(k)}(t), \tag{1}$$

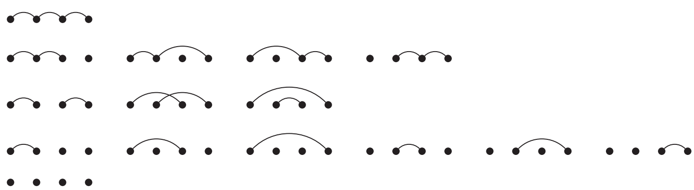
where $\mathrm{NW}_0^{(k)}(t) = 1$ *by convention.*

There are several partial results that lead to the discovery of (1).

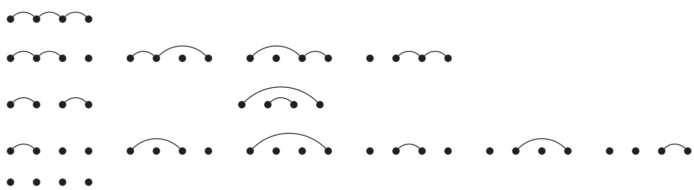
We illustrate the case of $(n, k) = (3, 2)$. The identity (1) for this case is

$$\mathrm{NC}_4^{(2)}(t) = t \sum_{i=0}^{3} \binom{3}{i} \mathrm{NW}_i^{(2)}(t).$$

There are 15 partitions in Π_4 whose arc diagrams are drawn below:



Close examination of the list reveals that there are 14 partitions in $\mathrm{NC}_4^{(2)}$ listed below:



Collecting their weights, the generating polynomial is $\mathrm{NC}_4^{(2)}(t) = t+6t^2+6t^3+t^4$.

The right hand side of the identity has four terms, involving $\mathrm{NW}_i^{(2)}(t)$ for $i = 0, 1, 2, 3$. These generating polynomials are shown below:

$\mathrm{NW}_0^{(2)}(t) = 1$

$\mathrm{NW}_1^{(2)}(t) = t$

$\mathrm{NW}_2^{(2)}(t) = t + t^2$

$\mathrm{NW}_3^{(2)}(t) = 3t^2 + t^3$

We can confirm that the above $\mathrm{NC}_4^{(2)}(t)$ and $\mathrm{NW}_i^{(2)}(t)$ for $i = 0, 1, 2, 3$ satisfy the identity.

The $k = 2$ and $t = 1$ case of the identity (1) is interesting. Enhanced 2-noncrossing partitions in Π_n are *noncrossing partial matchings* of $[n]$, i.e. noncrossing partitions for which the blocks have size one or two. Noncrossing partial matchings of $[n]$ are counted by the n-th *Motzkin number* $M_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i$, identity (1) reduces to

$$C_{n+1} = \sum_{i=0}^{n} \binom{n}{i} M_i, \tag{2}$$

which is well known. But a t-extension of (2), seems new:

$$C_{n+1}(t) = t \sum_{i=0}^{n} \binom{n}{i} M_i(t), \tag{3}$$

where $C_n(t)$ and $M_n(t)$ denote the generating functions of noncrossing partitions of $[n]$ and noncrossing partial matchings of $[n]$.

If k is sufficently large, i.e. $k > \frac{n+1}{2}$, then we have $\mathrm{NW}_n^{(k)} = \mathrm{NC}_n^{(k)} = \Pi_n$, and

$$\mathrm{NC}_{n+1}^{(k)}(t) = t\sum_{i=0}^{n}\binom{n}{i}\mathrm{NW}_i^{(k)}(t)$$

is equivalent to

$$\text{for all } m \geq 0, \quad S(n+1, m+1) = \sum_{i=0}^{n}\binom{n}{i}S(i,m),$$

where $S(a,b)$ denotes the *Stirling number of the second kind*.

Our objective is to prove that for all k,

$$\mathrm{NC}_{n+1}^{(k)}(t) = t\sum_{i=0}^{n}\binom{n}{i}\mathrm{NW}_i^{(k)}(t)$$

holds as a polynomial in t. For $t = 1$, the above identity has multiple proofs. But they, except what comes next, do not prove the above as a polynomial in t. We will first illustrate our bijective proof of (1), for $k = 2$,

$$\mathrm{NC}_{n+1}^{(2)}(t) = t\sum_{i=0}^{n}\binom{n}{i}\mathrm{NW}_i^{(2)}(t),$$

for noncrossing partitions, and then extend it to all k-noncrossing partitions.

The extension of our construction from $k = 2$ to general k is nontrivial. So we show our framework for the noncrossing partition case first.

From now on, we let Π_n denote the set of partitions of $\{0, 1, \ldots, n-1\}$ rather than partitions of $[n]$, for convenience's sake.

We give a combinatorial interpretation of identity (3),

$$C_{n+1}(t) = t\sum_{i=0}^{n}\binom{n}{i}M_i(t).$$

First we interpret the right hand side

$$t\sum_{i=0}^{n}\binom{n}{i}M_i(t)$$

as the generating function of all pairs (A, μ) such that A is a subset of $\{1, 2, \ldots, n\}$ and μ is a noncrossing matching whose nodes are elements of A placed on the line in the natural order. A pair (A, μ) is weighted by $t^{|\mu|+1}$, where $|\mu|$ is the number of blocks of μ. If A is the empty set, then μ is the empty matching with weight t.

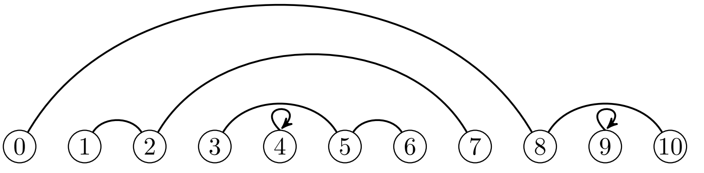
§3 Combinatorial bijections

We now define a combinatorial bijection Ψ from noncrossing partitions in Π_{n+1} to the set of all pairs (A, μ) in the above.

Let $n = 10$ and

$$\pi = \{\{0,8,10\},\{1,2,7\},\{3,5,6\},\{4\},\{9\}\}.$$

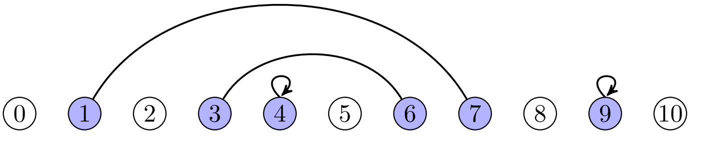
This π is a noncrossing partition:



Consider all blocks in π which do not contain 0: $\{\{1,2,7\},\{3,5,6\},\{4\},\{9\}\}$. From each block, delete all integers which are neither the smallest nor the largest in the block. Let the resulting set be μ, and let A be the union of all blocks in μ:

$$(A,\mu) = (\{1,3,4,6,7,9\},\{\{1,7\},\{3,6\},\{4\},\{9\}\})$$

The next figure shows the elements of A, in blue, and the matching μ.



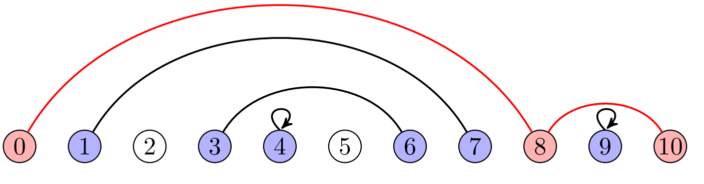
Let $\Psi(\pi) = (A,\mu)$. Clearly, this is weight-preserving.

The above procedure is reversible. Let (A,μ) be a pair such that A is a subset of $\{1,2,\ldots,n\}$ and μ is a noncrossing matching whose nodes are elements of A placed on the line in the natural order.

We will construct the corresponding partition π of $\{0,1,2,\ldots,n\}$ as follows. Interpret each block β in μ as an interval $I(\beta) = \{i : \min\{\beta\} \le i \le \max\{\beta\}\}$. Let the block of π containing 0 be

$$\{0,1,2,\ldots,n\} \setminus \cup_{\beta\in\mu} I(\beta).$$

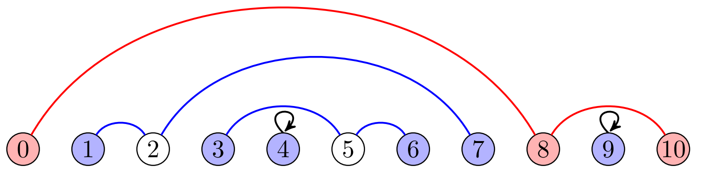
As an example, let $n = 10$ and $(A,\mu) = (\{1,3,4,6,7,9\},\{\{1,7\},\{3,6\},\{4\},\{9\}\})$. The block containing 0 is $\{0,8,10\}$, shown in red below.



Other blocks of π are obtained by extending blocks in μ by the rule:

> $i \in \{1,2,\ldots,n\} \setminus A$ belongs to the block originating from a block $\beta \in \mu$ if $I(\beta)$ is the smallest interval containing i.

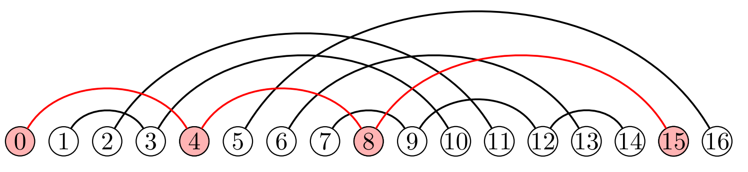
In our example, two blocks $\{1, 7\}$ and $\{3, 6\}$ are enlarged, shown in blue below.



So $\Psi^{-1}(\pi) = \{\{0, 8, 10\}, \{1, 2, 7\}, \{3, 5, 6\}, \{4\}, \{9\}\}$.

Since the block containing 0 is important in our discussion, we fix the following terminology.

Definition 9 (Red block, colored arc diagram) *In a partition P, the block containing 0 is called a* **red block**, *denoted by* red(P), *and other blocks are called* **black blocks**. *The elements in* red(P) *are colored red, and other elements are colored black. Arcs in arc diagram of P between red elements are colored red and other arcs are colored black. Such a colored version of arc diagram of P is called the colored arc diagram, denoted by $D(P)$.*



Let's recall what we want to prove, i.e. Theorem 8: For $n \geq 1$ and $k \geq 2$,

$$\mathrm{NC}_{n+1}^{(k)}(t) = t \sum_{i=0}^{n} \binom{n}{i} \mathrm{NW}_i^{(k)}(t), \tag{4}$$

where $\mathrm{NW}_0^{(k)}(t) = 1$ by convention.

To prove the above identity combinatorially, first we need to interpret the identity combinatorially. The right-hand side will be associated to $\mathrm{NBW}_n^{(k)}$ which is defined below. A partition is called **k-crossing** if it has at least one k-crossing. A k-crossing is called a **black k-crossing**, if all its arcs are black; a **red k-crossing**, otherwise. A weak k-crossing is called a **black weak k-crossing**, if all its arcs are black; a **red weak k-crossing**, otherwise.

Let's recall that

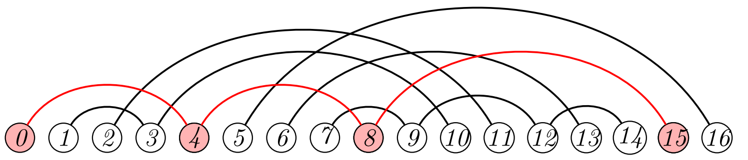
- $\mathrm{NC}_n^{(k)}$ is the set of all k-noncrossing partitions in Π_n.
- $\mathrm{NW}_n^{(k)}$ is the set of all partitions in Π_n which avoid enhanced k-crossings, i.e., have neither k-crossings nor weak k-crossings. (Enhanced k-noncrossing)

Definition 10 *Let $\mathrm{NBW}_n^{(k)}$ be the set of all partitions P in Π_n whose colored arc diagram, $D(P)$, has neither black k-crossings nor black weak k-crossings.*

An example follows.

Example 11 $P \in \mathrm{NBW}_{17}^{(3)}$ *and its colored diagram* $D(P)$*:*

$$P = \{\{0,4,8,15\},\{1,3,10\},\{2,11\},\{5,16\},\{6,13\},\{7,9,12,14\}\}$$



- *There are no black* 3*-crossings and no black weak* 3*-crossings.*
- *There are red* 3*-crossings and red weak* 3*-crossings.*

Decomposition of $\mathrm{NBW}_n^{(k)}$
For any subset A of $\{1,2,\ldots,n-1\}$, define a subset Π_A of $\Pi_n = \Pi_{\{0,1,\ldots,n-1\}}$ by

$$\Pi_A = \{P \in \Pi_n : \mathrm{red}(P) = \{0,1,\ldots,n-1\} \setminus A\}.$$

Π_n is partitioned into $\{\Pi_A\}_{A \subseteq \{1,2,\ldots,n-1\}}$, and there is a natural correspondence between Π_A and $\Pi_{|A|}$. If $A = \{a_1, a_2, \cdots, a_l\}$ with $a_1 < a_2 < \cdots < a_l$ then the correspondence is obtained by mapping a_i to $i-1$ for each i. This correspondence reduces the number of blocks by 1, since the red block is ignored. We define a subset $\mathrm{NBW}_A^{(k)}$ of $\mathrm{NBW}_n^{(k)}$ by

$$\mathrm{NBW}_A^{(k)} = \Pi_A \cap \mathrm{NBW}_n^{(k)}.$$

We can see that $\mathrm{NBW}_n^{(k)}$ is partitioned into

$$\{\mathrm{NBW}_A^{(k)}\}_{A \subseteq \{1,2,\ldots,n-1\}},$$

and there is a natural correspondence between $\mathrm{NBW}_A^{(k)}$ and $\mathrm{NW}_{|A|}^{(k)}$, i.e., the restriction of the natural correspondence between Π_A and $\Pi_{|A|}$.

Define a weight function w on Π_n by $w(P) = t^{|P|}$ for each $P \in \Pi_n$, where $|P|$ denotes the number of blocks in P. Since we have

$$\sum_{P \in \mathrm{NC}_{n+1}^{(k)}} w(P) = \mathrm{NC}_{n+1}^{(k)}(t)$$

and

$$\begin{aligned} \sum_{P \in \mathrm{NBW}_{n+1}^{(k)}} w(P) &= \sum_{A \subseteq \{1,\ldots,n\}} \sum_{P \in \mathrm{NBW}_A^{(k)}} w(P) \\ &= \sum_{A \subseteq \{1,\ldots,n\}} t \sum_{P \in \mathrm{NW}_{|A|}^{(k)}} w(P) \\ &= t \sum_{i=0}^{n} \binom{n}{n-i} \mathrm{NW}_i^{(k)}(t), \end{aligned}$$

identity (1) is equivalent to the following theorem.

Theorem 12 *For all n and k, there exists a weight-preserving combinatorial bijection* $\Phi : \mathrm{NBW}_{n+1}^{(k)} \to \mathrm{NC}_{n+1}^{(k)}$ *proving*

$$\sum_{P \in \mathrm{NBW}_{n+1}^{(k)}} w(P) = \sum_{P \in \mathrm{NC}_{n+1}^{(k)}} w(P).$$

First of all, let's begin with a rough plan of proof:

1. Change red nodes under black $(k-1)$-crossing into centers of black weak k-crossings.
2. Change red k-crossings into red nodes under black $(k-1)$-crossings.

We can describe the details but rigorous proofs for all steps are too complicated to introduce here. So the descriprion for the desired bijection is given without proof. In stead we illustrate the bijection by an example.

Since a $(k-1)$-noncrossing partition has no enhanced k-crossings and an enhanced k-noncrossing partition has no k-crossings,

$$\mathrm{NC}_n^{(k-1)} \subseteq \mathrm{NW}_n^{(k)} \subseteq \mathrm{NC}_n^{(k)}$$

for all $k \geq 3$. The combinatorial bijection, proving the above theorem,

$$\Phi : \mathrm{NBW}_{n+1}^{(k)} \to \mathrm{NC}_{n+1}^{(k)}$$

is constructed by the following steps:

1. Let $P = \{B_0, B_1, \ldots, B_l\} \in \mathrm{NBW}_{n+1}^{(k)}$ with $\mathrm{red}(P) = B_0$. If $P \in \mathrm{NBW}_{n+1}^{(k-1)}$ then P belongs to $\mathrm{NC}_{n+1}^{(k)}$ and we can set $\Phi(P) = P$. Otherwise, $P \in \mathrm{NBW}_{n+1}^{(k)} \setminus \mathrm{NBW}_{n+1}^{(k-1)}$.
2. Start with $D(P)$, the colored arc diagram of P.
3. If there exists a red node under a black $(k-1)$-crossing in $D(P)$, do **'enhanced left shift'** on $D(P)$, i.e.,
   - let a be the **smallest** such red node,
   - let $(i_1, j_1), (i_2, j_2), \ldots, (i_{k-1}, j_{k-1})$ be the **innermost** (that is to say the word $(j_1, j_2, \ldots, j_{k-1})$ is smallest in the lexicographic order) black $(k-1)$-crossing covering a,
   - change arcs forming a black $(k-1)$-crossing $(i_1, j_1), (i_2, j_2), \ldots, (i_{k-1}, j_{k-1})$ into arcs of a black weak k-crossing $(i_1, a), (i_2, j_1), \ldots, (i_{k-1}, j_{k-2}), (a, j_{k-1})$ with a as the center,
   - set $B_0 = B_0 \setminus \{a\}$, and let $\tilde{P}$ denote the resulting partition.

   Repeat this step until the colored arc diagram of $\tilde{P}$ has no red node under a black $(k-1)$-crossing. The resulting partition $\tilde{P}$ has no black k-crossing.
4. If $D(\tilde{P})$ has no red k-crossing, then set $\Phi(P) = \tilde{P}$; otherwise, do **'cyclic rotation'** on $D(\tilde{P})$, i.e.,
   - find the **rightmost** red arc in a k-crossing, say (i, j),
   - let $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ be the **greatest**, in the lexicographic order of $(j_1, j_2, \ldots, j_k)$, k-crossing with $(i_p, j_p) = (i, j)$ (here p is always greater than 1 in the process).

- change arcs forming a k-crossing $(i_1,j_1),(i_2,j_2),\dots,(i_p,j_p),\dots,(i_k,j_k)$ into arcs

$$(i_1,j_2),(i_2,j_3),\dots,(i_{p-1},j_p),(i_p,j_1),(i_{p+1},j_{p+1}),\dots,(i_k,j_k),$$

  where (i_p,j_1) is recolored red,
- color j_p black, j_1 red, and recolor the nodes in the blocks containing j_p and j_1 accordingly.

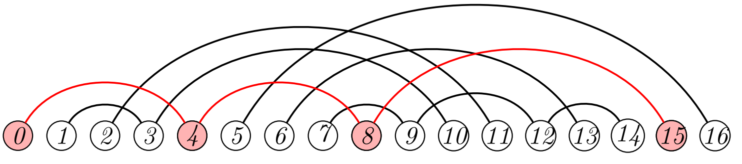
Repeat this process until the resulting colored arc diagram has no red k-crossing. The partition P' corresponding to the resulting colored arc diagram has no black k-crossing.

5. Finally we end up with a partition P' in $\mathrm{NC}^{(k)}_{n+1}$. Set $\Phi(P)=P'$.

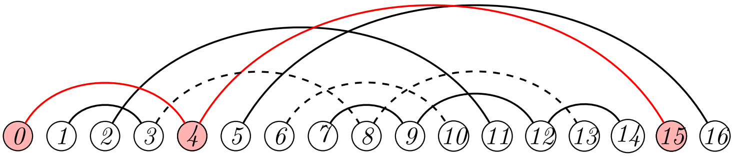
Note that if $P\in \mathrm{NBW}^{(k)}_{n+1}$ has a $(k-1)$-crossing then so does $\Phi(P)$ but the converse is not true. The reader is invited to check that Φ agrees with Ψ^{-1} when $k=2$, even though they are defined differently.

Example 13 *An example of* Φ *with* $(n,k)=(16,3)$ *and* $P\in\mathrm{NBW}^{(3)}_{17}$*:*

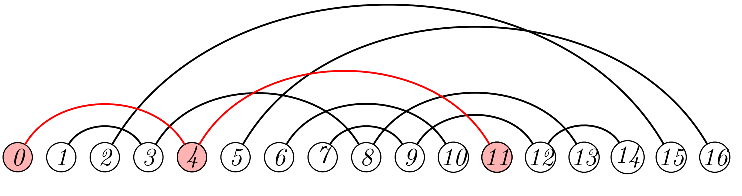
$$P=\{\{0,4,8,15\},\{1,3,10\},\{2,11\},\{5,16\},\{6,13\},\{7,9,12,14\}\}.$$



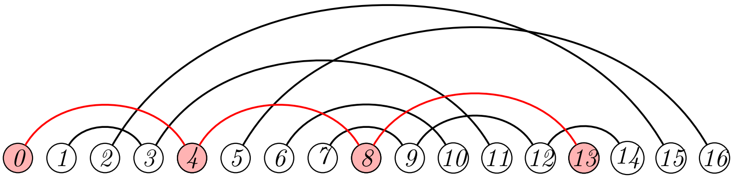
Red 8 *is under four black* 2*-crossings of which the innermost is* $(3,10),(6,13)$*. Make* 8 *the center of a black weak* 3*-crossing,* $(3,8),(6,10),(8,13)$*, and uncolor* 8*.*



Dashed arcs form the weak 3*-crossing. Arcs* $(2,11),(4,15),(5,16)$ *form a red* 3*-crossing. Do 'cyclic rotation':* $(2,11),(4,15),(5,16)\to(2,15),(4,11),(5,16)$.



Arcs $(3,8),(4,11),(5,16)$ *form a red* 3*-crossing. Do 'cyclic rotation':* $(3,8),(4,11),(5,16)\to(3,11),(4,8),(5,16)$.



The last colored arc diagram corresponds to $\Phi(P) \in \mathrm{NC}_{17}^{(3)}$*:*

$$\Phi(P) = (\{0,4,8,13\},\{1,3,11\},\{2,15\},\{5,16\},\{6,10\},\{7,9,12,14\}).$$

The crucial reason why Φ is reversible is that any 'cyclic rotation' to a red k-crossing leaves a trace, i.e., a red node under a black $(k-1)$-crossing. In fact, though we do not introduce it here, we can show that Φ is a bijection by defining its inverse explicitly.

The combinatorial bijections prove the main result but this is the only proof up to now. There is no algebraic or analytical proof yet. So we have the following question.

Problem 14 *Is there any generating function approach to* (1)*?*

Remark

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