Coproduct for the Yangian of type $A_2^{(2)}$

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§1. Finite Yangian

First, we recall the definition and some properties of the Yangian of finite dimensional simple Lie algebra.

Suppose that $\mathfrak{g}$ is a finite dimensional simple Lie algebra. Then, the Yangian $Y_h(\mathfrak{g})$ associated with $\mathfrak{g}$ is one kind of quantum group, that is,

1. $Y_h(\mathfrak{g})$ is a Hopf algebra associated with $\mathfrak{g}$ and a complex number h.
2. When $h = 0$, it coincides with the universal enveloping algebra of the current algebra $\mathfrak{g}[u]$.

Drinfeld found three different presentations of the finite Yangian; the RTT presentation, the Drinfeld J presentation, and the Drinfeld presentation. The first one is closely related to solutions of the Yang-Baxter equation. The second one is the original definition given by Drinfeld. By using the third one, we define Yangians of general symmetrizable Kac-Moody Lie algebras.

Let us recall the each presentations.

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§1.1. the RTT presentation

Definition 0.1. The Yangian for $\mathfrak{gl}_n$ is an associative algebra whose generators are $\{t_{i,j}^{(r)} | r = 1, \cdots, s, 1 \leq i, j \leq n\}$ with the following defining relations;

$$[t_{i,j}^{(r)}, t_{k,l}^{(s)}] = \delta_{k,j} t_{i,l}^{(r+s-1)} - \delta_{i,l} t_{k,j}^{(r+s-1)} + h \sum_{a=2}^{\min\{r,s\}} (t_{k,j}^{(a-1)} t_{i,l}^{(r+s-a-1)} - t_{k,j}^{(r+s-a)} t_{i,l}^{(a-1)})$$

The Yangian for $\mathfrak{gl}_n$ has a coproduct defined by

$$\Delta(t_{i,j}^{(r)}) = t_{i,j}^{(r)} \otimes 1 + 1 \otimes t_{i,j}^{(r)} + h \sum_{k=1}^{n} \sum_{s=1}^{r-1} t_{i,k}^{(s)} \otimes t_{k,j}^{(r-s)}.$$

As a quantum group, when $h = 0$, the Yangian for $\mathfrak{gl}_n$ is equal to the universal enveloping algebra of $\mathfrak{gl}_n \otimes \mathbb{C}[u]$.

Let us recall two applications of the Yangian for $\mathfrak{gl}_n$. First one gives the solutions of the quantum Yang-Baxter equation. Second one gives the defining relations of finite W-algebras of type A. In this report, we only write the result of the second one. This is the work of Brundan and Kleshechev ([BK]).

Theorem 0.2. *There exists a surjective homomorphism from subalgebras of the Yangian for $\mathfrak{gl}_n$, which are called shifted Yangians, to finite W-algebras of type A.*

§1.2. the Drinfeld J presentation

Suppose that $\mathfrak{g}$ is a finite dimensional simple Lie algebra.

Definition 0.3. Current algebra $\mathfrak{g}[u] = \mathfrak{g} \otimes \mathbb{C}[u]$ is a Lie algebra whose commutator relation is

$$\text{relation}: \ [x \otimes u^m, y \otimes u^n] = [x, y] \otimes u^{m+n}.$$

$U(\mathfrak{g}[u])$ has a minimalistic presentation as follows;

Proposition 0.4. *$U(\mathfrak{g}[u])$ is generated by $\{x, J(x) \mid x \in \mathfrak{g}\}$ with the following equa-*

tions;

$$
\begin{aligned}
&xy - yx = [x, y] \text{ for all } x, y \in \mathfrak{g},\\
&J(ax + by) = aJ(x) + bJ(y) \text{ for all } a, b \in \mathbb{C},\\
&J([x, y]) = [x, J(y)],\\
&[J(x), J([y, z])] + [J(z), J([x, y])] + [J(y), J([z, x])] = 0,\\
&[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] = 0,
\end{aligned}
$$

where $J(x)$ is corresponding to $x \otimes u$.

The definition of the Yangian is a deformation of this minimalistic presentation.

Definition 0.5. $Y_h(\mathfrak{g})$ is an associative algebra generated by $x, J(x)(x \in \mathfrak{g})$ subject to the following defining relations:

$$
\begin{aligned}
&xy - yx = [x, y] \text{ for all } x, y \in \mathfrak{g},\\
&J(ax + by) = aJ(x) + bJ(y) \text{ for all } a, b \in \mathbb{C},\\
&J([x, y]) = [x, J(y)],\\
&[J(x), J([y, z])] + [J(z), J([x, y])] + [J(y), J([z, x])]\\
&= h^2 \sum_{a,b,c \in \mathtt{A}} ([x, \xi_a], [[y, \xi_b], [z, \xi_c]])\{\xi_a, \xi_b, \xi_c\},\\
&[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]]\\
&= h^2 \sum_{a,b,c \in \mathtt{A}} \big(([x, \xi_a], [[y, \xi_b], [[z, w], \xi_c]])\\
&\qquad\qquad + ([z, \xi_a], [[w, \xi_b], [[x, y], \xi_c]])\big)\{\xi_a, \xi_b, J(\xi_c)\}
\end{aligned}
$$

where $\{\xi_a, \xi_b, \xi_c\} := \frac{1}{24} \sum_{\pi \in S_3} \xi_{\pi(a)} \xi_{\pi(b)} \xi_{\pi(c)}$.

It also has a Hopf algebra structure.

1. coproduct

$$
\begin{aligned}
\Delta_h(x) &:= x \otimes 1 + 1 \otimes x\\
\Delta_h(J(x)) &:= J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2} h[x \otimes 1, \Omega],
\end{aligned}
$$

2. antipode

$$
S_h(x) = -x, \quad S_h(J(x)) = -J(x) + \frac{1}{4} cx
$$

3. counit

$$\varepsilon_h(x) = \varepsilon_h(J(x)) = 0$$

where Ω is a Casimir element of $\mathfrak{g}$ and c is an eigenvalue of Ω in the adjoint representation of $\mathfrak{g}$.

When $h = 0$, it is equal to the universal enveloping algebra of the current algebra. Thus, $Y_h(\mathfrak{g})$ is a quantization of $\mathfrak{g}$.

By using the Drinfeld J-presentation, Ragoucy and Sorba ([RS]) have obtained the following theorem.

Theorem 0.6. *There exists a surjective homomorohism from the finite Yangian of type A to rectangular finite W-algebras of tyoe A.*

In Section 4, we give the analogy of this theorem in the affine (super) setting.

§2. Affine Yangian

Recently, the Yangian is gotten attention. One of the trigger is the result of Schiffmann and Vasserot.

Theorem 0.7. *There exists a homomorphism from the affine Yangian of $\widehat{\mathfrak{gl}}(1)$ to the universal evnveloping algebras of principal W-algebras of type A. Moreover, the image of this homomorphism is dense in the universal evnveloping algebra of principal W-algebras of type A.*

Unfortunately, the RTT presentation and the Drinfeld J presentation cannot be naturally extended. We need to introduce the third presentation, called the Drinfeld presentation.

§2.1. the Drinfeld presentation

Suppose that $(\mathfrak{g}, \mathfrak{h}, A = \{(a_{ij})\}_{i,j\in I}, \{x_i^{\pm}, h_i\}_{i\in I})$ is a symmetrizable Kac-Moody Lie algebra such that $(x_i^+, x_i^-) = 1$. We set Δ (resp. Δ^{re}, Δ^{im}, Δ^+, Δ^-) as the set of roots (resp. of real roots, of imaginary roots, of positive roots, of negative roots). We take $\{\alpha_i\}_{i\in I}$ as simple roots of $\mathfrak{g}$ and fix a complex number h

Definition 0.8. Yangian $Y_h(\mathfrak{g})$ is the associative algebra over $\mathbb{C}$ with generators $x_{i,s}^{\pm}$, $h_{i,s}$ ($i \in I$, $s \in \mathbb{Z}_{\geq 0}$) subject to the following defining relations:

$$[h_{i,s}, h_{j,r}] = 0, \tag{0.9}$$

$$[h_{i,0}, x_{j,s}^{\pm}] = \pm(\alpha_i, \alpha_j)x_{j,s}^{\pm}, \tag{0.10}$$

$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{ij}h_{i,r+s}, \tag{0.11}$$

$$[h_{i,r+1}, x_{j,s}^{\pm}] - [h_{i,r}, x_{j,s+1}^{\pm}] = \pm\frac{h(\alpha_i, \alpha_j)}{2}(h_{i,r}x_{j,s}^{\pm} + x_{j,s}^{\pm}h_{i,r}), \tag{0.12}$$

$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm\frac{h(\alpha_i, \alpha_j)}{2}(x_{i,r}^{\pm}x_{j,s}^{\pm} + x_{j,s}^{\pm}x_{i,r}^{\pm}), \tag{0.13}$$

$$\sum_{\sigma \in S_{1-a_{ij}}} [x_{i,r_{\sigma(1)}}^{\pm}, \cdots, [x_{i,r_{\sigma(1-a_{ij})}}^{\pm}, x_{j,s}^{\pm}]] = 0. \tag{0.14}$$

We remark some facts derived from the definition.

Lemma 0.15. (1) *When $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra, two definitions coincide.*

(2) · *When $h = 0$, $Y_0(\mathfrak{g})$ is equal to the universal enveloping algebra of the current algebra $[\mathfrak{g}, \mathfrak{g}][u]$.*

· *When $h \in \mathbb{C} \setminus \{0\}$, $Y_h(\mathfrak{g})$ is isomorphic to $Y_1(\mathfrak{g})$.*

Here after, we assume that $h = 1$.

(3) $Y_1(\mathfrak{g})$ *is generated by $x_{i,0}^{\pm}, h_{i,0}, h_{i,1}$.*

Proof. We only prove (3). Let us set $\hat{h}_{i,1} := h_{i,1} - \frac{1}{2}{h_{i,0}}^2$. Then we get

$$\begin{aligned} x_{i,s}^{\pm} &= \pm\frac{1}{(\alpha_i, \alpha_j)}[\hat{h}_{i,1}, x_{i,s-1}^{\pm}], \\ h_{i,s} &= [x_{i,s}^{+}, x_{i,0}^{-}]. \end{aligned}$$

□

After defining the Yangian of symmetrizable Kac-moody Lie algebras, it is natural to consider whether it has a coproduct or not. In the case of affine Lie algebra, it has been settled. Before defining the coproduct, let us prepare the concept of category $\mathcal{O}$.

Definition 0.16. The category $\mathcal{O}$ of modules over the Yangian $Y_1(\mathfrak{g})$ consists of all the modules V such that:

1. V is diagonizable with respect to $\mathfrak{h}$.
2. Each $\mathfrak{h}$-weight space V_μ is finite dimensional for all $\mu \in \mathfrak{h}$.
3. There exists $\lambda_1, \cdots, \lambda_k \in \mathfrak{h}^*$ such that if $V_\mu \neq 0$, then $\lambda_i - \mu \in \sum_{j \in I} \mathbb{Z}_{\geq 0}\alpha_j$ for some $1 \leq i \leq k$.

§2.2. The result of Boyarchenko and Levendorskiĭ

Boyarchenko and Levendorskiĭ defined the "Yangian" Y_h corresponding to $\hat{\mathfrak{sl}}_2$.

Definition 0.17. Y_h is an associative algebra with generators $x^{\pm}_{k,i}, h_{k,i} \quad (k \in \mathbb{Z}, i = 0, 1)$, c and the following commutation relations;

$$\text{c is a central element of } Y_h,$$
$$[h_{0,m}, x^{\pm}_{0,k}] = \pm 2x^{\pm}_{0,m+k}, \quad [x^{+}_{0,m}, x^{-}_{0,n}] = h_{0,m+n} + m\delta_{m+n,0}c,$$
$$[h_{0,m}, h_{0,n}] = 2mc\delta_{m,-n}, \quad [x^{\pm}_{0,m}, x^{\pm}_{0,n}] = 0, \quad [h_{1,0}, x^{\pm}_{0,k}] = \pm 2\delta_{k,0}x^{\pm}_{1,k}$$
$$h_{1,k} = [x^{+}_{1,k}, x^{-}_{0,0}] - \frac{1}{2}h \sum_{m,n \geq 0, m+n=k} h_{0,m}h_{0,n}$$
$$[h_{0,1}, h_0(u)] = -2[x^{-}_0(u)^{++}, x^{+}_0(u)^{+}]_+ + 2[x^{-}_0(u)^{-}, x^{+}_0(u)^{--}]_+$$
$$[x^{+}_1(u), x^{+}_0(v)] = (((x^{+}_0(u)^{+})^2 - ((x^{+}_0(u)^{+}--)^2)\delta(\frac{v}{u})$$
$$[x^{-}_1(u), x^{-}_0(v)] = (((x^{-}_0(u)^{++})^2 - ((x^{+}_0(u)^{-}-)^2)\delta(\frac{v}{u})$$
$$[x^{+}_{01}, x^{+}_1(u)] = x^{+}_0(u)^{+} + h_0(u)^{++}x^{+}_0(u)^{+} - x^{+}_0(u)^{--} + h_0(u)^{-}x^{+}_0(u)^{--}$$
$$[x^{-}_{01}, x^{-}_1(u)] = x^{-}_0(u)^{++} + h_0(u)^{+}x^{-}_0(u)^{++} - x^{-}_0(u)^{-} + h_0(u)^{--}x^{-}_0(u)^{-}$$
$$[h_{0,1}, h_1(u)^{+}] = -2[x^{+}_1(u)^{+}, x^{-}_0(u)^{++}]_+$$
$$[h_{0,1}, h_1(u)^{-}] = 2[x^{-}_1(u)^{--}, x^{+}_0(u)^{+}]_+$$
$$\begin{aligned}[h_0(u), [x^{+}_{0,1}, x^{-}_{0,1}] - h_{0,1}h_{0,0}] = {} & 2[x^{+}_1(u)^{+}, x^{-}_0(u)^{++}]_+ + 2[x^{-}_1(u)^{++}, x^{+}_0(u)^{++}]_+ \\ & - 2[x^{-}_1(u)^{-}, x^{+}_0(u)^{--}]_+ - 2[x^{+}_1(u)^{--}, x^{-}_0(u)^{-}]_+\end{aligned}$$
$$[h_{i,1}, [x^{+}_{i,1}, x^{-}_{i,1}]] = 0.$$

We have the following properties;

1. Y_0 coincides with $U(\hat{\mathfrak{sl}}_2[u])$.
2. There exists an algebra homomorphism $\Phi\colon \hat{\mathfrak{sl}}_2 \to Y_h$ determined by

$$\Phi(c) = c, \quad \Phi(x \otimes t^k) = x^{\pm}_{0,k}, \quad \Phi(h \otimes t^k) = h_{0,k}$$

3. There exists an algebra homomorphism $\Pi\colon Y(\mathfrak{sl}_2) \to Y_h$ determined by

$$\Pi(x^{\pm}_{0,0}) = x^{\pm}_{0,0}, \quad \Pi(\hat{h}_{0,1}) = h_{1,0}$$

4. Suppose that V and W are in the category $\mathcal{O}$ of modules over the Yangian Y_h. Then, $\Delta\colon Y_h \to \mathrm{End}(V \otimes W)$ determined by

$$\Delta(c) = c \otimes 1 + 1 \otimes c, \quad \Delta(d) = d \otimes 1 + 1 \otimes d,$$
$$\Delta(x^{\pm}_{0,k}) = x^{\pm}_{0,k} \otimes 1 + 1 \otimes x^{\pm}_{0,k}, \quad \Delta(h_{0,k}) = h_{0,k} \otimes 1 + 1 \otimes h_{0,k},$$
$$\Delta(h_{1,0}) = h_{1,0} \otimes 1 + 1 \otimes h_{1,0} - 2\sum_{j\geq 0} x^{-}_{0,-j} \otimes x^{+}_{0,j} + 2\sum_{j\geq 0} x^{+}_{0,-j} \otimes x^{-}_{0,j}$$

is an algebra homomorphism.

We remark that there exists another definition of the Yangian of type $A_1^{(1)}$ given by quiver varieties. This definition is as follows.

Definition 0.18. The affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\hat{\mathfrak{sl}}_2)$ is the algebra over $\mathbb{C}(\varepsilon_1,\varepsilon_2)$ generated by $x^{\pm}_{i,r}, h_{i,r}$ $(i \in \mathbb{Z}/2\mathbb{Z}, r \in \mathbb{Z}_{\geq 0})$ subject to the relations:

$$[h_{i,r}, h_{j,s}] = 0, \tag{0.19}$$

,

$$[x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{ij} h_{i,r+s}, \tag{0.20}$$

$$[h_{i,0}, x^{\pm}_{j,r}] = \pm a_{ij} x^{\pm}_{j,r}, \tag{0.21}$$

$$[h_{i,r+1}, x^{\pm}_{i,s}] - [h_{i,r}, x^{\pm}_{i,s+1}] = \pm(\varepsilon_1+\varepsilon_2)(h_{i,r}x^{\pm}_{i,s} + x^{\pm}_{i,s}h_{i,r}), \tag{0.22}$$

$$\begin{aligned}&[h_{i,r+2}, x^{\pm}_{i+1,s}] - 2[h_{i,r+1}, x^{\pm}_{i+1,s+1}] + [h_{i,r}, x^{\pm}_{i+1,s+2}]\\ &\quad \pm (\varepsilon_1+\varepsilon_2)(h_{i,r+1}x^{\pm}_{i+1,s} + x^{\pm}_{i+1,s}h_{i,r+1})\\ &\qquad \mp (\varepsilon_1+\varepsilon_2)(h_{i,r}x^{\pm}_{i+1,s+1} + x^{\pm}_{i+1,s+1}h_{i,r}) + \varepsilon_1\varepsilon_2[h_{i,r}, x^{\pm}_{i+1,s}] = 0,\end{aligned} \tag{0.23}$$

$$[x^{\pm}_{i,r+1}, x^{\pm}_{i,s}] - [x^{\pm}_{i,r}, x^{\pm}_{i,s+1}] = \pm(\varepsilon_1+\varepsilon_2)(x^{\pm}_{i,r}x^{\pm}_{i,s} + x^{\pm}_{i,s}x^{\pm}_{i,r}), \tag{0.24}$$

$$\begin{aligned}&[x^{\pm}_{i,r+2}, x^{\pm}_{i+1,s}] - 2[x^{\pm}_{i,r+1}, x^{\pm}_{i+1,s+1}] + [x^{\pm}_{i,r}, x^{\pm}_{i+1,s+2}]\\ &\quad \pm (\varepsilon_1+\varepsilon_2)(x^{\pm}_{i,r+1}x^{\pm}_{i+1,s} + x^{\pm}_{i+1,s}x^{\pm}_{i,r+1})\\ &\qquad \mp (\varepsilon_1+\varepsilon_2)(x^{\pm}_{i,r}x^{\pm}_{i+1,s+1} + x^{\pm}_{i+1,s+1}x^{\pm}_{i,r}) + \varepsilon_1\varepsilon_2[x^{\pm}_{i,r}, x^{\pm}_{i+1,s}] = 0,\end{aligned} \tag{0.25}$$

$$\sum_{w\in\mathfrak{S}_{1-a_{ij}}}[x^{\pm}_{i,r_{w(1)}},[x^{\pm}_{i,r_{w(2)}},\dots,[x^{\pm}_{i,r_{w(1-a_{ij})}},x^{\pm}_{j,s}]\dots]]=0\ (i\neq j),\tag{0.26}$$

where

$$a_{ij}=\begin{cases}2 \text{ if } (i,j)=(0,0),(1,1),\\ -2 \text{ if } (i,j)=(0,1),(1,0).\end{cases}$$

We do not know whether the defnition of Boyarchenko and Levendorskiĭ is equal to the one given by quiver varieties. Precisely, in the case of type $A_1^{(1)}$, the problem has not been settled yet.

§2.3. The result of Guay, Nakajima, and Wendland

Guay, Nakajima, and Wendland solved the problem of the existance of the coproduct in the case of affine Lie algebras except of types $A_1^{(1)}$ and $A_2^{(2)}$. Their main result is as follows.

Theorem 0.27 ([GNW, Theorem 4.11])**.** *Suppose that $\mathfrak{g}$ is affine Lie algebra except of types $A_1^{(1)}$ and $A_2^{(2)}$. And we also assume V, W are representations of $Y_1(\mathfrak{g})$ in the category $\mathcal{O}$. Then, $\Delta\colon Y_1(\mathfrak{g})\to \mathrm{End}_{\mathbb{C}}(V\otimes W)$ given by*

$$\begin{aligned}
&\Delta(h_{i,0}):=h_{i,0}\otimes 1+1\otimes h_{i,0}\\
&\Delta(x^{\pm}_{i,0}):=x^{\pm}_{i,0}\otimes 1+1\otimes x^{\pm}_{i,0}\\
&\Delta(h_{i,1}):=h_{i,1}\otimes 1+1\otimes h_{i,1}+h_{i,0}\otimes h_{i,0}-\sum_{\alpha\in\Delta_+}(\alpha,\alpha_i)x^k_{-\alpha}\otimes x^k_{\alpha}
\end{aligned}$$

is an algebra homomorphism, where x^k_α is a basis of $\mathfrak{g}_\alpha$ such that $(x^k_\alpha,x^l_{-\alpha})=\delta_{k,l}$. Moreover, Δ satisfies coassociativity.

The outline of the proof is as follows;

1. construction of the minimalistic presentation of $Y_1(\mathfrak{g})$.
2. construction of the operator $J(h_i)$, $J(x_i^{\pm})$ and showing their properties.
3. computation of the compability with the defining relations by the properties of the root system.

First, let us recall the minimalistic presentation of $Y_1(\mathfrak{g})$.

Theorem 0.28. *Suppose that for any $i,j\in I$ with $i\neq j$, the matrix $\begin{pmatrix}a_{ii} & a_{ij}\\ a_{ji} & a_{jj}\end{pmatrix}$ is* *invertible. Moreover, assume also that there exists a pair of indices $i, j \in I$ such that $a_{ij} = -1$. Then, $Y_1(\mathfrak{g})$ is isomorphic to the associative algebra over $\mathbb{C}$ with generators $x^{\pm}_{i,0}$, $h_{i,0}$, $h_{i,1}$ subject to the following defining relations:*

$$[h_{i,s}, h_{j,r}] = 0 \quad (0 \leq r, s \leq 1), \tag{0.29}$$

$$[h_{i,0}, x^{\pm}_{j,s}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,s} \quad (s = 0, 1), \tag{0.30}$$

$$[x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{ij} h_{i,r+s} \quad (0 \leq r + s \leq 1), \tag{0.31}$$

$$[\hat{h}_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,1}, \tag{0.32}$$

$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\frac{(\alpha_i, \alpha_j)}{2}(x^{\pm}_{i,0}x^{\pm}_{j,0} + x^{\pm}_{j,0}x^{\pm}_{i,0}), \tag{0.33}$$

$$\mathrm{ad}(x^{\pm}_{i,0})^{1-a_{ij}}(x^{\pm}_{j,0}) = 0. \tag{0.34}$$

If $\mathfrak{g}$ is a finite dimensional simple Lie algebra except for sl_2 and an affine Lie algebra except for type $A^{(1)}_1$ and type $A^{(2)}_2$, then $\mathfrak{g}$ satisfies

(1) For any $i, j \in I$ with $i \neq j$, the matrix $\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ is invertible.

(2) There exists a pair of indices $i, j \in I$ such that $a_{ij} = -1$.

Then, the Yangian of affine Lie algebras except of types $A^{(1)}_1$ and $A^{(2)}_2$ has this minimalistic presentation. Next, let us recall the definitions of $J(h_i)$ and $J(x^{\pm}_i)$.

Let us set two operators acting on representations in the category $\mathcal{O}$:

$$J(h_i) = h_{i,1} + v_i, v_i := \frac{1}{2}\sum_{\alpha \in \Delta_+}(\alpha, \alpha_i)x_{-\alpha}x_{\alpha} - \frac{1}{2}{h_i}^2,$$

$$J(x^{\pm}_i) = x^{\pm}_i + w^{\pm}_i, w^{\pm}_i := \pm\frac{1}{(\alpha_i, \alpha_j)}[v_i, x^{\pm}_i] + \frac{1}{2}(x^{\pm}_i h_i + h_i x^{\pm}_i).$$

When $\mathfrak{g}$ is a finite dimensional simple Lie algebra,

$$\begin{aligned} Y(\mathfrak{g}) = \langle x, J(x)\rangle &\to Y(\mathfrak{g}) = \langle x^{\pm}_{i,0}, h_{i,1}\rangle \\ x^{\pm}_{i,0} \qquad &\mapsto x^{\pm}_{i,0} \\ J(h_i) \qquad &\mapsto h_{i,1} + \frac{1}{2}\sum_{\alpha \in \Delta_+}(\alpha, \alpha_i)x_{-\alpha}x_{\alpha} - \frac{1}{2}{h_i}^2 \end{aligned}$$

By using this operators, we can rewrite some defining relations of $Y_1(\mathfrak{g})$.

1.

$$[h_{i,0}, h_{j,1}] = 0 \Longleftrightarrow [h_{i,0}, J(h_j)] = 0$$

2.

$$[h_{i,0}, x^{\pm}_{j,1}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,1} \Longleftrightarrow [h_{i,0}, J(x^{\pm}_j)] = \pm(\alpha_i, \alpha_j)J(x^{\pm}_j)$$

3.

$$\begin{aligned}[x^{+}_{i,1}, x^{-}_{j,0}] = [x^{+}_{i,0}, x^{-}j, 1] = \delta_{i,j}h_{i,1} \\ \Longleftrightarrow [J(x^{+}_i), x^{-}_j] = [x^{+}_i, J(x^{-}_j)] = \delta_{i,j}J(h_i)\end{aligned}$$

4.

$$[\hat{h}_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,1} \Longleftrightarrow [J(h_i), x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)J(x^{\pm}_j)$$

5.

$$\begin{aligned}&[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\frac{(\alpha_i, \alpha_j)}{2}(x^{\pm}_{i,0}x^{\pm}_{j,0} + x^{\pm}_{j,0}x^{\pm}_{i,0}) \\ &\Longleftrightarrow [J(x^{\pm}_i), x^{\pm}_j] = [x^{\pm}_i, J(x^{\pm}_j)]\end{aligned}$$

By rewriting defining relations, it is easy to show that Δ is compatible with these relations. Thus, it is enough to check that Δ is compatible with $[h_{i,1}, h_{j,1}] = 0$. In order to prove this, we prepare more relations of $J(h_i)$.

Definition 0.35. α: a positive real root

s_i: the simple reflection corresponding to a simple real root α_i

w: an element of the Weyl group of $\mathfrak{g}$, such that $\alpha = w(\alpha_i)$.

$w = s_1 \cdots s_n$: a reduced expression of w

$$\begin{aligned}x^{\pm}_{\alpha} &:= \tau_{i_1} \cdots \tau_{i_n-1}(x^{\pm}_{i_m}), \\ J(x^{\pm}_{\alpha}) &= \tau_{i_1} \cdots \tau_{i_n-1}J(x^{\pm}_{i_m}),\end{aligned}$$

where $\tau_i := \exp(\mathrm{ad}(x^{+}_{i,0}))\exp(-\mathrm{ad}(x^{-}_{i,0}))\exp(\mathrm{ad}(x^{+}_{i,0}))$.

Then, when $\alpha \in \Delta^{\mathrm{re}}$, we get

$$[J(h_i), x_{\alpha}] = (\alpha, \alpha_i)J(x_{\alpha})$$

By using this relation, we show that Δ is compatible with $[h_{i,1}, h_{j,1}] = 0$ by direct computation. In ordetr to prove this, we use the fact that $(\alpha, \delta) = 0$ when $\mathfrak{g}$ is of affine type. Thus, we do not know whether the Yangians of other symmetrizable Kac-MOody Lie algebras have a coproduct or not.

§3. Main Result

FInally, we state the main result.

Theorem 0.36 ([U1]). *Suppose that $\mathfrak{g}$ is of type $A_2^{(2)}$. And we also assume V, W are representations of $Y_1(\mathfrak{g})$ in the category $\mathcal{O}$. Then, $\Delta\colon Y_1(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V \otimes W)$ given by*

$$\begin{aligned}
&\Delta(h_{i,0}) := h_{i,0} \otimes 1 + 1 \otimes h_{i,0}\\
&\Delta(x_{i,0}^{\pm}) := x_{i,0}^{\pm} \otimes 1 + 1 \otimes x_{i,0}^{\pm}\\
&\Delta(h_{i,1}) := h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + h_{i,0} \otimes h_{i,0} - \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i) x_{-\alpha}^k \otimes x_\alpha^k
\end{aligned}$$

is an algebra homomorphism,.

The outline of the proof is similar to the one of Guay, Nakajima, and Wendland.

1. construction of the minimalistic presentation of $Y_1(\mathfrak{g})$.
2. construction of the operator $J(h_i)$, $J(x_i^{\pm})$ and showing their properties.
3. computation of the compability with the defining relations by the properties of the root system.

Here after, we assume that $\mathfrak{g}$ is of type $A_2^{(2)}$ and fix

$$\begin{aligned}
&I := \{0, 1\},\\
&A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix},\\
&\alpha_0 + 2\alpha_1 = \delta,\\
&(\alpha_1, \alpha_1) = 1,\\
&\Delta = \{\pm 2\alpha_1 + (2n-1)\delta, \pm\alpha_1 + n\delta, n\delta | n \in \mathbb{Z}\}.
\end{aligned}$$

First, we give the minimalistic presentation. Since A is not invertible, the Yangian of type $A_2^{(2)}$ does not have the minimalistic presentation given by Guay, Nakajima, and Wendland. We construct another minimalitstic presentation.

Theorem 0.37. *The affine Yangian $Y_1(\mathfrak{g})$ is isomorphic to the associative algebra* *over $\mathbb{C}$ with generators $x^{\pm}_{i,0}$, $h_{i,0}$, $h_{i,1}$ subject to the following defining relations:*

$$[h_{i,s}, h_{j,r}] = 0 \quad (0 \leq r, s \leq 1), \tag{0.38}$$

$$[h_{i,0}, x^{\pm}_{j,s}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,s} \quad (s = 0, 1), \tag{0.39}$$

$$[x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{ij} h_{i,r+s} \quad (0 \leq r + s \leq 1), \tag{0.40}$$

$$[\hat{h}_{i,1}, x^{\pm}_{j,0}] = \pm(\alpha_i, \alpha_j)x^{\pm}_{j,1}, \tag{0.41}$$

$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\frac{(\alpha_i, \alpha_j)}{2}(x^{\pm}_{i,0}x^{\pm}_{j,0} + x^{\pm}_{j,0}x^{\pm}_{i,0}), \tag{0.42}$$

$$\mathrm{ad}(x^{\pm}_{i,0})^{1-a_{ij}}(x^{\pm}_{j,0}) = 0, \tag{0.43}$$

$$[x^{\pm}_{0,2}, x^{\pm}_{1,0}] - [x^{\pm}_{0,1}, x^{\pm}_{1,1}] = \pm\frac{(\alpha_0, \alpha_1)}{2}(x^{\pm}_{0,1}x^{\pm}_{1,0} + x^{\pm}_{1,0}x^{\pm}_{0,1}), \tag{0.44}$$

where $x^{\pm}_{0,2} := \frac{1}{(\alpha_0,\alpha_0)}[h_{0,1} - \frac{1}{2}{h_{i,0}}^2, [h_{0,1} - \frac{1}{2}{h_{i,0}}^2, x^{+}_{0,0}]]$.

In the similar way as that of Guay, Nakajima, and Wendland, we can prove Δ is compatible with defining relations of minimalistic presentation except of the last one. Thus, it is enough to check that Δ is compatible with the last defining relation. Next, we prove the properties of $J(h_i)$ and $J(x^{\pm}_i)$. In order to prove the compatibility with the last relation, we need to prepare the following relations;

Lemma 0.45. *The following equations hold:*

$$[J(h_i), x_{(2n+1)\delta}] = 0,$$
$$[J(x^{\pm}_i), x_\beta] = J([x^{\pm}_i, x_\beta]) \quad (\textit{if} \quad \beta + \alpha_i \textit{ is a real root}).$$

We prove this relation by using the own properties of type $A^{(2)}_2$. By using this relation, we can prove that Δ is compatible with the defining relations of the minimalistic presentation.

§4. Affine Super Yangian

The proof of the existance of the coproduct for the Yangian of type $A^{(2)}_2$ has an application to the affine super case. We can define the affine super Yangians as follows; (citeU2)

Definition 0.46. Suppose that $m, n \geq 2$ and $m \neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is the associative superalgebra over $\mathbb{C}$ generated by $x^{+}_{i,r}, x^{-}_{i,r}, h_{i,r}$ $(i \in$

$\mathbb{Z}/(m+n)\mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$) with parameters $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ subject to the relations:

$$[h_{i,r}, h_{j,s}] = 0, \tag{0.47}$$

$$[x^+_{i,r}, x^-_{j,s}] = \delta_{ij} h_{i,r+s}, \tag{0.48}$$

$$[h_{i,0}, x^\pm_{j,r}] = \pm a_{ij} x^\pm_{j,r}, \tag{0.49}$$

$$[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] = \pm a_{ij} \frac{\varepsilon_1 + \varepsilon_2}{2} \{h_{i,r}, x^\pm_{j,s}\} - m_{ij} \frac{\varepsilon_1 - \varepsilon_2}{2} [h_{i,r}, x^\pm_{j,s}], \tag{0.50}$$

$$[x^\pm_{i,r+1}, x^\pm_{j,s}] - [x^\pm_{i,r}, x^\pm_{j,s+1}] = \pm a_{ij} \frac{\varepsilon_1 + \varepsilon_2}{2} \{x^\pm_{i,r}, x^\pm_{j,s}\} - m_{ij} \frac{\varepsilon_1 - \varepsilon_2}{2} [x^\pm_{i,r}, x^\pm_{j,s}], \tag{0.51}$$

$$\sum_{w \in \mathfrak{S}_{1+|a_{ij}|}} [x^\pm_{i,r_{w(1)}}, [x^\pm_{i,r_{w(2)}}, \dots, [x^\pm_{i,r_{w(1-|a_{ij}|)}}, x^\pm_{j,s}] \dots]] = 0 (i \neq j), \tag{0.52}$$

$$[x^\pm_{i,r}, x^\pm_{i,s}] = 0 \ (i = 0, m), \tag{0.53}$$

$$[[x^\pm_{i-1,r}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0 \ (i = 0, m), \tag{0.54}$$

where

$$a_{ij} = \begin{cases} -1 & \text{if } (i,j) = (0,1), (1,0), \\ 1 & \text{if } (i,j) = (0, m+n-1), (m+n-1, 0), \\ 2 & \text{if } i = j \leq m-1, \\ -2 & \text{if } i = j \geq m+1, \\ -1 & \text{if } i = j \pm 1 \text{ and } \max\{i,j\} \leq m, \\ 1 & \text{if } i = j \pm 1 \text{ and } \min\{i,j\} \geq m+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$m_{i,j} = \begin{cases} -1 & \text{if } (i,j) = (0,1), (1,0), \\ 1 & \text{if } (i,j) = (0, m+n-1), (m+n-1, 0), \\ a_{i,i+1} & \text{if } i = j-1, \\ -a_{i,i-1} & \text{if } i = j+1, \\ 0 & \text{otherwise,} \end{cases} .$$

and the generators $x^\pm_{m,r}$ and $x^\pm_{0,r}$ are odd and all other generators are even.

It also has a minimalistic presentation whose generators are $x^\pm_{i,0}$ and $h_{i,1}$ and has a super coproduct defined by the same formula as one of the affine Yangian (see [U2]). By using the affine super Yangian, we have the similar result as that of [RS].

Theorem 0.55. *There exists a surjective homomorphism from the affine super Yangian to the universal enveloping algebra of rectangular W-superalgebras of type A. In particular, we obtain a surjective homomorphism from the affine Yangians of type A*

to the universal enveloping algebra of the rectangular W-superalgebras of type A.

In the case when $l \geq 2$ (resp. $l = 1$). the theorem is proven in [U3] (resp. [U4]).

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