On the Feigin-Tipunin conjecture

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1 Introduction

The triplet W-algebra (= type A_1 logarithmic W-algebra) ([AM1]-[AM3], [FGST1]-[FGST3], [NT], [TW], ...) is one of the most famous examples of C_2-cofinite but irrational vertex operator algebra, and it relates to many interesting objects such as the tails of colored Jones polynomials and false theta functions [BM1, CCFGH, MN], quantum groups at root of unity [CGR, FGR, NT], and the quantum geometric Langlands program [CG, Cr1]. We can immediately generalize the definition of the triplet W-algebra to type ADE cases, and we call them the type ADE logarithmic W-algebras $W(p)_Q$. However, very little is known about the properties and the representation theory of the higher rank generalizations of the triplet W-algebra.

In [FT], without detailed proofs, they claimed that $W(p)_Q$ and its irreducible modules are constructed as the spaces of global sections of some homogeneous vector bundles over the flag variety, and we call it Feigin-Tipunin conjecture. In [S1, S2], the author proved it partially and obtained some of new results on the type ADE logarithmic W-algebras.

In this paper, with some comments and remarks, we gather results that will given in [S1, S2]. We give the geometric construction of the type ADE logarithmic W-algebra $W(p)_Q$ that claimed in [FT]. This construction reveals us the G-module structure and the character formula of $W(p)_Q$. Moreover, under the assumption of simpleness of $W(p)_Q$, we also completely determine the $\mathcal{W}^k(\mathfrak{g})$-module structure of $W(p)_Q$. Finally, applying this result to the cases of type A_2 with small $p \in \mathbb{Z}_{\geq 2}$, we prove the C_2-cofiniteness of $W(p)_Q$ in these cases under the assumption of simpleness.

1.1 Setting

Let $\mathfrak{g}$ be a simply-laced simple Lie algebra of rank l, and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ the triangular decomposition, $\mathfrak{h}$ the Cartan subalgebra, $\mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h}$ the Borel subalgebra, G, H, and B the semisimple, simply-connected, complex algebraic groups corresponding to $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$, respectively. We adopt the standard numbering for the simple roots $\{\alpha_1, \ldots, \alpha_l\}$ of $\mathfrak{g}$ as in [B] and denote by $\{\omega_1, \ldots, \omega_l\}$ the corresponding fundamental weights, and denote by Π denotes the set of simple roots. Let Q be the root lattice of $\mathfrak{g}$, P the weight lattice of $\mathfrak{g}$, P_+ the set of dominant integral weights of $\mathfrak{g}$. Denote by $(\cdot, \cdot)$ the normalized invariant form of $\mathfrak{g}$, W the Weyl group of $\mathfrak{g}$ generated by the simple reflections $\{\sigma_i\}_{i=1}^{l}$, (c_{ij}) the Cartan matrix of $\mathfrak{g}$

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and (c^{ij}) the inverse matrix to (c_{ij}), ρ the half sum of positive roots, h the Coxeter number of $\mathfrak{g}$, Ω the abelian group P/Q. We choose the representatives of generators of Ω in P in the following way: for A_l, D_l, E_6, E_7, E_8, we choose $\{0,\omega_1,\dots,\omega_l\}$, $\{0,\omega_1,\omega_{l-1},\omega_l\}$, $\{0,\omega_1,\omega_3\}$, $\{0,\omega_2\}$, $\{0\}$, respectively. We fix an integer $p\in\mathbb{Z}_{\geq 2}$.

Let $V_{\sqrt{p}Q}=\bigoplus_{\alpha\in Q}\mathcal{F}_{\sqrt{p}\alpha}$ be the lattice vertex operator algebra associated to the rescaled root lattice $\sqrt{p}Q$, where $\mathcal{F}_{\sqrt{p}\alpha}=\mathcal{U}(\hat{\mathfrak{h}}^{<0})\otimes|\sqrt{p}\alpha\rangle$ is the Fock module of the Heisenberg vertex operator algebra $\mathcal{F}_0=\mathcal{U}(\hat{\mathfrak{h}}^{<0})\otimes|0\rangle$.

We choose the shifted conformal vector ω of $V_{\sqrt{p}Q}$ as

$$\omega=\frac{1}{2}\sum_{1\leq i,j\leq l}c^{ij}(\alpha_i)_{(-1)}\alpha_j+Q_0(\rho)_{(-2)}|0\rangle\in\mathcal{F}_0\subseteq V_{\sqrt{p}Q}, \tag{1}$$

where $Q_0=\sqrt{p}-\frac{1}{\sqrt{p}}$. The central charge c of ω is given by

$$c=l+12(\rho,\rho)(2-p-\frac{1}{p})=l+h\dim\mathfrak{g}(2-p-\frac{1}{p}). \tag{2}$$

For $n\in\mathbb{Z}$, we use the traditional notation L_n for the Virasoro operator $\omega_{(n+1)}$.

Irreducible modules over $V_{\sqrt{p}Q}$ are classified by elements of the abelian group $\Lambda=\frac{1}{\sqrt{p}}P/\sqrt{p}Q$ ([D]). For each equivalence class $\langle\lambda\rangle\in\Lambda$, we choose the unique representative $\lambda\in\frac{1}{\sqrt{p}}P$ of $\langle\lambda\rangle\in\Lambda$ as

$$\lambda=-\sqrt{p}\hat{\lambda}+\bar{\lambda}=-\sqrt{p}\hat{\lambda}+\sum_{j=1}^{l}\frac{s_j}{\sqrt{p}}\omega_j, \tag{3}$$

where $0\leq s_j\leq p-1$, $\hat{\lambda}\in\Omega$ and the representatives of generators of Ω are given in above: for A_l, D_l, E_6, E_7, E_8, we have $\{0,\omega_1,\dots,\omega_l\}$, $\{0,\omega_1,\omega_{l-1},\omega_l\}$, $\{0,\omega_1,\omega_3\}$, $\{0,\omega_2\}$, $\{0\}$, respectively. For $\lambda\in\frac{1}{\sqrt{p}}P$, denote by $V_{\sqrt{p}Q+\lambda}$ the irreducible $V_{\sqrt{p}Q}$-module

$$V_{\sqrt{p}Q+\lambda}=\bigoplus_{\alpha\in Q}\mathcal{F}_{\sqrt{p}\alpha+\lambda}. \tag{4}$$

corresponding to $\langle\lambda\rangle\in\Lambda$, where $\mathcal{F}_{\sqrt{p}\alpha+\lambda}$ is the Fock module over $\mathcal{F}_0$ with the highest weight vector $|\sqrt{p}\alpha+\lambda\rangle$. For $\mu\in\frac{1}{\sqrt{p}}P$, the conformal weight Δ_μ of $|\mu\rangle$ is

$$\Delta_\mu=\frac{1}{2}|\mu-Q_0\rho|^2+\frac{c-l}{24}=\frac{1}{2}|\mu|^2-Q_0(\mu,\rho). \tag{5}$$

1.2 Screening and narrow screening

For $1\leq i\leq l$, $\alpha\in Q$ and $\lambda\in\Lambda$, we consider the *screening operators*

$$f_i=|\sqrt{p}\alpha_i\rangle_{(0)}\in\mathrm{Hom}(\mathcal{F}_{-\sqrt{p}\alpha+\lambda},\mathcal{F}_{-\sqrt{p}(\alpha+\alpha_i)+\lambda}). \tag{6}$$

For $\sigma\in W$ and $\mu\in\frac{1}{\sqrt{p}}P$, set

$$\sigma\star\mu=\sigma(\mu+\frac{1}{\sqrt{p}}\rho)-\frac{1}{\sqrt{p}}\rho. \tag{7}$$

Then we have the following W-action on Λ:

$$\sigma*\lambda=-\sqrt{p}\hat{\lambda}+\sigma\star\bar{\lambda}. \tag{8}$$

In order to define the *narrow screening operators* $F_{i,\lambda} \in \mathrm{Hom}(\mathcal{F}_{-\sqrt{p}\alpha+\lambda}, \mathcal{F}_{-\sqrt{p}\alpha+\sigma_i*\lambda})$ for $1 \leq i \leq l$, we consider the following element in $\mathrm{Hom}(V_{\sqrt{p}Q}, V_{\sqrt{p}Q-\frac{\alpha_i}{\sqrt{p}}}) \otimes \mathbb{C}[[z^{\pm}]]$:

$$F_i(z) = e^{-\frac{\alpha_i}{\sqrt{p}}} z^{-\frac{(\alpha_i)_{(0)}}{\sqrt{p}}} \exp\big(\sum_{n<0} \frac{z^{-n}}{n} \frac{(\alpha_i)_{(n)}}{\sqrt{p}}\big) \exp\big(\sum_{n>0} \frac{z^{-n}}{n} \frac{(\alpha_i)_{(n)}}{\sqrt{p}}\big) c_{-\frac{\alpha_i}{\sqrt{p}}}. \tag{9}$$

Here the element $e^{-\frac{\alpha_i}{\sqrt{p}}} \in \mathrm{Hom}(V_{\sqrt{p}Q}, V_{\sqrt{p}Q-\frac{\alpha_i}{\sqrt{p}}})$ is defined by

$$\begin{cases} e^{-\frac{\alpha_i}{\sqrt{p}}} |\sqrt{p}\mu\rangle = |-\frac{\alpha_i}{\sqrt{p}} + \sqrt{p}\mu\rangle, \\ [h_{(n)}, e^{-\frac{\alpha_i}{\sqrt{p}}}] = \delta_{n,0}(h, -\frac{\alpha_i}{\sqrt{p}}) e^{-\frac{\alpha_i}{\sqrt{p}}}, \end{cases} \tag{10}$$

for $\mu \in Q$ and $h_{(n)} \in \mathcal{U}(\hat{\mathfrak{h}})$, and the element $c_{-\frac{\alpha_i}{\sqrt{p}}} \in \mathrm{Hom}(V_{\sqrt{p}Q}, V_{\sqrt{p}Q-\frac{\alpha_i}{\sqrt{p}}})$ is defined by

$$c_{-\frac{\alpha_i}{\sqrt{p}}} s|\sqrt{p}\mu\rangle = \epsilon'(-\alpha_i, \mu) s|\sqrt{p}\mu\rangle. \tag{11}$$

Here $s \in \mathcal{U}(\hat{\mathfrak{h}}^{<0})$ and $\epsilon' : Q \times Q \to \mathbb{C}^{\times}$ is the 2-cocycle defined by

$$\epsilon(\alpha_i, \alpha_j) = \begin{cases} (-1) & i = j, \\ (-1)^{(\alpha_i, \alpha_j)} & i < j, \\ 1 & i > j. \end{cases} \tag{12}$$

For $\alpha \in Q$, the narrow screening operator is given by the z^{-1} coefficient of $F_i(z)$

$$F_{i,0} = \int F_i(z) dz \in \mathrm{Hom}(\mathcal{F}_{-\sqrt{p}\alpha}, \mathcal{F}_{-\sqrt{p}\alpha-\frac{\alpha_i}{\sqrt{p}}}). \tag{13}$$

Denote by $\mathcal{F}_0^i$ the rank 1 Heisenberg vertex algebra generated by α_i, and $\mathcal{F}_0^{i,\perp}$ the rank $l-1$ Heisenberg vertex algebra generated by $\{\omega_j\}_{1 \leq j \neq i \leq l}$, respectively. Then for $a \in \mathcal{F}_0^{i,\perp}$ and $n \in \mathbb{Z}$, we have $F_{i,0} a_{(n)} = a_{(n)} F_{i,0}$. By applying the multiplication of narrow screening operators in the case of type A_1 (see [CRW, NT]) to $F_{i,0}|_{\mathcal{F}_0^i}$, for $\alpha \in Q$ and $\lambda \in \Lambda$ such that $0 \leq s_i \leq p-2$, we have the non-trivial map

$$F_{i,\lambda} = \int_{[\Gamma_{s_i+1}]} F_i(z_1) \dots F_i(z_{s_i+1}) \mathrm{d}z_1 \dots \mathrm{d}z_{s_i+1} \in \mathrm{Hom}(\mathcal{F}_{-\sqrt{p}\alpha+\lambda}, \mathcal{F}_{-\sqrt{p}\alpha+\sigma_i*\lambda}),$$

where the cycle $[\Gamma_{s_i+1}]$ such that $F_{i,\lambda}$ to be non-trivial is uniquely determined up to normalization. For convenience, we set $F_{i,\lambda} = 0$ for $\lambda \in \Lambda$ such that $s_i = p-1$.

Clearly, the screening and narrow screening operators are differential operators on $V_{\sqrt{p}Q}$ because they are zero modes. In other words, they satisfy the Leibniz rule

$$f_i u_{(n)} v = (f_i u)_{(n)} v + u_{(n)} f_i v, \tag{14}$$

$$\begin{aligned} F_{i,0} a_{(n)} b &= (F_{i,0} a)_{(n)} b + a_{(n)} F_{i,0} b \qquad (15) \\ &= \sum_{m \geq 0} \frac{(-1)^{-n-m-1}}{m!} T^m b_{(n+m)} F_{i,0} a + a_{(n)} F_{i,0} b, \end{aligned}$$

where $n \in \mathbb{Z}$, $u, v \in V_{\sqrt{p}Q+\lambda}$ and $a, b \in V_{\sqrt{p}Q}$. Moreover, a straightforward calculation shows that

$$[f_i, L_n] = [F_{i,\lambda}, L_n] = 0 \tag{16}$$

and

$$[f_i, F_{j,\lambda}] = 0 \tag{17}$$

for $1 \leq i, j \leq l$ and $n \in \mathbb{Z}$. In particular, (16) means that f_i and $F_{i,\lambda}$ preserve the conformal grading.

1.3 Logarithmic W-algebra

Since every $F_{i,0}$ satisfies (15), we have the vertex operator subalgebra

$$W(p)_Q = \bigcap_{i=1}^{l} \ker F_{i,0}|_{V_{\sqrt{p}Q}} \subseteq V_{\sqrt{p}Q}. \tag{18}$$

By (16), ω is a conformal vector of $W(p)_Q$. This vertex operator algebra is called the *logarithmic W-algebra* associated to Q and p. In particular, in the case of type A_1, $W(p)_Q$ is the *triplet W-algebra* ([AM1]-[AM3], [FGST1]-[FGST3], [NT], [TW], ...).

For $1 \leq i \leq l$, we consider the following operator $h_{i,\lambda}$ acting on $V_{\sqrt{p}Q+\lambda}$:

$$h_{i,\lambda} = -\frac{1}{\sqrt{p}}(\alpha_i)_{(0)} + \frac{1}{\sqrt{p}}(\alpha_i, \bar{\lambda})\,\mathrm{id}\,. \tag{19}$$

Theorem 1 ([FT, Theorem 4.1])**.**

1. *The operators $\{f_i, h_{i,\lambda}\}_{i=1}^{l}$ give rise to an action of $\mathfrak{b}$ on $V_{\sqrt{p}Q+\lambda}$.*
2. *The action of $\mathfrak{b}$ in* (1) *is integrable.*

For $\lambda \in \Lambda$, we consider the homogeneous vector bundle

$$\xi_\lambda = G \times_B V_{\sqrt{p}Q+\lambda} \tag{20}$$

over the flag variety G/B, where the action of B on G is given by the right multiplication and that on $V_{\sqrt{p}Q,\lambda}$ is given by Theorem 1. We can easily show that the space of global sections $H^0(\xi_0)$ inherits the vertex operator algebra structure from $V_{\sqrt{p}Q}$, and each $H^0(\xi_\lambda)$ is a $H^0(\xi_0)$-module as in the same way.

1.4 Results

Definition 1. For $\lambda \in \Lambda$ and $\sigma \in W$, set

$$\epsilon_\lambda(\sigma) = \frac{1}{\sqrt{p}}(\sigma \star \bar{\lambda} - \overline{\sigma * \lambda}). \tag{21}$$

Let J be a subset of nodes of the Dynkin diagram of G and $\lambda \in \Lambda$. The pairing (J, λ) is *good* if $\epsilon_\lambda(\sigma_j) = -\alpha_j$ for any $j \in J$ or $(\epsilon_\lambda(\sigma_j), \alpha_i) = -\delta_{i,j}$ for any $i, j \in J$. In particular, when $J = \Pi$, we call λ is good if (Π, λ) is good.

Remark 1. If $|J| = 1$, (J, λ) is good.

The following three theorems will be given in [S1].

Theorem 2. 1. *For $p \in \mathbb{Z}_{\geq 2}$, we have the vertex operator algebra isomorphism*

$$H^0(\xi_0) \simeq W(p)_Q.$$

In particular, the group G acts on $W(p)_Q$ as an automorphism group.

2. *More generally, if λ is good, we have the $W(p)_Q$-module and G-module isomorphism*

$$H^0(\xi_\lambda) \simeq \bigcap_{i=1}^{l} \ker F_{i,\lambda}|_{V_{\sqrt{p}Q+\lambda}}.$$

3. If λ is not good, $H^0(\xi_\lambda)$ is properly embedded into $\bigcap_{i=1}^{l} \ker F_{i,\lambda}|_{V_{\sqrt{p}Q+\lambda}}$.

Theorem 3. *We have the vertex operator algebra isomorphism*

$$\mathcal{W}^k(\mathfrak{g}) \simeq \bigcap_{i=1}^{l} \ker f_i|_{\mathcal{F}_0},$$

where $\mathcal{W}^k(\mathfrak{g})$ is the affine W-algebra [FF] of level $k = p - h$

Theorem 4.

1. *Let $\mathcal{R}_\mu$ be the irreducible $\mathfrak{g}$-module with the highest weight $\mu \in P_+$. Then we have the $W(p)_Q$-module and G-module isomorphism*

$$H^0(\xi_\lambda) \simeq \bigoplus_{\alpha \in P_+ \cap Q} \mathcal{R}_{\alpha+\hat{\lambda}} \otimes \mathcal{W}_{-\sqrt{p}\alpha+\lambda} \subseteq V_{\sqrt{p}Q+\lambda} \tag{22}$$

where $\mathcal{W}_{-\sqrt{p}\alpha+\lambda} = \bigcap_{i=1}^{l}(\ker f_i|_{\mathcal{F}_0}| - \sqrt{p}\alpha + \lambda\rangle)$ and $\ker f_i|_{\mathcal{F}_0}| - \sqrt{p}\alpha + \lambda\rangle$ is the $\ker f_i|_{\mathcal{F}_0}$-module generated by the highest weight vector $| - \sqrt{p}\alpha + \lambda\rangle$. In particular, we have

$$W(p)_Q \simeq \bigoplus_{\alpha \in P_+ \cap Q} \mathcal{R}_\alpha \otimes \mathcal{W}_{-\sqrt{p}\alpha},$$

and $\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{R}_0 \otimes \mathcal{W}_0$ is the vertex operator full subalgebra of $W(p)_Q$.

2. *Let us fix $\lambda \in \Lambda$, and a minimal expression of $w_0 = \sigma_{i_N} \ldots \sigma_{i_1}$. If $(\epsilon_\lambda(\sigma_{i_k} \ldots \sigma_{i_1}), \alpha_{i_{k+1}}) = 0$ for $1 \le k \le N-1$, then we have $H^k(\xi_\lambda) = 0$ for $k \ge 1$. In particular, if $\bar{\lambda} = 0$, then $H^k(\xi_\lambda) = 0$ for $k \ge 1$. Moreover, we have the character formula*

$$\begin{aligned}\mathrm{Tr}_{H^0(\xi_\lambda)}(q^{L_0-\frac{c}{24}} z_1^{h_{1,\lambda}} \cdots z_l^{h_{l,\lambda}}) &= \sum_{\alpha \in P_+ \cap Q} \chi^{\mathfrak{g}}_{\alpha+\hat{\lambda}}(z) \Big(\sum_{\sigma \in W} (-1)^{l(\sigma)} \frac{q^{\frac{1}{2}|\sqrt{p}\sigma(\alpha+\rho+\hat{\lambda}) - \bar{\lambda} - \frac{1}{\sqrt{p}}\rho|^2}}{\eta(q)^l} \Big) \\ &= \sum_{\alpha \in P_+ \cap Q} \chi^{\mathfrak{g}}_{\alpha+\hat{\lambda}}(z)\, \mathrm{Tr}_{H^0_{DS,\alpha+\hat{\lambda}}(\mathbb{V}_{p,\sqrt{p}\bar{\lambda}})}(q^{L_0-\frac{c}{24}}),\end{aligned}$$

where $\chi^{\mathfrak{g}}_\beta(z)$ be the Weyl character of $\mathcal{R}_\beta$, $l(\sigma)$ the length of $\sigma \in W$, $\eta(q)$ the Dedekind eta function, and $H^0_{DS,\alpha+\hat{\lambda}}(\mathbb{V}_{p,\sqrt{p}\bar{\lambda}})$ is the $\mathcal{W}^k(\mathfrak{g})$-module defined in [ArF].

Remark 2. The author believe that the assumption $(\epsilon_\lambda(\sigma_{i_k} \ldots \sigma_{i_1}), \alpha_{i_{k+1}}) = 0$ for $1 \le k \le N-1$ in Theorem 4.2 is not necessary: i.e. he expect that $H^k(\xi_\lambda) = 0$ and the character formula above hold for all $\lambda \in \Lambda$ and $k \ge 1$. However, because of some technical difficulty in the proof of vanishing of higher cohomologies, he proved them on the restricted cases.

The following three theorems will be given in [S2].

Theorem 5. *If $H^0(\xi_\lambda)$ is an irreducible $W(p)_Q$-module, then $\mathcal{W}_{-\sqrt{p}\alpha+\lambda} \simeq \mathcal{W}^k(\mathfrak{g})| - \sqrt{p}\alpha + \lambda\rangle$. In other words, $\bigcap_{i=1}^{l}(\ker f_i|_{\mathcal{F}_0}| - \sqrt{p}\alpha + \lambda\rangle) = (\bigcap_{i=1}^{l} \ker f_i|_{\mathcal{F}_0})| - \sqrt{p}\alpha + \lambda\rangle$. Moreover, when $\lambda = 0$, $\mathcal{W}_{-\sqrt{p}\alpha}$ is the irreducible $\mathcal{W}^k(\mathfrak{g})$-module.*

Definition 2. 1. For $\alpha \in P_+ \cap Q$, let H_α be a nonzero element of $\mathcal{R}_{\alpha,0} \otimes \mathbb{C}| - \sqrt{p}\alpha\rangle$, where $\mathcal{R}_{\alpha,0}$ is the space of zero-weight vectors of $\mathcal{R}_\alpha$.

2. Let $\{W_i\}_{i=2}^{l+1}$ be strong generators of $\mathcal{W}^k(\mathfrak{g})$ such that $\Delta_{W_i} = i$. We use the notation

$$(W_i)_n = (W_i)_{(n+i-1)} \tag{23}$$

for $n \in \mathbb{Z}$. However, we often use the notation not $(W_2)_n$ but L_n for traditional reason. Moreover, for a fixed $\alpha \in P_+ \cap Q$, we can assume that

$$(W_i)_0 |-\sqrt{p}\alpha\rangle = 0 \tag{24}$$

for $3 \le i \le l+1$ by considering the new strong generators

$$\{\omega\} \cup \{W_i - \nabla_{\alpha,i}\omega\}_{i=3}^{l+1} \tag{25}$$

of $\mathcal{W}^k(\mathfrak{g})$, where $\nabla_{\alpha,i} \in \mathbb{C}$ is defined by $(W_i)_0|-\sqrt{p}\alpha\rangle = \nabla_{\alpha,i}|-\sqrt{p}\alpha\rangle$.

3. For $a \in W(p)_Q \simeq \bigoplus_{\alpha\in P_+\cap Q} \mathcal{R}_\alpha \otimes \mathcal{W}_{-\sqrt{p}\alpha}$, denote by $\tilde{a} \in \mathcal{W}^k(\mathfrak{g})$ be the $\mathcal{W}^k(\mathfrak{g})$-component of a.

Theorem 6. *1. For the projection to the C_2-algebra $\pi : W(p)_Q \to R_{W(p)_Q} = W(p)_Q/C_2(W(p)_Q)$, we have $\dim \pi(W(p)_Q \backslash \mathcal{W}^k(\mathfrak{g})) < \infty$. In other words, if $a \in W(p)_Q \backslash \mathcal{W}^k(\mathfrak{g})$, then $\pi(a)$ is nilpotent. In particular, for $\alpha \in P_+ \cap Q$, $\alpha \neq 0$, $\pi(H_\alpha)$ is nilpotent.*

2. If $W(p)_Q$ is simple, then $W(p)_Q$ is strongly generated by $\{W_i\}_{i=2}^{l+1}$ and finitely many elements in $W(p)_Q \setminus \mathcal{W}^k(\mathfrak{g})$. In particular, if $W(p)_Q$ is simple and all $\pi(W_i)$ are nilpotent, then $W(p)_Q$ is C_2-cofinite.

3. For $\alpha \in P_+ \cap Q$, $\{\widetilde{(H_\alpha)_{(N)}H_{\alpha'}}\}_{N\in\mathbb{Z}}$ satisfy the following conditions:

(a) For $m \ge 0$, we have

$$\widetilde{(H_\alpha)_{(N+m)}H_{\alpha'}} = \frac{L_m \widetilde{(H_\alpha)_{(N)}H_{\alpha'}}}{(m+1)(\Delta_{-\sqrt{p}\alpha}-1) - N + \delta_{m,0}\Delta_{-\sqrt{p}\alpha}}. \tag{26}$$

(b) Moreover, for $3 \le i \le l+1$, $n \ge i-1$ and $N \in \mathbb{Z}$, we have

$$\sum_{k=0}^{i-1}(-1)^k \binom{i-1}{k} \frac{(W_i)_{n-k}L_k \widetilde{(H_\alpha)_{(N)}H_{\alpha'}}}{(k+1)(\Delta_{-\sqrt{p}\alpha}-1) - N + \delta_{k,0}\Delta_{-\sqrt{p}\alpha}} = 0. \tag{27}$$

4. In the cases of types A_1 or A_2, the conditions (26) and (27) determines $\{\widetilde{(H_\alpha)_{(N)}H_{\alpha'}}\}_{N\in\mathbb{Z}}$ uniquely up to scalar. Moreover, if $W(p)_Q$ is simple, then the conditions (26) and (27) determines $\{\widetilde{(H_\alpha)_{(N)}H_{\alpha'}}\}_{N\in\mathbb{Z}}$ uniquely up to nonzero scalar.

Remark 3. Theorem 6 claims that if $W(p)_Q$ is simple, the conditions (26) and (27) give an algorithm that enables us to calculate the nilpotent ideal in $\pi(\mathcal{W}^k(\mathfrak{g}))$ much easier than direct calculation. Applying it to the cases of type A_2 with small p, we obtain the following:

Theorem 7. *Let us consider the cases when $\mathfrak{g} = \mathfrak{sl}_3$ and $p = 2, 3, 4$. If $W(p)_Q$ is simple, then $W(p)_Q$ is C_2-cofinite.*

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