TRANSFORMATION FORMULAE AND ASYMPTOTIC EXPANSIONS FOR DOUBLE NON-HOLOMORPHIC EISENSTEIN SERIES OF TWO COMPLEX VARIABLES (SUMMARIZED VERSION)

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ABSTRACT. It is shown that complete asymptotic expansions exist for the double non-holomorphic Eisenstein series $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ having two variables $\boldsymbol{s}\in\mathbb{C}^2$ and two parameters $\boldsymbol{z}=(z_1,z_2)\in\mathfrak{H}^+\times\mathfrak{H}^-$, when both $|z_1-z_2|\to+\infty$ and $|z_1-z_2|\to 0$ (Theorems 1 and 3).

1. INTRODUCTION

Throughout the paper, s and $\boldsymbol{s}=(s_1,s_2)$ are complex variables with $s=\sigma+\sqrt{-1}t$ and $s_i=\sigma+\sqrt{-1}t_i$ $(i=1,2)$, z and $\boldsymbol{z}=(z_1,z_2)$ complex parameters with $z=x+\sqrt{-1}y$ and $z_i=x_i+\sqrt{-1}y_i$ $(i=1,2)$, and $\widetilde{\mathbb{C}^\times}$ denotes the universal covering of $\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$. Note that $\arg w\in]-\infty,+\infty[$ is uniquely determined for any $w\in\widetilde{\mathbb{C}^\times}$. The principal branches of the upper and lower half-planes are denoted respectively by

$$\mathfrak{H}^+=\{z\in\widetilde{\mathbb{C}^\times}\mid 0<\arg z<\pi\}\qquad\text{and}\qquad\mathfrak{H}^-=\{z\in\widetilde{\mathbb{C}^\times}\mid -\pi<\arg z<0\}. \tag{1.1}$$

We write $\langle\boldsymbol{s}\rangle=s_1+s_2$ for any $\boldsymbol{s}\in\mathbb{C}^2$ hereafter, and introduce the bilateral double series of the form

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})=\sideset{}{'}\sum_{m,n=-\infty}^{\infty}(m+nz_1)^{-s_1}(m+nz_2)^{-s_2}, \tag{1.2}$$

converging absolutely for $\mathrm{Re}\langle\boldsymbol{s}\rangle>2$, where a more accurate definition of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ will be given in (1.8) and (1.9) below. The primed summation symbols hereafter indicate omission of the (possibly emerging) singular terms such as 0^{-s}. Let $\zeta(s)$ denote the Riemann zeta-function, and $\overline{z}$ the complex conjugate of z. When $\boldsymbol{z}=(z_1,z_2)\in\mathfrak{H}^\pm\times\mathfrak{H}^\pm$, $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ is called the double *holomorphic* Eisenstein series, e.g.,

$$\widetilde{\zeta_{\mathbb{Z}^2}}((s,0);(z,z_2))=\sideset{}{'}\sum_{m,n=-\infty}^{\infty}(m+nz)^{-s}=F(s;z)\qquad(\sigma>2) \tag{1.3}$$

is the classical holomorphic Eisenstein series of one complex variable. When $\boldsymbol{z}=(z_1,z_2)\in\mathfrak{H}^\pm\times\mathfrak{H}^\mp$, $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ is called the double *non-holomorphic* Eisenstein series, e.g.,

$$\begin{aligned}\widetilde{\zeta_{\mathbb{Z}^2}}((s,s);(z,\overline{z}))&=\sideset{}{'}\sum_{m,n=-\infty}^{\infty}(m+nz)^{-s}(m+n\overline{z})^{-s}=\sideset{}{'}\sum_{m,n=-\infty}^{\infty}|m+nz|^{-2s}\\&=\zeta_{\mathbb{Z}^2}(s;z)\qquad(\sigma>1)\end{aligned} \tag{1.4}$$

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is the classical (one variable) Epstein zeta-function, and further, e.g.,

$$\widetilde{\zeta_{\mathbb{Z}^2}}((s+k,s);(z,\overline{z})) = \sum_{m,n=-\infty}^{\infty}{}' (m+nz)^{-s-k}(m+n\overline{z})^{-s} = 2\zeta(2s)E_k(s;z), \tag{1.5}$$

giving the *non-holomorphic* Eisenstein series $E_k(s;z)$ of weight $k \in 2\mathbb{Z}$, defined by

$$E_k(s;z) = \frac{1}{2}\sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty} (cz+d)^{-k}|cz+d|^{-2s} \qquad (\sigma > 1-k/2). \tag{1.6}$$

The *holomorphic* case of (1.2) has recently been studied by the authors [9], who established complete asymptotic expansions for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ as both $|z_1 - z_2| \to 0$ and $|z_1 - z_2| \to +\infty$ through the poly-sector $\boldsymbol{z} \in \mathfrak{H}^{\pm} \times \mathfrak{H}^{\pm}$. The present paper proceeds further with our previous study to show that similar asymptotic series still exist for (*non-holomorphic*) $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ as $|z_1 - z_2| \to +\infty$ through the poly-sector $\boldsymbol{z} \in \mathfrak{H}^{+} \times \mathfrak{H}^{-}$ (Theorem 1); these can be switched to the counterpart expansions as $|z_1 - z_2| \to 0$ through the same poly-sector (Theorem 3) by means of the quasi-modular relation in (6.5) below; several applications of our main results will also be presented (Theorems 2, 4 and Corollaries 1.1–1.4). The detailed proofs will appear in a forthcoming article [10].

We remark here that the recent progress of research into the present direction has been made, e.g., by Matsumoto [15], and the first and second authors [5][6][7][8][9][16], especially on asymptotic aspects of various Eisenstein series, while it has also been made on the relevant (bilateral) double series, e.g., by Gangl-Kaneko-Zagier [3], Komori-Matsumoto-Tsumura [11][12], Kaneko-Tasaka [4], Tasaka [18] and Lim [14] (mostly) from a point of view of modular forms and functions; we refer the reader, e.g., to [9, Sect. 1] for further details.

We now proceed to make an accurate definition of the *non-holomorphic* Eisenstein series. Suppose throughout the following that

$$\boldsymbol{z} = (z_1, z_2) \in \mathfrak{H}^{+} \times \mathfrak{H}^{-}, \tag{1.7}$$

and set

$$\widetilde{\zeta_{\mathbb{Z}^2}}^{\pm}(\boldsymbol{s};\boldsymbol{z}) = \sum_{m,n=-\infty}^{\infty}{}' (m+nz_1)^{-s_1}(m+nz_2)^{-s_2} \qquad (\mathrm{Re}\langle\boldsymbol{s}\rangle > 2), \tag{1.8}$$

where in each summand $\arg(m+nz_1)$ falls within $]-\pi,\pi]$ and $\arg(m+nz_2)$ within $[-\pi,\pi[$ in $\widetilde{\zeta_{\mathbb{Z}^2}}^{+}(\boldsymbol{s};\boldsymbol{z})$, while $\arg(m+nz_1)$ within $[-\pi,\pi[$ and $\arg(m+nz_2)$ within $]-\pi,\pi]$ in $\widetilde{\zeta_{\mathbb{Z}^2}}^{-}(\boldsymbol{s};\boldsymbol{z})$. Then the main object of study is defined, from a symmetric point of view, as the arithmetical mean

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z}) = \frac{1}{2}\Big\{\widetilde{\zeta_{\mathbb{Z}^2}}^{+}(\boldsymbol{s};\boldsymbol{z}) + \widetilde{\zeta_{\mathbb{Z}^2}}^{-}(\boldsymbol{s};\boldsymbol{z})\Big\}, \tag{1.9}$$

which will be continued to a meromorphic function in the whole $\boldsymbol{s}$-space $\mathbb{C}^2$ (see Theorem 1).

In what follows, several notations and terminology are prepared to describe our main results. Define first the 'pivotal parameter' η by

$$\eta = \eta(\boldsymbol{z}) = \frac{1}{2}e^{-\pi\sqrt{-1}/2}(z_1 - z_2), \tag{1.10}$$

which with $0 < \arg(z_1 - z_2) < \pi$, by (1.7), shows that

$$|\arg\eta| = \left|\arg(z_1 - z_2) - \frac{\pi}{2}\right| < \frac{\pi}{2}. \tag{1.11}$$

The introduction of η here is oriented toward the non-holomorphic case $\boldsymbol{z} = (z, \overline{z}) \in \mathfrak{H}^+ \times \mathfrak{H}^-$ or $\boldsymbol{z} = (\overline{z}, z) \in \mathfrak{H}^- \times \mathfrak{H}^+$, where

$$y = \frac{1}{2} e^{-\pi\sqrt{-1}/2}(z - \overline{z}) \quad \text{or} \quad y = \frac{1}{2} e^{\pi\sqrt{-1}/2}(\overline{z} - z)$$

holds; this suggests that η may be regarded as a 'complexification' of the real variable $y = \operatorname{Im} z$. Next let $e(s) = e^{2\pi\sqrt{-1}s}$ in the sequel, $\Gamma(s)$ the gamma function, $(s)_n = \Gamma(s+n)/\Gamma(s)$ for $n \in \mathbb{Z}$ the rising factorial, and write

$$\Gamma\binom{\alpha_1, \ldots, \alpha_m}{\beta_1, \ldots, \beta_n} = \frac{\prod_{h=1}^m \Gamma(\alpha_h)}{\prod_{k=1}^n \Gamma(\beta_k)}$$

for $\alpha_h, \beta_k \in \mathbb{C}$ $(h = 1, \ldots, m;\ k = 1, \ldots, n)$. Further let

$$U(\alpha; \gamma; Z) = \frac{1}{\Gamma(\alpha)\{e(\alpha) - 1\}} \int_\infty^{(0+)} e^{-wZ} w^{\alpha-1}(1+w)^{\gamma-\alpha-1} dw \tag{1.12}$$

for any $(\alpha, \gamma) \in \mathbb{C}^2$ and for $|\arg Z| < \pi/2$ be Kummer's confluent hypergeometric function of the second kind (cf. [2, p.255, 6.5.(2)]), where the path of integration is a contour which starts from ∞, proceeds along the real axis to a small $\delta > 0$, encircles the origin counterclockwise, and returns to ∞; $\arg w$ varies from 0 to 2π along the contour. The domain of Z here can be extended to the whole sector $|\arg Z| < 3\pi/2$ by rotating appropriately the path in (1.12) (cf. [2, p.273, 6.11.2.(9)]). We write $\sigma_w(l) = \sum_{0<h|l} h^w$, and define for any $(r, s) \in \mathbb{C}^2$ and for $z \in \mathfrak{H}^+$ the function

$$\Phi_{r,s}(e(z)) = \sum_{h,k=1}^\infty h^r k^s e(hkz) = \sum_{l=1}^\infty \sigma_{r-s}(l) l^s e(lz), \tag{1.13}$$

which was first introduced and studied by Ramanujan [17] (see also [1, Chap. 4]) for the purpose of giving various evaluations of the classical (integer weight) Eisenstein series $E_k(z)$ $(k = 2, 4, 6, \ldots)$.

2. Asymptotic expansions for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ as $\eta \to \infty$

Throughout the following sections, at each occurrence of the pairing use of the indices $\{i, j\} = \{1, 2\}$ within an individual formula, the index j is always to be assigned as $j \in \{1, 2\} \setminus \{i\}$ $(i = 1, 2)$ for brevity of description. We then proceed to state our first main result.

Theorem 1 ([10, Theorem 1]). *The formula*

$$\begin{aligned}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z}) &= 2\cos^2\Big\{\frac{\pi}{2}(s_1 - s_2)\Big\}\zeta\big(\langle\boldsymbol{s}\rangle\big) + 4\pi\cos\Big\{\frac{\pi}{2}(s_1 - s_2)\Big\} \\ &\quad\times \Gamma\binom{\langle\boldsymbol{s}\rangle - 1}{s_1, s_2}\zeta\big(\langle\boldsymbol{s}\rangle - 1\big)(2\eta)^{1-\langle\boldsymbol{s}\rangle} + \widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s}; \boldsymbol{z})\end{aligned} \tag{2.1}$$

holds upon providing the meromorphic continuation of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ to the whole $\boldsymbol{s}$-space $\mathbb{C}^2$ except the hyper-planes $\langle\boldsymbol{s}\rangle = 2 - k$ $(k = 0, 1, \ldots)$, and to all $\boldsymbol{z} \in \mathfrak{H}^+ \times \mathfrak{H}^-$, where

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s}; \boldsymbol{z}) = 2(2\pi)^{\langle\boldsymbol{s}\rangle}\cos\Big\{\frac{\pi}{2}(s_1 - s_2)\Big\}\Big\{\frac{T_1(\boldsymbol{s}; \boldsymbol{z})}{\Gamma(s_1)} + \frac{T_2(\boldsymbol{s}; \boldsymbol{z})}{\Gamma(s_2)}\Big\}, \tag{2.2}$$

and $T_i(\boldsymbol{s};\boldsymbol{z})$ $(i=1,2)$ *are represented for any integer* $N\geq 0$ *as*

$$T_i(\boldsymbol{s};\boldsymbol{z})=S_{i,N}(\boldsymbol{s};\boldsymbol{z})+R_{i,N}(\boldsymbol{s};\boldsymbol{z}), \tag{2.3}$$

the right sides of which provide the holomorphic continuation of $\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s};\boldsymbol{z})$ *in the region* $-N<\sigma_i<N+1$. *Here*

$$S_{i,N}(\boldsymbol{s};\boldsymbol{z})=\sum_{n=0}^{N-1}\frac{(-1)^n(s_j)_n(1-s_i)_n}{n!}\Phi_{s_i-n-1,-s_j-n}(e((-1)^jz_i))(4\pi\eta)^{-s_j-n} \tag{2.4}$$

for $i=1,2$, *both giving the asymptotic series in the descending order of* η *as* $\eta\to\infty$ *through* $|\arg\eta|<\pi/2$; *the remainders* $R_{i,N}(\boldsymbol{s};\boldsymbol{z})$ $(i=1,2)$ *satisfy*

$$R_{i,N}(\boldsymbol{s};\boldsymbol{z})=O(e^{-2\pi|\operatorname{Im}z_i|}|\eta|^{-\sigma_j-N}), \tag{2.5}$$

as $\eta\to\infty$ *through* $|\arg\eta|\leq\pi/2-\delta$ *for any small* $\delta>0$, *while* $\boldsymbol{z}\in\mathfrak{H}^+\times\mathfrak{H}^-$ *moves within* $|\operatorname{Im}z_i|\geq y_0>0$ $(i=1,2)$, *where the O-constants may depend on* $\boldsymbol{s}$, y_0, N *and* δ. *Furthermore, we have*

$$R_{i,N}(\boldsymbol{s};\boldsymbol{z})=\frac{(-1)^N(s_j)_N(1-s_i)_N}{(N-1)!}\sum_{h,k=1}^{\infty}h^{\langle\boldsymbol{s}\rangle-1}e((-1)^jhkz_i) \tag{2.6}$$
$$\times\int_0^1\xi^{-s_j-N}(1-\xi)^{N-1}U(s_j+N;\langle\boldsymbol{s}\rangle;4\pi hk\eta/\xi)d\xi$$

for $i=1,2$, *where the case* $N=0$ *should read without the factor* $(-1)!$ *and the* ξ-*integrations.*

Since $U(0;\gamma;Z)=1$ by (1.12), the case $\boldsymbol{s}=(s,0)$ and $\boldsymbol{z}=(z,z_2)$ of Theorem 1 when $N=0$ reduces to the following transformation formula for one variable holomorphic Eisenstein series $F(s;z)$ (see (1.3)).

Corollary 1.1 (Lewittes [13])**.** *Let* $F(s;z)$ *be defined in the sense of (1.9). Then for any* $s\in\mathbb{C}$ *and* $z\in\mathfrak{H}^+$ *we have the transformation formula*

$$F(s;z)=2\cos^2\Big(\frac{\pi s}{2}\Big)\zeta(s)+\frac{2(2\pi)^s}{\Gamma(s)}\cos\Big(\frac{\pi s}{2}\Big)\sum_{l=1}^{\infty}\sigma_{s-1}(l)e(z).$$

Theorem 1 implies all the known transformation formulae and asymptotic expansions for $F(s;z)$, for $\zeta_{\mathbb{Z}^2}(s;z)$, and for $E_k(s;z)$, but not for (holomorphic) $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$; it is highly desirable to invent a unified method of treatments which naturally covers both the holomorphic and non-holomorphic cases.

3. Singularities of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$

We shall make in this section a consideration on the possibilities of the polar and null sets, together with their crossings, arising from $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$.

In view of (2.1), we set

$$f(\boldsymbol{s};\boldsymbol{z})=\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})-\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s};\boldsymbol{z}).$$

Then the possibility of the poles of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ comes from the factors $\zeta(\langle\boldsymbol{s}\rangle)$, $\zeta(\langle\boldsymbol{s}\rangle-1)$ and $\Gamma(\langle\boldsymbol{s}\rangle-1)$ in $f(\boldsymbol{s};\boldsymbol{z})$; these are collected to form the (possible) polar set $\mathcal{P}$ as

$$\mathcal{P}=\big\{\boldsymbol{s}\in\mathbb{C}^2\mid\langle\boldsymbol{s}\rangle=1-2k\text{ or }\langle\boldsymbol{s}\rangle=2-2k\ (k\in\mathbb{Z}_{\geq0})\big\}.$$

On the other hand, the possibility of the zeros of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ comes from $\cos\{\pi(s_1-s_2)/2\}$ and $\Gamma\binom{1}{s_1,s_2}$ in $f(\boldsymbol{s};\boldsymbol{z})$; these are collected to form the (possible) null sets $\mathcal{N}_j$ $(j=1,2)$ as

$$\mathcal{N}_1=\{\boldsymbol{s}\in\mathbb{C}^2 \mid s_1-s_2=2l+1 \ \ (l\in\mathbb{Z})\},\quad \mathcal{N}_2 \ =\{\boldsymbol{s}\in\mathbb{C}^2 \mid s_1\in\mathbb{Z}_{\le 0} \text{ or } s_2\in\mathbb{Z}_{\le 0}\}.$$

The crossings of the polar sets and the null sets above give rise to the indeterminate singularities of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$, e.g.,

$$\mathcal{P}\cap\mathcal{N}_1=\left\{\begin{matrix}\boldsymbol{s}=(3/2-k+l,1/2-k-l)\in(\frac{1}{2}\mathbb{Z})^2,\\ \boldsymbol{s}=(1-k+l,-k-l)\in\mathbb{Z}^2\end{matrix}\ \middle|\ k\in\mathbb{Z}_{\ge 0},\ l\in\mathbb{Z}\right\},$$

$$\mathcal{P}\cap\mathcal{N}_2=\{\boldsymbol{s}=(m_1,m_2)\in\mathbb{Z}^2 \mid \ m_1\in\mathbb{Z}_{\le 0} \text{ or } m_2\in\mathbb{Z}_{\le 0}\}.$$

are the indeterminate singularities. Note in particular that

$$\mathcal{P}\cap\mathcal{N}_1\supset(\mathbb{Z}_{\ge 1})^2 \qquad\text{and}\qquad \mathcal{P}\cap\mathcal{N}_2\supset(\mathbb{Z}_{\le 0})^2 \tag{3.1}$$

It is in general known that the limiting processes toward such singularities encounter serious indeterminate situations in treating functional (limit) values. The presence of this kind of indeterminate nature in the neighbour of such singularities is crucial; however, it is in fact possible to extract the characteristics of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ by means of appropriate limiting evaluations, whose limits are taken through certain specific directions.

4. Limiting evaluations at the points $\boldsymbol{s}=\boldsymbol{m}\in\mathbb{Z}^2$

The observation made in (3.1) suggests that certain limiting evaluations are possible from Theorem 1 at the indeterminate singularities, especially when $\boldsymbol{s}\to\boldsymbol{m}\in(\mathbb{Z}_{\ge 1})^2$ and $\boldsymbol{s}\to-\boldsymbol{m}\in(\mathbb{Z}_{\le 0})^2$, for which we shall show several of their instances.

Let B_n $(n\in\mathbb{Z}_{\ge 0})$ denote the n*th* Bernoulli number (cf. [2, p.35, 1.13.(1)]), $\gamma_0=-\Gamma'(1)$ the 0*th* Euler-Stieltjes constant (cf. [2, p.34, 1.12.(19)]), and write $\boldsymbol{0}=(0,0)$, $\boldsymbol{e}_1=(1,0)$, $\boldsymbol{e}_2=(0,1)$ and $\boldsymbol{1}=(1,1)$. We use the customary notation $(q;q)_\infty=\prod_{k=1}^{\infty}(1-q^k)$ for $|q|<1$, and set $q_i=e((-1)^j z_i)$ $(i=1,2)$.

Corollary 1.2. *The following formulae hold at the points* $\boldsymbol{s}=\boldsymbol{m}\in(\mathbb{Z}_{\ge 1})^2$ *with* $\boldsymbol{m}=(m_1,m_2)$:

i) *if* $\boldsymbol{m}=\boldsymbol{1}$,

$$\lim_{\varepsilon\to 0}\left\{\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{1}+\varepsilon\boldsymbol{1};\boldsymbol{z})-\frac{\pi/\eta}{\varepsilon}\right\}=\frac{\pi^2}{3}+\frac{2\pi}{\eta}\{\gamma_0-\log(2\eta)\}+\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{1};\boldsymbol{z}),$$

where

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{1};\boldsymbol{z})=\frac{2\pi}{\eta}\{\Phi_{0,-1}(q_1)+\Phi_{0,-1}(q_2)\}=-\frac{2\pi}{\eta}\log\{(q_1;q_1)_\infty(q_2;q_2)_\infty\}$$

giving a two variable analogue of Kronecker's limit formula for $\zeta_{\mathbb{Z}^2}(s;z)$ *as* $s\to 1$;

ii) *if* $\langle\boldsymbol{m}\rangle=m_1+m_2$ *is even with* $\langle\boldsymbol{m}\rangle\ge 4$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};\boldsymbol{z})=-\frac{(2\pi i)^{\langle\boldsymbol{m}\rangle}B_{\langle\boldsymbol{m}\rangle}}{\langle\boldsymbol{m}\rangle!}+\frac{4\pi(-1)^{(m_1-m_2)/2}(\langle\boldsymbol{m}\rangle-2)!}{(m_1-1)!(m_2-1)!}\zeta(\langle\boldsymbol{m}\rangle-1)(2\eta)^{1-\langle\boldsymbol{m}\rangle}$$
$$+\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{m};\boldsymbol{z}),$$

where

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{m};\boldsymbol{z})=2(2\pi)^{\langle\boldsymbol{m}\rangle}(-1)^{(m_1-m_2)/2}\left\{\frac{S_{1,m_1}(\boldsymbol{m};\boldsymbol{z})}{(m_1-1)!}+\frac{S_{2,m_2}(\boldsymbol{m};\boldsymbol{z})}{(m_2-1)!}\right\} \tag{4.1}$$

with $S_{i,m_i}(\boldsymbol{m};\boldsymbol{z})$ $(i=1,2)$ *being given in (4.3) below;*

iii) *if* $\langle\boldsymbol{m}\rangle$ *is odd with* $\langle\boldsymbol{m}\rangle\geq 3$,

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{m};\boldsymbol{z})=0,$$

and further for $i=1,2$,

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_i}(\boldsymbol{m};\boldsymbol{z})=\frac{(-1)^{m_i}\pi^{\langle\boldsymbol{m}\rangle-1}B_{\langle\boldsymbol{m}\rangle-1}\eta^{1-\langle\boldsymbol{m}\rangle}}{(m_1-1)!(m_2-1)!(\langle\boldsymbol{m}\rangle-1)!}+\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{m};\boldsymbol{z}),$$

where

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{m};\boldsymbol{z})=\frac{1}{2}(2\pi)^{\langle\boldsymbol{m}\rangle+1}(-1)^{(m_i-m_j+1)/2}\left\{\frac{S_{i,m_i}(\boldsymbol{m};\boldsymbol{z})}{(m_i-1)!}+\frac{S_{j,m_j}(\boldsymbol{m};\boldsymbol{z})}{(m_j-1)!}\right\}. \tag{4.2}$$

Here in (4.1) and (4.2), for $i=1,2$,

$$S_{i,m_i}(\boldsymbol{m};\boldsymbol{z})=\sum_{n=0}^{m_i-1}\binom{m_i-1}{n}(m_j)_n\Phi_{m_i-n-1,-m_j-n}(q_j)(4\pi\eta)^{-m_j-n}. \tag{4.3}$$

Corollary 1.3. *The following formulae hold at the points* $\boldsymbol{s}=\boldsymbol{e}_i$ $(i=1,2)$:

i) *if* $\boldsymbol{s}=\boldsymbol{e}_1$,

$$\lim_{\varepsilon\to 0}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{e}_1+\varepsilon\boldsymbol{1};\boldsymbol{z})=0,$$

and further for $i=1,2$,

$$\lim_{\varepsilon\to 0}\left\{\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_i}(\boldsymbol{e}_1+\varepsilon\boldsymbol{1};\boldsymbol{z})+\frac{\pi^2}{\varepsilon}\right\}=(-1)^i\pi^2\{\gamma_0-\log(2\eta)\}+\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{e}_1;\boldsymbol{z}),$$

where

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{e}_1;\boldsymbol{z})=(-1)^i4\pi^2\Phi_{0,0}(q_1)=(-1)^i4\pi^2\sum_{k=1}^{\infty}\frac{q_1^k}{1-q_1^k};$$

ii) *if* $\boldsymbol{s}=\boldsymbol{e}_2$,

$$\lim_{\varepsilon\to 0}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{e}_2+\varepsilon\boldsymbol{1};\boldsymbol{z})=0,$$

and further for $i=1,2$,

$$\lim_{\varepsilon\to 0}\left\{\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_i}(\boldsymbol{e}_2+\varepsilon\boldsymbol{1};\boldsymbol{z})-\frac{\pi^2}{\varepsilon}\right\}=(-1)^j\pi^2\{\gamma_0-\log(2\eta)\}+\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{e}_2;\boldsymbol{z}),$$

where

$$\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{e}_2;\boldsymbol{z})=(-1)^j4\pi^2\Phi_{0,0}(q_2)=(-1)^j4\pi^2\sum_{k=1}^{\infty}\frac{q_2^k}{1-q_2^k}.$$

Corollary 1.4. *The following formulae hold at the points* $\boldsymbol{s}=-\boldsymbol{m}\in(\mathbb{Z}_{\leq 0})^2$ *with* $\boldsymbol{m}=(m_1,m_2)$;

i) *if* $\boldsymbol{m}=\boldsymbol{0}$,

$$\lim_{\varepsilon\to 0}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{0}+\varepsilon\boldsymbol{1};\boldsymbol{z})=0,$$

and further for $i=1,2$,

$$\lim_{\varepsilon\to 0}\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_j}(\boldsymbol{0}+\varepsilon\boldsymbol{1};\boldsymbol{z})=-\log 2\pi+\frac{2}{3}\pi\eta+\frac{\partial\widetilde{\zeta_{\mathbb{Z}^2}}^{*}}{\partial s_i}(\boldsymbol{0};\boldsymbol{z}),$$

where

$$\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}^*}{\partial s_i}(\mathbf{0};\boldsymbol{z}) = 2\Phi_{-1,0}(q_i) = -2\log(q_i;q_i)_\infty,$$

giving a two variable analogue of Kronecker's limit formula for $(\partial/\partial s)\zeta_{\mathbb{Z}^2}(s;z)$ *at* $s=0$ (*see, e.g.,* [5, (2.13) and (2.15)]);

ii) *if* $\langle \boldsymbol{m}\rangle$ *is even with* $\langle \boldsymbol{m}\rangle \geq 2$,

$$\lim_{\varepsilon\to 0}\widetilde{\zeta_{\mathbb{Z}^2}}(-\boldsymbol{m}+\varepsilon\mathbf{1};\boldsymbol{z}) = 0,$$

and further for $i=1,2$,

$$\lim_{\varepsilon\to 0}\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_i}(-\boldsymbol{m}+\varepsilon\mathbf{1};\boldsymbol{z}) = \frac{(-1)^{\langle \boldsymbol{m}\rangle/2}\langle \boldsymbol{m}\rangle!}{(2\pi)^{\langle \boldsymbol{m}\rangle}}\zeta(\langle \boldsymbol{m}\rangle+1) + 4\pi(-1)^{(m_1-m_2)/2} \times \frac{B_{\langle \boldsymbol{m}\rangle+2}m_1!m_2!}{(\langle \boldsymbol{m}\rangle+2)!}(2\eta)^{1+\langle \boldsymbol{m}\rangle} + \frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}^*}{\partial s_i}(-\boldsymbol{m};\boldsymbol{z}),$$

where

$$\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}^*}{\partial s_i}(-\boldsymbol{m};\boldsymbol{z}) = \frac{2(-1)^{m_i}m_i!}{(2\pi)^{\langle \boldsymbol{m}\rangle}}S_{i,m_j+1}(-\boldsymbol{m};\boldsymbol{z}) \tag{4.4}$$

with $S_{i,m_j+1}(-\boldsymbol{m};\boldsymbol{z})$ $(i=1,2)$ *being given in (4.6) below;*

iii) *if* $\langle \boldsymbol{m}\rangle$ *is odd,*

$$\lim_{\varepsilon\to 0}\widetilde{\zeta_{\mathbb{Z}^2}}(-\boldsymbol{m}+\varepsilon\mathbf{1};\boldsymbol{z}) = 0 \qquad \textit{and} \qquad \lim_{\varepsilon\to 0}\frac{\partial \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_i}(-\boldsymbol{m}+\varepsilon\mathbf{1};\boldsymbol{z}) = 0$$

for $i=1,2$, *and further*

$$\lim_{\varepsilon\to 0}\frac{\partial^2 \widetilde{\zeta_{\mathbb{Z}^2}}}{\partial s_1\partial s_2}(-\boldsymbol{m}+\varepsilon\mathbf{1};\boldsymbol{z}) = -\frac{\pi^2 B_{\langle \boldsymbol{m}\rangle+1}}{\langle \boldsymbol{m}\rangle+1} + \frac{\partial^2 \widetilde{\zeta_{\mathbb{Z}^2}}^*}{\partial s_1\partial s_2}(-\boldsymbol{m};\boldsymbol{z}),$$

where

$$\frac{\partial^2 \widetilde{\zeta_{\mathbb{Z}^2}}^*}{\partial s_1\partial s_2}(-\boldsymbol{m};\boldsymbol{z}) = \frac{1}{2(2\pi\sqrt{-1})^{\langle \boldsymbol{m}\rangle-1}}\left\{m_1!S_{1,m_2+1}(-\boldsymbol{m};\boldsymbol{z}) + m_2!S_{2,m_1+1}(-\boldsymbol{m};\boldsymbol{z})\right\}. \tag{4.5}$$

Here in (4.4) and (4.5), for $i=1,2$,

$$S_{i,m_j+1}(-\boldsymbol{m};\boldsymbol{z}) = \sum_{n=0}^{m_j}\binom{m_j}{n}(1+m_i)_n\Phi_{-m_i-n-1,m_j-n}(q_i)(4\pi\eta)^{m_j-n}. \tag{4.6}$$

5. Functional equations for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$

We shall show in this section that the functional equation exists for $\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s};\boldsymbol{z})$ (see (2.1) and (2.2)) under the restriction on certain hyper-planes in the $\boldsymbol{s}$-space $\mathbb{C}^2$.

Let (u) for any $u\in\mathbb{R}$ denote the vertical straight path from $u-i\infty$ to $u+i\infty$, and write $\widehat{\boldsymbol{s}}=(s_2,s_1)$ for any $\boldsymbol{s}=(s_1,s_2)\in\mathbb{C}^2$. Then it is crucial in proving Theorem 1 to apply the Mellin-Barnes type integral expressions for $T_i(\boldsymbol{s};\boldsymbol{z})$ $(i=1,2)$ in (2.3); these assert that

$$T_i(\boldsymbol{s};\boldsymbol{z}) = \frac{1}{2\pi\sqrt{-1}}\int_{(u_i)}\Gamma\binom{s_j+w,-w,1-\langle \boldsymbol{s}\rangle-w}{s_j,1-s_i} \times \Phi_{\langle \boldsymbol{s}\rangle+w-1,w}(e((-1)^j z_i))(4\pi\eta)^w dw, \tag{5.1}$$

being valid for $0<\sigma_i<1$, where u_i $(i=1,2)$ satisfy $-\sigma_j<u_i<\min(0,1-\sigma_1-\sigma_2)$. Substituting the variable $w=1-\langle\boldsymbol{s}\rangle+w'$ in (5.1), and then using the primary symmetry, by (1.13),

$$\Phi_{r,s}(e(z))=\Phi_{s,r}(e(z)),$$

for any $z\in\mathfrak{H}^+$ and any $(r,s)\in\mathbb{C}^2$, we obtain the following result by analytic continuation.

Proposition 1. *The (partial) functional equations*

$$T_i(\boldsymbol{s};\boldsymbol{z})=(4\pi\eta)^{1-\langle\boldsymbol{s}\rangle}T_i(\mathbf{1}-\widehat{\boldsymbol{s}};\boldsymbol{z})\qquad(i=1,2)\tag{5.2}$$

hold for all $\boldsymbol{s}\in\mathbb{C}^2$ *and for all* $\boldsymbol{z}\in\mathfrak{H}^+\times\mathfrak{H}^-$.

Letting

$$T^*(\boldsymbol{s};\boldsymbol{z})=\frac{T_1(\boldsymbol{s};\boldsymbol{z})}{\Gamma(s_1)}+\frac{T_2(\boldsymbol{s};\boldsymbol{z})}{\Gamma(s_2)},\tag{5.3}$$

one can see from (2.2) and (5.2) that the functional equation

$$T^*(\boldsymbol{s};\boldsymbol{z})=(4\pi\eta)^{1-\langle\boldsymbol{s}\rangle}\chi^*(\boldsymbol{s})T^*(\mathbf{1}-\widehat{\boldsymbol{s}};\boldsymbol{z})\tag{5.4}$$

holds if

$$\Gamma\binom{1-s_2}{s_1}=\chi^*(\boldsymbol{s})=\Gamma\binom{1-s_1}{s_2}\Longleftrightarrow\sin(\pi s_1)=\sin(\pi s_2)$$
$$\Longleftrightarrow s_1+s_2=2m+1\quad\text{or}\quad s_1-s_2=2m\quad(\text{for any } m\in\mathbb{Z}).$$

It is therefore seen from (2.2), (5.2) and (5.4), upon setting $\chi(\boldsymbol{s};\boldsymbol{z})=(\pi/\eta)^{\langle\boldsymbol{s}\rangle-1}\chi^*(\boldsymbol{s})$, that the following functional equation for $\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s};\boldsymbol{z})$ is valid.

Theorem 2. *The functional equation*

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s};\boldsymbol{z})=\chi(\boldsymbol{s};\boldsymbol{z})\widetilde{\zeta_{\mathbb{Z}^2}}^*(\mathbf{1}-\widehat{\boldsymbol{s}};\boldsymbol{z})$$

holds on the hyper-planes $s_1+s_2=2m+1$ *and* $s_1-s_2=2m$ *for any* $m\in\mathbb{Z}$*, where*

$$\chi(\boldsymbol{s};\boldsymbol{z})=\left(\frac{\pi}{\eta}\right)^{\langle\boldsymbol{s}\rangle-1}\Gamma\binom{1-s_1}{s_2}.$$

6. Asymptotic expansions for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ as $\eta\to 0$

We shall show in this section that asymptotic expansions for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$ as $\eta\to\infty$ in Theorem 1 can be transferred to those as $\eta\to 0$ through the quasi-modular relation (6.5) below.

We first introduce for convenience of description the new parameters τ_i $(i=1,2)$ as

$$\tau_1=e^{-\pi\sqrt{-1}/2}z_1\qquad\text{and}\qquad\tau_2=e^{\pi\sqrt{-1}/2}z_2.$$

One can see from (1.1) and (1.7) that

$$|\arg\tau_1|=\left|\arg z_1-\frac{\pi}{2}\right|<\frac{\pi}{2}\qquad\text{and}\qquad|\arg\tau_2|=\left|\arg z_2+\frac{\pi}{2}\right|<\frac{\pi}{2},$$

and further on the pivotal parameter η in (1.10) that

$$\eta=\eta(\boldsymbol{z})=\frac{1}{2}e^{-\pi\sqrt{-1}/2}(z_1+e^{\pi\sqrt{-1}}z_2)=\frac{1}{2}(\tau_1+\tau_2).$$

Let φ be the (fractional) transformation defined by

$$\mathfrak{H}^+\times\mathfrak{H}^-\ni\boldsymbol{z}=(z_1,z_2)\overset{\varphi}{\longmapsto}(-1/z_1,-1/z_2)=-\mathbf{1}/\boldsymbol{z}\in\mathfrak{H}^+\times\mathfrak{H}^-,\tag{6.1}$$

which is, in terms of (τ_1, τ_2), equivalent to

$$\mathfrak{H}^+ \times \mathfrak{H}^- \ni \boldsymbol{z} = (i\tau_1, -i\tau_2) \overset{\varphi}{\longmapsto} (i/\tau_1, -i/\tau_2) = -\mathbf{1}/\boldsymbol{z} \in \mathfrak{H}^+ \times \mathfrak{H}^-.$$

Hence observing the correspondence $(i\infty, -i\infty) \overset{\varphi}{\longmapsto} (0,0)$ or $(0,0) \overset{\varphi}{\longmapsto} (i\infty, -i\infty)$ (with an extended sense of φ), one finds on the movement of $\boldsymbol{z} \in \mathfrak{H}^+ \times \mathfrak{H}^-$ that

$$\text{(6.2)} \qquad \begin{aligned} &\boldsymbol{z} \text{ tends to } (0,0) \text{ through } \mathfrak{H}^+ \times \mathfrak{H}^- \\ &\iff -\mathbf{1}/\boldsymbol{z} \text{ tends to } (i\infty, -i\infty) \text{ through } \mathfrak{H}^+ \times \mathfrak{H}^-, \end{aligned}$$

which is, in terms of (τ_1, τ_2), equivalent to

$$\text{(6.3)} \qquad \begin{aligned} &(\tau_1, \tau_2) \text{ tends to } (0,0) \text{ through } |\arg \tau_i| < \pi/2 \ (i = 1, 2) \\ &\iff (1/\tau_1, 1/\tau_2) \text{ tends to } (\infty, \infty) \text{ through } |\arg \tau_i| < \pi/2 \ (i = 1, 2) \end{aligned}$$

Note further that the new pivotal parameter $\check{\eta}$, transformed from η by (6.1), is of the form

$$\text{(6.4)} \qquad \check{\eta} = \eta(-\mathbf{1}/\boldsymbol{z}) = \frac{1}{2} e^{-\pi\sqrt{-1}/2} \Big(\frac{e^{\pi\sqrt{-1}/2}}{\tau_1} + e^{\pi\sqrt{-1}} \cdot \frac{e^{-\pi\sqrt{-1}/2}}{\tau_2} \Big) = \frac{1}{2} \Big(\frac{1}{\tau_1} + \frac{1}{\tau_2} \Big).$$

The asymptotic expansions for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z})$ as $\eta \to \infty$ (in Theorem 1) can therefore be switched to those as $\eta \to 0$ through the quasi-modular relation

$$\text{(6.5)} \qquad \widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z}) = \frac{e^{-\pi\sqrt{-1}(s_1 - s_2)/2}}{\tau_1^{s_1} \tau_2^{s_2}} \widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z})$$

for all $\boldsymbol{s} \in \mathbb{C}^2$ except the hyper-planes $\langle \boldsymbol{s} \rangle = 2 - k$ $(k = 0, 1, \ldots)$, and in the poly-sector $|\arg \tau_i| < \pi/2$ $(i = 1, 2)$; Theorem 1 applied for $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z})$ described with the transformed parameter $\check{\eta}$ implies the following result.

Theorem 3 ([10, Theorem 3])**.** *The formula*

$$\begin{aligned} \widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s}; \boldsymbol{z}) = \frac{e^{-\pi\sqrt{-1}(s_1 - s_2)/2}}{\tau_1^{s_1} \tau_2^{s_2}} \Big[&2 \cos^2 \Big\{ \frac{\pi}{2}(s_1 - s_2) \Big\} \zeta(\langle \boldsymbol{s} \rangle) + 4\pi \cos \Big\{ \frac{\pi}{2}(s_1 - s_2) \Big\} \\ &\times \Gamma \binom{\langle \boldsymbol{s} \rangle - 1}{s_1, s_2} \zeta(\langle \boldsymbol{s} \rangle - 1) \Big(\frac{2\tau_1\tau_2}{\tau_1 + \tau_2} \Big)^{\langle \boldsymbol{s} \rangle - 1} + \widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z}) \Big] \end{aligned}$$

holds in the whole $\boldsymbol{s}$*-space* $\mathbb{C}^2$ *except the hyper-planes* $\langle \boldsymbol{s} \rangle = 2 - k$ $(k = 0, 1, \ldots)$*, and in the poly-sector* $|\arg \tau_i| < \pi/2$ $(i = 1, 2)$*, where*

$$\widetilde{\zeta_{\mathbb{Z}^2}}^*(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z}) = 2(2\pi)^{\langle \boldsymbol{s} \rangle} \cos \Big\{ \frac{\pi}{2}(s_1 - s_2) \Big\} \Big\{ \frac{T_1(\boldsymbol{s}, -\mathbf{1}/\boldsymbol{z})}{\Gamma(s_1)} + \frac{T_2(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z})}{\Gamma(s_2)} \Big\},$$

and $T_i(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z})$ $(i = 1, 2)$ *are represented for any integer* $N \geq 0$ *as*

$$T_i(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z}) = S_{i,N}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z}) + R_{i,N}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z})$$

in the region $-N < \sigma_i < N + 1$*. Here*

$$S_{i,N}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z}) = \sum_{n=0}^{N-1} \frac{(-1)^n (s_j)_n (1 - s_i)_n}{n!} \Phi_{s_i - n - 1, -s_j - n}(e^{-2\pi/\tau_i}) \Big\{ \frac{\tau_1\tau_2}{2\pi(\tau_1 + \tau_2)} \Big\}^{s_j + n}$$

for $i = 1, 2$*, both giving the asymptotic series in the ascending order of* τ_i *as* $\tau_i \to 0$ *through the poly-sector* $|\arg \tau_i| < \pi/2$*; the remainders* $R_{i,N}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z})$ $(i = 1, 2)$ *satisfy*

$$R_{i,N}(\boldsymbol{s}; -\mathbf{1}/\boldsymbol{z}) = O \Big\{ e^{-2\pi/|\tau_i|} \Big(\frac{|\tau_1\tau_2|}{|\tau_1| + |\tau_2|} \Big)^{\sigma_j + N} \Big\}$$

as $\tau_i \to 0$ *through* $|\arg\tau_i| \leq \pi/2-\delta$ *for any small* $\delta > 0$*, where the O-constants may depend on* $\boldsymbol{s}$*,* y_0*,* N *and* δ*.*

7. Laplace-Mellin transform of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$

We shall show in this section the asymptotic expansions for the Laplace-Mellin transform of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z})$.

Let α be a complex number with $\operatorname{Re}\alpha > 0$, Y a real positive parameter, and write $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{d} = \boldsymbol{e}_1 - \boldsymbol{e}_2 = (1,-1)$. Then the formulation of the Laplace-Mellin transform here is of the form

$$\mathcal{LM}^{\alpha}_{Y;y}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{x}+\sqrt{-1}y\boldsymbol{d}) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{x}+\sqrt{-1}yY\boldsymbol{d})y^{\alpha-1}e^{-y}dy$$

with the normalization gamma multiple, where the factor $y^{\alpha-1}$ is inserted to secure the convergence of the integral as $y \to 0^+$, while e^{-y} has an effect to extract the portion of $\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{x}+\sqrt{-1}y\boldsymbol{d})$ corresponding to $y = O(Y)$. Let ${}_2F_1\left({\alpha,\beta \atop \gamma};Z\right)$ denote the Gauß' hypergeometric function defined for any $(\alpha,\beta,\gamma) \in \mathbb{C}^2 \times (\mathbb{C}\setminus\mathbb{Z}_{\leq 0})$ by

$${}_2F_1\left({\alpha,\beta \atop \gamma};Z\right) = \sum_{k=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}Z^n \qquad (|Z|<1),$$

which can be continued to a holomorphic function of Z in the sector $|\arg(1-Z)| < \pi$ by Euler's formula (cf. [2, p.56, 2.1.1.(2); p.59, 2.1.3.(10)]).

Theorem 4. *Let* α *be a complex number with* $\operatorname{Re}\alpha > 1$*,* x_j $(j=1,2)$ *and* Y *any positive real parameter. Then the formula*

$$\begin{aligned}\mathcal{LM}^{\alpha}_{Y;y}\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{x}+\sqrt{-1}y\boldsymbol{d}) &= 2\cos^2\left\{\frac{\pi}{2}(s_1-s_2)\right\}\zeta(\langle\boldsymbol{s}\rangle) + 4\pi\cos\left\{\frac{\pi}{2}(s_1-s_2)\right\}\\ &\quad\times\Gamma\left({\langle\boldsymbol{s}\rangle-1,\alpha+1-\langle\boldsymbol{s}\rangle \atop s_1,s_2,\alpha}\right)(2Y)^{1-\langle\boldsymbol{s}\rangle}\\ &\quad+\mathcal{LM}^{\alpha}_{Y;y}\widetilde{\zeta_{\mathbb{Z}^2}}^{*}(\boldsymbol{s};\boldsymbol{x}+\sqrt{-1}y\boldsymbol{d})\end{aligned}$$

holds, where

$$\begin{aligned}\mathcal{LM}^{\alpha}_{Y;y}\widetilde{\zeta_{\mathbb{Z}^2}}^{*}(\boldsymbol{s},\boldsymbol{x}+\sqrt{-1}y\boldsymbol{d}) &= 2(2\pi)^{\langle\boldsymbol{s}\rangle}\cos\left\{\frac{\pi}{2}(s_1-s_2)\right\}\\ &\quad\times\left\{\frac{T_1^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y)}{\Gamma(s_1)}+\frac{T_2^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y)}{\Gamma(s_2)}\right\},\end{aligned}$$

and $T_i^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y)$ $(i=1,2)$ *are represented for any integer* $N \geq 0$ *as*

$$T_{i,N}^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y) = S_{i,N}^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y) + R_{i,N}^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y)$$

in the region $\sigma_1+\sigma_2 < \operatorname{Re}\alpha$*. Here*

$$\begin{aligned}&S_{i,N}^{\alpha}(\boldsymbol{s},\boldsymbol{x};Y)\\ &\quad=\Gamma\left({\alpha+1-\langle\boldsymbol{s}\rangle \atop \alpha+1-s_i}\right)\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n(\alpha+1-\langle\boldsymbol{s}\rangle)_n}{n!(\alpha+1-s_i)_n}\\ &\qquad\times{}_2F_1\left({\alpha+n,s_j \atop \alpha+n+1-s_i};-1\right)\Phi_{\langle\boldsymbol{s}\rangle-1-\alpha-n,-\alpha-n}(e((-1)^j x_i))(2\pi Y)^{-\alpha-n}\end{aligned}$$

for $i=1,2$, both giving the asymptotic series in the descending order of Y as $Y\to+\infty$; the reminders $R^{\alpha}_{i,N}(\boldsymbol{s},\boldsymbol{x};Y)$ $(i=1,2)$ satisfy

$$R^{\alpha}_{i,N}(\boldsymbol{s},\boldsymbol{x};Y)=O(Y^{-\operatorname{Re}\alpha-N})$$

as $Y\to+\infty$, where the O-constant may depend on $\boldsymbol{s}$, α and N. Furthermore for $i=1,2$, we have

$$\begin{aligned}&R^{\alpha}_{i,N}(\boldsymbol{s},\boldsymbol{x};Y)\\&=\frac{(-1)^N(\alpha)_N}{(N-1)!}\Gamma\binom{\alpha+1-\langle\boldsymbol{s}\rangle}{\alpha+1-s_i}\sum_{h,k=1}^{\infty}h^{\langle\boldsymbol{s}\rangle-1}e((-1)^j hkx_i)\int_0^{\infty}\xi^{-\alpha-N}(1-\xi)^{N-1}\\&\quad\times(1+2\pi hkY/\xi)^{-\alpha-N}{}_2F_1\left(\begin{matrix}\alpha+N,s_j\\ \alpha+N+1-s_i\end{matrix};\frac{1-2\pi hkY/\xi}{1+2\pi hkY/\xi}\right)d\xi.\end{aligned}$$

8. Outline of the proof

We shall show in this section the outline of the proof of Theorem 1. For this, introduce first the two variable analogue of the bilateral Hurwitz zeta-function defined for any $\boldsymbol{z}\in\mathfrak{H}^+\times\mathfrak{H}^-$ by

$$\widetilde{\zeta}_{\mathbb{Z}}(\boldsymbol{s};\boldsymbol{z})=\sum_{m=-\infty}^{\infty}(m+z_1)^{-s_1}(m+z_2)^{-s_2}\qquad(\operatorname{Re}\langle\boldsymbol{s}\rangle>1).$$

Then the most crucial step of the proof of Theorem 1 is to establish the following result.

Proposition 2 (cf. [10]). *The transformation formula*

$$\begin{aligned}\widetilde{\zeta}_{\mathbb{Z}}(\boldsymbol{s};\boldsymbol{z})=e^{-\pi\sqrt{-1}(s_1-s_2)/2}\Bigg[&2\pi\Gamma\binom{\langle\boldsymbol{s}\rangle-1}{s_1,s_2}(2\eta)^{1-\langle\boldsymbol{s}\rangle}\\&+(2\pi)^{\langle\boldsymbol{s}\rangle}\Bigg\{\frac{1}{\Gamma(s_1)}\sum_{l=1}^{\infty}l^{\langle\boldsymbol{s}\rangle-1}e(lz_1)U(s_2;\langle\boldsymbol{s}\rangle;4\pi l\eta)\\&+\frac{1}{\Gamma(s_2)}\sum_{l=1}^{\infty}l^{\langle\boldsymbol{s}\rangle-1}e(-lz_2)U(s_1;\langle\boldsymbol{s}\rangle;4\pi l\eta)\Bigg\}\Bigg]\end{aligned}\tag{8.1}$$

holds for all $\boldsymbol{s}\in\mathbb{C}^2$ except the hyper-planes $\langle\boldsymbol{s}\rangle=1-k$ $(k=0,1,\ldots)$, and all $\boldsymbol{z}\in\mathfrak{H}^+\times\mathfrak{H}^-$; the right side provides the meromorphic continuation of $\widetilde{\zeta}_{\mathbb{Z}}(\boldsymbol{s};\boldsymbol{z})$ to the whole $\boldsymbol{s}$-space $\mathbb{C}^2$.

A central rôle in deriving Proposition 2 is played by the following formula.

Lemma 1 (cf. [10]). *For any $\alpha,\beta\in\mathbb{C}$ the Mellin-Barnes formula*

$$\begin{aligned}(1-Z)^{-\alpha}=&\frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{C}_1}\Gamma\left(\begin{matrix}\alpha+w,1-\beta+w,\beta-w,-w\\ \alpha,\beta,1-\beta\end{matrix}\right)Z^w dw\\&+\frac{Z^{\beta}}{2\pi\sqrt{-1}}\int_{\mathcal{C}_2}\Gamma\left(\begin{matrix}\alpha+\beta+w,1+w,-\alpha-w,-w\\ \alpha,\beta,1-\alpha\end{matrix}\right)(1-Z)^w dw\end{aligned}$$

holds for $|\arg Z|<2\pi$ and $|\arg(1-Z)|<2\pi$, where the path $\mathcal{C}_1$ proceeds upward to separate the poles of the integrand at $w=-\alpha-n$, $\beta-1-n$ $(n=0,1,\ldots)$ from those at $w=n$, $\beta-1+n$ $(n=0,1,\ldots)$, while $\mathcal{C}_2$ at $w=-1-n$, $-\alpha-\beta-n$ $(n=0,1,\ldots)$ from those at $w=n$, $-\alpha+n$ $(n=0,1,\ldots)$.

Proof of Theorem 1. It follows from the series rearrangements of the defining series in (1.8) with (1.9) that

$$\widetilde{\zeta_{\mathbb{Z}^2}}(\boldsymbol{s};\boldsymbol{z}) = 2\cos^2\Bigl\{\frac{\pi}{2}(s_1-s_2)\Bigr\}\zeta(\langle\boldsymbol{s}\rangle) + \bigl\{1+e^{\pi\sqrt{-1}(s_1-s_2)}\bigr\} \times \sum_{n=1}^{\infty} n^{-\langle\boldsymbol{s}\rangle}\sum_{r=0}^{n-1}\widetilde{\zeta_{\mathbb{Z}}}\Bigl(\boldsymbol{s};\boldsymbol{z}+\frac{r}{n}\mathbf{1}\Bigr),$$

where the last r-sum of the bilateral Hurwitz zeta-functions can be transformed by (8.1); this concludes Theorem 1. □

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