LARGE DEVIATION PRINCIPLE FOR ARITHMETIC MEAN OF CONTINUED FRACTION DIGITS

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[1] Each irrational number $x \in (0,1)$ has the continued fraction expansion

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}},$$

where each a_i is positive integers. Khinchin [4] noted that for λ-almost every $x \in (0,1)$ the inequality $a_n(x) > n \log n$ holds for infinitely many $n \in \mathbb{N}$ as a consequence of Borel-Bernstein's theorem [1, 2, 3], and thus

$$\limsup_{n\to\infty} \frac{a_n}{n} = \infty \quad \lambda\text{-a.e.},$$

and S_n/n does not converge where $S_n = a_1 + a_2 + \cdots + a_n$. In fact,

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty \quad \lambda\text{-a.e.} \tag{0.1}$$

Hence, none of the classical limit theorems in probability, such as the weak and strong laws of large numbers and central limit theorems hold for the sum. In this paper we establish the level-1 Large Deviation Principle (LDP), i.e., show the existence of a rate function which estimates exponential probabilities with which the time average S_n/n stays away from ∞.

Main Theorem. *Let $\psi\colon \mathbb{N}\backslash\{0\} \to \mathbb{R}$ satisfy $\lim_{n\to\infty} \psi(n) = \infty$ and $\int \psi(a_1(x))dx = \infty$. There exists $\alpha^- \in \mathbb{R}$ such that the following holds:*

- *for every $\alpha \in \mathbb{R}$ the limit*

$$J(\alpha) := -\lim_{n\to\infty} \frac{1}{n} \log \lambda \left\{ x \in (0,1) \setminus \mathbb{Q} \colon \frac{1}{n} \sum_{i=1}^{n} \psi(a_i(x)) \leq \alpha \right\}$$

exists and is finite if and only if $\alpha \geq \alpha^-$. The function $\alpha \in [\alpha^-, \infty) \mapsto J(\alpha)$ is lower semi-continuous, strictly positive, convex, strictly monotone decreasing and $J(\alpha) \to 0$ as $\alpha \to \infty$;

[1]Part of the contents of this paper has been taken from the introduction of [5].

- *for every* $\alpha \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda \left\{ x \in (0,1) \setminus \mathbb{Q} \colon \frac{1}{n} \sum_{i=1}^{n} \psi(a_i(x)) \geq \alpha \right\} = 0.$$

To obtain the result for the arithmetic mean of digits, take $\psi(n) = n$. For a proof of the main theorem, see [5]. The arithmetic mean of digits of the backward continued fraction expansion has a totally different behavior, see [6].

References

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