ON THE p-ADIC BEHAVIORS OF STIRLING NUMBERS OF THE FIRST AND SECOND KINDS

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ABSTRACT. Let n and k be positive integers. The Stirling number of the first kind, denoted by $s(n,k)$, counts the number of permutations of n elements with k disjoint cycles. The Stirling number of the second kind, denoted by $S(n,k)$, is defined as the number of ways to partition a set of n elements into exactly k nonempty subsets. We have $s(n,k)=(n-1)!H(n-1,k-1)$ with $H(n-1,k-1)$ being the $(k-1)$-th symmetric function of $1, \frac{1}{2}, ..., \frac{1}{n-1}$ and

$$S(n,k)=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

Let p be a prime and $v_p(n)$ stand for the p-adic valuation of n, i.e., $v_p(n)$ is the biggest nonnegative integer r with p^r dividing n. Divisibility properties of Stirling numbers of the first and second kinds have been studied from a number of different perspectives. In this survey paper, we mainly review some old and recent new results on the p-adic behaviors of Stirling numbers of both kinds. Meanwhile, we also propose some problems and conjectures to promote the research in this area in the future.

1. Introduction

The Stirling numbers are common topics in number theory and combinatorics. Let $\mathbb{N}$ denote the set of natural numbers. The Stirling number of the first kind, denoted by $s(n,k)$ (with a lower-case "s"), counts the number of permutations of n elements with k disjoint cycles. The Stirling number of the second kind, denoted by $S(n,k)$ (with a capital "S"), is defined for $n\in\mathbb{N}$ and positive integer $k\le n$ as the number of ways to partition a set of n elements into exactly k non-empty subsets. One can characterize the Stirling numbers $s(n,k)$ of the first kind and the Stirling numbers $S(n,k)$ of the second kind by

$$(x)_n=\sum_{k=0}^{n}s(n,k)x^k$$

and

$$x^n=\sum_{k=0}^{n}(-1)^{n-k}S(n,k)(x)_k,$$

respectively, where $(x)_n$ is the rising factorial $(x)_n := x(x+1)...(x+n-1)$. The Stirling numbers of the first and second kinds can be considered to be inverses of one another:

$$\sum_{l=0}^{\max(j,k)}(-1)^{l-j}s(l,j)S(k,l)=\delta_{jk}$$

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and

$$\sum_{l=0}^{\max(j,k)} (-1)^{k-l} S(l,j)s(k,l) = \delta_{jk},$$

where δ_{jk} is the Kronecker delta.

The Stirling number $S(n,k)$ of the second kind satisfies the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k),$$

with initial condition: $S(0,0) = 1$ and $S(n,0) = 0$ for $n \geq 1$. There is also an explicit formula in terms of binomial coefficients given by

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n. \tag{1.1}$$

Given a prime p and a nonzero integer m, there exist unique integers a and r with $p \nmid a$ and $r \geq 0$, such that $m = ap^r$. The number r is called the *p-adic valuation* of m, and denoted by $v_p(m) := r$. Define $v_p(0) := \infty$. If $x = \frac{m_1}{m_2}$, where m_1 and m_2 are integers and $m_2 \neq 0$, then we define $v_p(x) := v_p(m_1) - v_p(m_2)$ (see, for example, [14]). It is easy to see that the arithmetic function v_p is completely additive. That is, $v_p(mn) = v_p(m) + v_p(n)$ holds for all positive integers m and n. Furthermore, we have

$$v_p(x+y) \geq \min\{v_p(x), v_p(y)\},$$

and if $v_p(x) \neq v_p(y)$, then $v_p(x+y) = \min\{v_p(x), v_p(y)\}$. This property is known as the *isosceles triangle principle* [14].

Divisibility properties of Stirling numbers of the second kind have been studied from a number of different perspectives. It is known that for each fixed k, the sequence $\{S(n,k) : n \geq k\}$ is periodic modulo prime powers. The length of this period has been studied by Carlitz [3] and Kwong [16]. Chan and Manna [4] characterized $S(n,k)$ modulo prime powers in terms of binomial coefficients. The numbers $\min\{v_p(k!S(n,k)) : m \leq k \leq n\}$ are important in algebraic topology. Some work on evaluating $v_p(k!S(n,k))$ has been published.

This is a survey paper. We mainly review some old and recent new results on the p-adic behaviors of Stirling numbers of the first and second kinds. Meanwhile, we also propose some problems and conjectures to promote the research in this area in the future.

This paper is organized as follows. First of all, in Section 2, we reveal some p-adic properties of Stirling numbers of the second kind. We also recall some problems and conjectures in this topic. Finally, in Section 3, we review some old and recent new results on the p-adic valuations of Stirling numbers of the first kind. Several conjectures are also raised in the last section.

2. p-Adic valuations of Stirling numbers $S(n,k)$ of the second kind

The study of p-adic valuations of Stirling numbers of the second kind is important in algebraic topology and full with challenging problems (see, for instance, [1]-[12], [17]-[19], [21], [27], [28]). Lengyel [17] studied the 2-adic valuations of $S(n,k)$ and conjectured, proved by Wannemacker [24], that $v_2(S(2^n,k)) = s_2(k) - 1$, where $s_2(k)$ means the base 2 digital sum of k. Lengyel showed that if $1 \leq k \leq 2^n$, then $v_2(S(c2^n,k)) = s_2(k) - 1$ for any positive integer c. Meanwhile, Lengyel proved that $v_2(S(c2^n,k)) \geq s_2(k) - 1$ if $c \geq 1$ is an odd integer and $1 \leq k \leq 2^{n+1}$. Amdeberhan, Manna and Moll [2] studied the 2-adic valuations of Stirling numbers of the second kind, and also conjectured that

$v_2(S(4n, 5)) \neq v_2(S(4n+3, 5))$ if and only if $n \in \{32j+7 : j \in \mathbb{N}\}$. In 2012, Hong, Zhao and Zhao [12] proved that this conjecture is true. Namely, they showed the following result.

Theorem 2.1. [12] *Let $n \geq 1$ be an integer. Then $v_2(S(4n, 5)) \neq v_2(S(4n+3, 5))$ if and only if $n \in \{32j+7 : j \in \mathbb{N}\}$.*

Furthermore, Hong, Zhao and Zhao [12] confirmed another conjecture of Amdeberhan, Manna and Moll [2] raised in 2008 by showing the following result.

Theorem 2.2. [12] *Let n and k be nonnegative integers with $k \leq 2^n$. Then*

$$v_2(S(2^n+1, k+1)) = s_2(k) - 1.$$

In 2014, Zhao, Hong and Zhao [27] showed that a more general result true. That is, we have

Theorem 2.3. [27] *Let $n, a, b, c \in \mathbb{N}$ with $0 < a < 2^{n+1}$, $b2^{n+1} + a \leq c2^n$ and $c \geq 1$ being odd. Then $v_2(S(c2^n, b2^{n+1}+a)) \geq s_2(a) - 1$.*

If one picks $b = \frac{c-1}{2}$ and $1 \leq a \leq 2^n$, then the lower bound in Theorem 2.3 above is arrived as the following result shows.

Theorem 2.4. [27] *Let $a, c, n \in \mathbb{N}$ with $c \geq 1$ being odd, $n \geq 2$ and $1 \leq a \leq 2^n$. Then*

$$v_2(S(c2^n, (c-1)2^n + a)) = s_2(a) - 1.$$

In 2009, Lengyel [18] studied the 2-adic valuations of the difference $S(c2^{n+1}, k) - S(c2^n, k)$ with $1 \leq k \leq 2^n$ and $c \geq 1$ odd. In the meantime, Lengyel [18] posed the following conjecture.

Conjecture 2.5. [18] *Let $n, k, a, b \in \mathbb{N}$, $c \geq 1$ being odd and $3 \leq k \leq 2^n$. Then*

$$v_2(S(c2^{n+1}, k) - S(c2^n, k)) = n + 1 - f(k) \tag{2.1}$$

and

$$v_2(S(a2^n, k) - S(b2^n, k)) = n + 1 + v_2(a-b) - f(k) \tag{2.2}$$

for some function $f(k)$ which is independent of n.

Note that Lengyel [18] proved in 2009 that (2.1) is true for any integer k with $s_2(k) \leq 2$. As usual, for any real number x, we let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer no less than x and the biggest integer no more than x, respectively. We have the following result.

Theorem 2.6. [27] *Let $n, k, a, b \in \mathbb{N}$, $c \geq 1$ being odd, $3 \leq k \leq 2^n$, and $a > b$. If k is not a power of 2 minus 1, then*

$$v_2(S(a2^n, k) - S(b2^n, k)) = n + v_2(a-b) - \lceil \log_2 k \rceil + s_2(k) + \delta(k),$$

where $\delta(4) = 2$, $\delta(k) = 1$ if $k > 4$ is a power of 2, and $\delta(k) = 0$ otherwise. In particular,

$$v_2(S(c2^{n+1}, k) - S(c2^n, k)) = n - \lceil \log_2 k \rceil + s_2(k) + \delta(k).$$

By Theorem 2.6, we know that Conjecture 2.5 is true except when k is a power of 2 minus 1. In 2014, one cannot prove Conjecture 2.5 for the remaining case that k is a power of 2 minus 1 because one encountered difficulties in strengthening the Junod's congruence about the Bell polynomials. But in 2015, Zhao, Zhao and Hong [28] can treat with the remaining case by introducing a new method. They proved the following result.

Theorem 2.7. [28] *Let $c, m, n \in \mathbb{N}$ with $c \geq 1$ being odd and $2 \leq m \leq n$. Then*

$$\nu_2(S(c2^{n+1}, 2^m - 1) - S(c2^n, 2^m - 1)) = n + 1$$

except when $n = m = 2$ and $c = 1$, in which case one has $\nu_2(S(8,3) - S(4,3)) = 6$.

Remark. From Theorem 2.7, one knows that the first part of Conjecture 2.5 (i.e. (2.1)) holds except for the case that $n = m = 2$ and $c = 1$, in which case it is not true. But the truth of the second part of Conjecture 2.5 (i.e. (2.2)) remains open when $a > b$ and k is equal to a power of 2 minus 1.

In closing this section, let us describe an interesting and fantastic conjecture of Clark [5] raised in 1995. Let p be a prime. For any positive integer n and k, one defines

$$T_p(n,k) := \sum_{\substack{i=0 \\ p \nmid i}}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

By formula (3.1) we can see that $k!S(n,k) - T_p(n,k)$ is divisible at least by p^n. In this sense, $T_p(n,k)$ also known as Stirling-like numbers [5]. In 1990, Davis [7] gave a method for calculating $v_2(T_2(n,5))$ and $v_2(T_2(n,6))$. Then Clarke [5] generalized this result by applying Hensel's lemma on the p-adic integers. Furthermore, Clarke suggested the following conjecture.

Conjecture 2.8. [5] *Let n and k be nonnegative integers such that $n \geq k+1$ and let p be an odd prime. Then*

$$v_p(T_p(n, n-k)) < n.$$

This is equivalent to the following conjecture.

Conjecture 2.9. [5] *Let n and k be nonnegative integers such that $n \geq k+1$ and let p be an odd prime. Then*

$$v_p((n-k)!S(n, n-k)) < n.$$

Evidently, the truth of Conjecture 2.10 implies that the truth of the following conjecture also proposed by Clarke in [5].

Conjecture 2.10. [5] *Let n and k be nonnegative integers such that $n \geq k+1$ and let p be an odd prime. Then*

$$v_p((n-k)!S(n, n-k)) = v_p(T(n, n-k)).$$

Clarke [5] proved the truth of Conjecture 2.9 (hence Conjecture 10) for the cases that $1 \leq k \leq 4$. In [10], Feng and Qiu presented a formula for $v_p(S(n, n-k))$ in terms of n and k. For the case that $1 \leq k \leq 7$, they arrived at an explicit formula for $v_p(S(n, n-k))$ in terms of n. Then they used these formulae to show the following result.

Theorem 2.11. [10] *Let p be an odd prime. Let k be an integer such that $0 \leq k \leq 7$. For any positive integer n with $n \geq k+1$, one has*

$$v_p((n-k)!S(n, n-k)) < n.$$

From Theorem 2.11, one can read that Conjecture 2.9 (hence Conjecture 10) holds for the case when $0 \leq k \leq 7$. Although the method of Feng and Qiu [10] can be used to check Conjectures 2.9 and 2.10 for more smaller integers $k \geq 8$, it seems to be hard to check the truth of Conjectures 2.9 and 2.10 for all positive integers $k \leq n-1$. One needs some new ideas and approachs attacking Conjectures 2.9 and 2.10.

3. p-Adic valuations of Stirling numbers $s(n,k)$ of the first kind

We note that, unlike for the Stirling number $S(n,k)$ of the second kind, there is no easy way to use an explicit formula for $s(n,k)$, the Stirling number of the first kind, and this fact makes it more difficult to deal with certain characteristics of these numbers from the point of view of congruential and divisibility properties. In fact, the Stirling numbers of the first kind exhibit very different characteristics from the Stirling numbers of the the second kind.

On the other hand, $H(n,k)$ is closely related to the Stirling number of the first kind by the following identity

$$s(n,k) = (n-1)!H(n-1,k-1), \tag{3.1}$$

where $H(n-1,k-1)$ stands for the $(k-1)$-th symmetric function of $1, \frac{1}{2}, \ldots, \frac{1}{n-1}$. Actually, since

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} s(n,k)x^k,$$

we can easily obtain that

$$s(n,k) = \sum_{\substack{(i_1,\ldots,i_{n-k})\in\mathbb{Z}^{n-k}\\ 1\le i_1<\cdots<i_{n-k}\le n-1}} i_1\cdots i_{n-k}.$$

Hence $s(n,1) = (n-1)!$, and if $k \geq 2$, then

$$\begin{aligned} s(n,k) &= (n-1)! \sum_{\substack{(j_1,\ldots,j_{k-1})\in\mathbb{Z}^{k-1}\\ 1\le j_1<\cdots<j_{k-1}\le n-1}} \frac{1}{j_1\cdots j_{k-1}} \\ &= (n-1)!H(n-1,k-1) \end{aligned}$$

as (3.1) desired. So (3.1) is true.

For every fixed integer $k \geq 1$, Lengyel proved that, unlike for Stirling numbers of the second kind, the p-adic order of $s(n,k)$ becomes arbitrarily large for large values of n.

Theorem 3.1. [19] *For any prime p and any integer $k \geq 1$, we have*

$$\lim_{n\to\infty} v_p(s(n,k)) = \infty.$$

Actually, Lengyel showed *conditionally* the following result.

Theorem 3.2. [19] *For any prime pand integer $k \geq 1$, one has*

$$\lim_{n\to\infty} \frac{v_p(s(n,k))}{n} = \frac{1}{p-1}.$$

Also Lengyel proved the following result.

Theorem 3.3. [19] *For any integer $n \geq 1$, we have $v_2(s(2^n,3)) = 2^n - 3n + 3$.*

Based on the theory of Newton polygons, Komatsu and Young showed the following results.

Theorem 3.4. [15] *Let k be a nonnegative integer and p be a prime. Suppose that n is of the form $n = kp^r + m$, where $0 \leq m < p^r$. Then $v_p((s(n+1,k+1)) = v_p(n!) - v_p(k!) - kr$. Equivalently, one may write*

$$v_p((s(kp^r+m+1,k+1)) = k\Big(\frac{p^r-1}{p-1} - r\Big) + v_p(m!).$$

Theorem 3.5. [15] *Let k be a positive integer and p be a prime. Then*
$$v_p(s(kp^r,k))=k\Big(\frac{p^r-1}{p-1}-r\Big)$$
and
$$v_p(s(kp^r+1,k+1))=k\Big(\frac{p^r-1}{p-1}-r\Big).$$
Particularly, one has for $0\le m\le n$ that
$$v_p(s(p^n,p^m))=\frac{p^n-p^m}{p-1}-p^m(n-m)$$
and
$$v_p(s(p^n+1,p^m+1))=\frac{p^n-p^m}{p-1}-p^m(n-m).$$

Theorem 3.6. [15] *Let p be an odd prime and $0\le m\le n$. Then*
$$v_p\Big(s\Big(\frac{p^n+1}{2},\frac{p^m+1}{2}\Big)\Big)=\frac{1}{2}\Big(\frac{p^n-p^m}{p-1}-p^m(n-m)\Big).$$

Theorem 3.7. [15] *Let k be a positive integer and p be a prime. Let a positive integer $n\ge k$ be given and define r so that $kp^r\le n<(k+1)p^r$. Then*
$$v_p((s(n+1,k+1))=v_p(n!)-v_p(k!)-kr.$$

Theorem 3.8. [15] *For any positive integer k and prime p, we have*
$$\liminf_{n\to\infty}\frac{v_p(s(n,k))}{n}=\frac{1}{p-1}.$$

As mentioned above, Lengyel conjectured, proved by Wannemacker, the 2-adic valuation of $S(2^n,k)$ is $d_2(k)-1$, i.e. $v_2(S(2^n,k))=s_2(k)-1$, where $s_2(k)$ represents the base 2 digital sum of k. In addition, Lengyel and Hong et al further studied the properties of $v_2(S(n,k))$. In 2019, Qiu and Hong [23] arrived at an explicit formula for the 2-adic valuation $v_2(s(2^n,k))$ of the Stirling numbers of the first kind $s(2^n,k)$.

Theorem 3.9. [23] *For any integers n,m and k such that $2\le m\le n$, $2\le k\le 2^{m-1}+1$, we have*
$$v_2(s(2^n,2^m-k))=2^n-2^m-(n-m)\Big(2^m-2\Big\lfloor\frac{k}{2}\Big\rfloor\Big)+m-2-v_2\Big(\Big\lfloor\frac{k}{2}\Big\rfloor\Big)+(n-1)\epsilon_k,$$
where $\epsilon_k=0$ if k is even, and $\epsilon_k=1$ if k is odd.

For the Stirling numbers of the second kind, Hong, Zhao and Zhao [12] confirmed in 2012 a conjecture of Amdeberhan, Manna and Moll [2] raised in 2008 by showing that
$$v_2(S(2^n+1,k+1))=v_2(S(2^n,k)).$$
By using Theorem 3.9, Qiu and Hong [23] established the following analogous result for the Stirling numbers of the first kind.

Theorem 3.10. [23] *For arbitrary positive integers n and k such that $k\le 2^n$, we have*
$$v_2(s(2^n+1,k+1))=v_2(s(2^n,k)).$$

Another consequence of Theorem 3.9 is the following interesting result.

Corollary 3.11. [23] *For arbitrary positive integers n and k such that $k\le 2^n$, we have*
$$v_2(s(2^n,k))\le v_2(s(2^n,1)).$$

From Theorem 3.9, Corollary 3.11 and (3.1), we can derive an upper bound for $v_2(H(2^n,k))$ as follows.

Corollary 3.12. [23] *For arbitrary positive integers n and k such that $k \le 2^n$, we have*

$$v_2(H(2^n,k)) \le -n.$$

Clearly, Corollary 3.12 confirms partially a conjecture of Leonetti and Sanna raised in 2017 [20].

Lengyel also proved the following result.

Theorem 3.13. [19] *For any integer $n \ge 1$, we have*

$$v_3(s(3^n,2)) = \frac{3^n+3}{2} - 2n, v_3(s(3^n,3)) = \frac{3^n+3}{2} - 3n, v_3(s(2\cdot 3^n,2)) = 3^n - 2n - 1.$$

Recently, Qiu, Feng and Hong [22] arrive at successfully an explicit formula for the 3-adic valuation of the Stirling numbers of the first kind $s(a3^n,k)$ with $2 \le k \le a3^n$.

Theorem 3.14. [22] *Let $a \in \{1,2\}$. For any integers n, m and k such that $1 \le m \le n$ and $2 \le k \le 2a3^{m-1}+1 < a3^m$, we have*

$$v_3(s(a3^n, a3^m - k)) = \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - k) + m - 1 - v_3\left(\left\lfloor\frac{k}{2}\right\rfloor\right) + (m + v_3(k))\epsilon_k,$$

where $\epsilon_k = 0$ if k is even, and $\epsilon_k = 1$ if k is odd.

By using Theorem 3.14, Qiu, Feng and Hong [22] prove the following results.

Theorem 3.15. [22] *Let $a \in \{1,2\}$. For any positive integers n and k such that $1 \le k \le a3^n$, we have*

$$v_3(s(a3^n+1, k+1)) \begin{cases} = v_3(s(a3^n,k)), & \text{if } 2 \mid (k-a), \\ \ge v_3(s(a3^n,k+1)) + n, & \text{if } 2 \nmid (k-a) \end{cases}$$

Theorem 3.16. [22] *For arbitrary positive integers n and k such that $k \le 3^n$, we have*

$$v_3(s(3^n,k)) \le \begin{cases} v_3(s(3^n,1)) = \frac{3^n-2n-1}{2}, & \text{if } n \ge 3, \\ v_3(s(3^2,3^2-3)) = 4, & \text{if } n = 2, \\ v_3(s(3,2)) = 1, & \text{if } n = 1. \end{cases}$$

Theorem 3.17. [22] *For arbitrary positive integers n and k such that $k \le 2\cdot 3^n$, we have*

$$v_3(s(2\cdot 3^n,k)) \le \begin{cases} v_3(s(2\cdot 3^n,1)) = 3^n - n - 1, & \text{if } n \ge 2, \\ v_3(s(2\cdot 3, 2\cdot 3 - 3)) = 2, & \text{if } n = 1. \end{cases}$$

From Theorems 3.15 to 3.17, we can derive an upper bound for $v_3(H(a3^n,k))$ as follows.

Corollary 3.18. [22] *Let $a \in \{1,2\}$. For arbitrary positive integers n and k such that $n \ge 3$, $k \le a3^n$ and $2 \mid (k-a)$, we have $v_3(H(a3^n,k)) \le -n$.*

Clearly, Corollary 3.18 confirms partially the conjecture of Leonetti and Sanna [20] raised in 2017. We also observe the following two facts:

Remark 3.19. For arbitrary positive integers n and m such that $2 \le m \le n$, we note that $v_2(s(2^n,2^m)) = 2^n - 2^m - 2^m(n-m)$. For any integer k with $2 \le k \le 2^{m-1}+1$, we have

$$v_2(s(2^n,2^m-k)) = v_2(s(2^n,2^m)) + \left(2\left\lfloor\frac{k}{2}\right\rfloor - 1\right)(n-m) + n - 2 - v_2\left(\left\lfloor\frac{k}{2}\right\rfloor\right) + (n-1)\epsilon_k.$$

Particularly, we have $v_2(s(2^n,2^n-k)) = n - 2 - v_2\left(\left\lfloor\frac{k}{2}\right\rfloor\right) + (n-1)\epsilon_k$.

Remark 3.20. Let $a \in \{1,2\}$. For any positive integers n and m such that $m \le n$, we note that $v_3(s(a3^n, a3^m)) = \frac{a}{2}(3^n - 3^m) - a3^m(n-m)$. For any integer k with $2 \le k \le 2a3^{m-1} + 1 < a3^m$, we have

$$\begin{aligned} & v_3(s(a3^n, a3^m - k)) \\ =& v_3(s(a3^n, a3^m)) + \Big(2\Big\lfloor\frac{k}{2}\Big\rfloor - 1\Big)(n-m) + n - 1 - v_3\Big(\Big\lfloor\frac{k}{2}\Big\rfloor\Big) + (n + v_3(k))\epsilon_k. \end{aligned}$$

Particularly, we have $v_3(s(a3^n, a3^n - k)) = n - 1 - v_3\big(\lfloor\frac{k}{2}\rfloor\big) + (n + v_3(k))\epsilon_k$.

As usual, for any integer k and prime p, let $\langle k\rangle$ denote the integer such that $0 \le \langle k\rangle \le p-2$ and $k \equiv \langle k\rangle \pmod{p-1}$. Let $\epsilon_k := 0$ if k is even and $\epsilon_k := 1$ if k is odd. The n-th Bernoulli number B_n is defined by the Maclaurin series as

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

By the von Staudt-Clausen theorem (see, for instance, [13]), we know that if n is even, then

$$B_n + \sum_{(p-1)|n} \frac{1}{p} \in \mathbb{Z},$$

where the sum is over all primes p such that $(p-1)|n$. The von Staudt-Clausen theorem tells us that $v_p(B_l) \ge 0$ for all even integers l with $2 \le l \le p-3$. In particular, if $v_p(B_l) = 0$ for all even integers l with $2 \le l \le p-3$, then p is called a *regular prime*. Note that 3 is a regular prime. If p is not regular it is called *irregular*. The first irregular primes are 37 and 59. We point that the cardinality and density of regular primes are largely unknown. Although one can argue heuristically that asymptotically more than half of primes should be regular, it is not even known that there are infinitely many regular primes, while infinitely many primes are known to be irregular. The interested readers are refereed to [13] and [25] for the history and basic facts on the regular and irregular primes. For the general p-adic valuations of $s(n,k)$, by using Wahsington's congruences on the reciprocal power sums [26], Hong and Qiu [11] showed the following results.

Theorem 3.21. [11] *Let $p \ge 5$ be a prime and let a and k be integers such that $1 \le a \le p-1$ and $2 \le k \le ap-2$. Let ϵ_k be defined by $\epsilon_k := 0$ if k is even and $\epsilon_k := 1$ if k is odd.*

(i). *If $k \equiv \epsilon_k \pmod{p-1}$, then $v_p(s(ap, ap-k)) = (v_p(k)+1)\epsilon_k$.*

(ii). *If $2 \le k \le a(p-1) - 1$ and $k \not\equiv \epsilon_k \pmod{p-1}$, then*

$$v_p(s(ap, ap-k)) \ge (v_p(k)+1)\epsilon_k + 1,$$

with the equality holding if and only if $v_p\big(B_{2\lfloor\frac{\langle k\rangle}{2}\rfloor}\big) = 0$, where $\langle k\rangle$ means the least positive integer such that $\langle k\rangle \equiv k \pmod{p-1}$. In particular, if p is regular, then $v_p(s(ap, ap-k)) = (v_p(k)+1)\epsilon_k + 1$.

(iii). *If $a \ge 4$ and $a(p-1)+2 \le k \le ap-2$, then $v_p(s(ap, ap-k)) \ge a + k - ap$.*

Theorem 3.22. [11] *Let $p \ge 5$ be a prime and let a and n be positive integers such that $(a,p) = 1$. Let k be an odd integer with $1 \le k \le ap^n - 1$. Then*

$$\begin{aligned} & v_p(s(ap^n, ap^n - k)) \\ & \begin{cases} = v_p(s(ap^n, ap^n - k + 1)) + v_p(ap^n - k) + n, & \text{if } v_p(s(ap^n, ap^n - k + 1)) \le 2n - 1; \\ \ge v_p(ap^n - k) + 3n, & \text{if } v_p(s(ap^n, ap^n - k + 1)) \ge 2n. \end{cases} \end{aligned}$$

Notice that Lengyel [19] proved the following interesting result.

Theorem 3.23. [19] *For any prime p, any integer $a \geq 1$ with $(a, p) = 1$, and any even number $k \geq 2$ with the condition: There exists $n_1 \in \mathbb{Z}^+ : n_1 > 3\log_p k + \log_p a$ such that $v_p(s(ap^{n_1}, ap^{n_1} - k)) < n_1$, or $k = 1$ with $n_1 = 1$, one has for $n \geq n_1$ that*

$$v_p(s(ap^{n+1}, ap^{n+1} - k)) = v_p(s(ap^n, ap^n - k)) + 1.$$

Meanwhile, for any odd $k \geq 3$, Lengyel conjectured in 2015 that for any integer $n \geq n_1(p, k)$ with some sufficiently large $n_1(p, k)$, one has

$$v_p(s(ap^{n+1}, ap^{n+1} - k)) = v_p(s(ap^n, ap^n - k)) + 2. \tag{3.2}$$

Now by Theorems 3.22 and 3.23, one yields the following analogous result which proves partially Lengyel's Conjecture (3.2).

Theorem 3.24. [11] *Let $p \geq 5$ be a prime and let a be a positive integer such that $(a, p) = 1$. Let $k \geq 3$ be an odd integer with the condition: $\exists\ n_1 \in \mathbb{Z}^+ : n_1 > 3\log_p(k-1) + \log_p a$ such that$v_p(s(ap^{n_1}, ap^{n_1} - (k-1))) < n_1$. Then for any positive integer n with $n \geq n_1$ one has $v_p(s(ap^{n+1}, ap^{n+1} - k)) = v_p(s(ap^n, ap^n - k)) + 2$. Furthermore, we have $v_p(s(ap^n, ap^n - k)) = v_p(s(ap^{n_1}, ap^{n_1} - k)) + 2(n - n_1)$.*

In [23], Qiu and Hong gave a formula for $v_2(s(2^n, k))$ with k being an integer such that $1 \leq k \leq 2^n$. In [22], Qiu, Feng and Hong presented a formula for $v_3(s(a3^n, k))$ with k being an integer such that $1 \leq k \leq a3^n$, where $a \in \{1, 2\}$. In [11], Hong and Qiu arrived at an exact expression or a lower bound of $v_p(s(ap, k))$ with a and k being integers such that $1 \leq a \leq p-1$ and $1 \leq k \leq ap$. It is natural to consider the p-adic valuation of the Stirling number $s(ap^n, k)$, where a, n and k being integers such that $1 \leq a \leq p-1, n \geq 2$ and $1 \leq k \leq ap^n$. For any odd prime p and any positive integer k, recall that ϵ_k is defined by $\epsilon_k := 0$ if k is even and $\epsilon_k := 1$ if k is odd, and $\langle k \rangle$ denotes the integer such that $0 \leq \langle k \rangle \leq p-2$ and $k \equiv \langle k \rangle \pmod{p-1}$. We propose the following conjecture.

Conjecture 3.25. *Let p be an odd prime. Let a, n, m and k be positive integers such that $1 \leq a \leq p-1$, $1 \leq m \leq n$ and $2 \leq k \leq ap^n - 2$. Then each of the following is true:*

(i). *If $2 \leq k \leq a(p-1)p^{m-1} + 1 < ap^m$, then*

$$v_p(s(ap^n, ap^m - k)) = \frac{a}{p-1}(p^n - p^m) - (n-m)(ap^m - k) + m + (m + v_p(k))\epsilon_k + T_k,$$

where

$$T_k := \begin{cases} -1 - v_p(\lfloor \frac{k}{2} \rfloor), & \text{if } k \equiv \epsilon_k \pmod{p-1}; \\ v_p(B_{2\lfloor \frac{\langle k \rangle}{2} \rfloor}), & \text{if } k \not\equiv \epsilon_k \pmod{p-1}. \end{cases}$$

(ii). *If $a \geq 4$ and $a(p-1) + 2 \leq k \leq ap - 2$, then*

$$v_p(s(ap^n, ap - k)) \geq \frac{a}{p-1}(p^n - p) - (n-1)(ap - k) + a + k - ap.$$

From Theorem 3.21, we can see that for all primes $p \geq 5$ part (ii) of Conjecture 3.25 is true when $n = 1$ and part (i) of Conjecture 3.25 also holds for $n = 1$ and $k \equiv \epsilon_k \pmod{p-1}$. By the main result in [22], we know that Conjecture 3.25 is true when $p = 3$. Letting $m = n$, Conjecture 3.25 becomes the following conjecture.

Conjecture 3.26. *Let p be an odd prime. Let a, n and k be positive integers such that $1 \leq a \leq p-1$ and $2 \leq k \leq a(p-1)p^{n-1} + 1$. Then*

$$v_p(s(ap^n, ap^n - k)) = \begin{cases} n + (n + v_p(k))\epsilon_k - 1 - v_p(\lfloor \frac{k}{2} \rfloor), & \text{if } k \equiv \epsilon_k \pmod{p-1}; \\ n + (n + v_p(k))\epsilon_k + v_p(B_{2\lfloor \frac{\langle k \rangle}{2} \rfloor}), & \text{if } k \not\equiv \epsilon_k \pmod{p-1}. \end{cases}$$

On the other hand, Corollary 4 in [15] gives us that

$$v_p(s(ap^n, ap^m)) = \frac{a}{p-1}(p^n - p^m) - a(n-m)p^m.$$

So we suggest the following conjecture as the conclusion of this paper.

Conjecture 3.27. *Let p be a prime. Let a, n, m and k be positive integers such that $1 \le a \le p-1$, $1 \le m \le n$ and $2 \le k \le a(p-1)p^{m-1} + 1 < ap^m$. Then*

$$v_p(s(ap^n, ap^m - k)) = v_p(s(ap^n, ap^m)) + v_p(s(ap^n, ap^n - k)) + \big(2\Big\lfloor \frac{k}{2} \Big\rfloor - 1\big)(n-m).$$

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