The locations of the zeros of certain weakly holomorphic modular form.

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1 Introduction

Let $p = 1, 2, 3, 5,$ or 7, $\Gamma_0^+(p) = \Gamma_0(p) \cup \begin{pmatrix} 0 & \frac{-1}{\sqrt{p}} \\ \sqrt{p} & 0 \end{pmatrix} \Gamma_0(p)$ be the Fricke Group of level p, and $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the complex upper half plane. It is known that a fundamental domain for $\Gamma_0^+(p)$ is given by

$$\mathbb{F}^+(p) = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq \frac{1}{\sqrt{p}},\ \left| z + \frac{1}{2} \right| \geq \frac{1}{2\sqrt{p}},\ -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\}$$
$$\cup \left\{ z \in \mathbb{H} \;\middle|\; |z| > \frac{1}{\sqrt{p}},\ \left| z - \frac{1}{2} \right| > \frac{1}{2\sqrt{p}},\ 0 < \mathrm{Re}(z) < \frac{1}{2} \right\}.$$

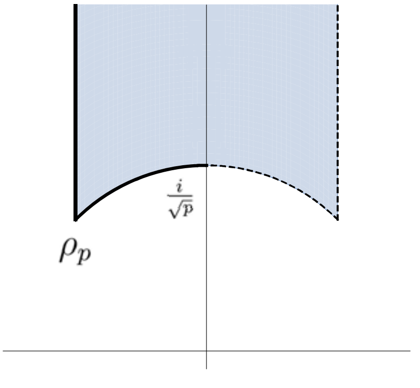

Figure 1: $\mathbb{F}^+(p)$ $(p = 1, 2, 3)$

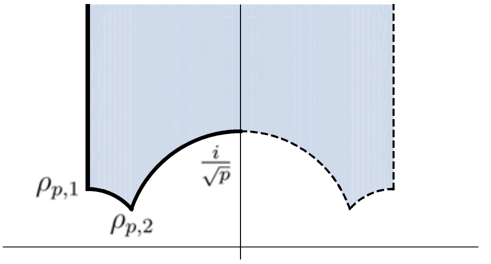

Figure 2: $\mathbb{F}^{+}(p)$ $(p = 5, 7)$

Here, we put $\rho_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\rho_2 = -\frac{1}{2} + \frac{i}{2}$, $\rho_3 = -\frac{1}{2} + \frac{i}{2\sqrt{3}}$, $\rho_{5,1} = -\frac{1}{2} + \frac{i}{2\sqrt{5}}$, $\rho_{5,2} = -\frac{2}{5} + \frac{i}{5}$, $\rho_{7,1} = -\frac{1}{2} + \frac{i}{2\sqrt{7}}$, and $\rho_{7,2} = -\frac{5}{14} + \frac{\sqrt{3}}{14}i$.

In 1970, Rankin and Swinnerton-Dyer studied the location of the zeros of the Eisenstein series $E_k(z) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$ of weight $k \geq 4$ for the full modular group $SL_2(\mathbb{Z}) = \Gamma_0^+(1)$, where $q = e^{2\pi i z}$, B_k is the kth Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. They showed that all the zeros of E_k on $\mathbb{F}_1^+$ lie on the lower boundary arc [6]. Their method can be applied other holomorphic modular form. In recent years the locations of the zeros of certain holomorphic modular forms have been studied in several cases. In the cases of the Eisenstein series for $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$ have been studied by Miezaki, Nozaki, and Shigezumi [5], for $\Gamma_0^+(5)$ and $\Gamma_0^+(7)$ by Shigezumi [7]. Here, the Eisenstein series of weight $k \geq 4$ for $\Gamma_0^+(p)$ is defined by

$$E_{p,k}^{+}(z) = \frac{1}{1 + p^{\frac{k}{2}}}(E_k(z) + p^{\frac{k}{2}}E_k(pz)).$$

In 2008, Duke and Jenkins considered weakly holomorphic modular forms for $SL_2(\mathbb{Z})$ [3]. They constructed the natural basis of the space of weakly holomorphic modular form and proved that the zeros of almost all elements in the natural basis on $\mathbb{F}_1^+$ lie on the lower boundary arc. By using their method, Choi and Im studied the zeros of certain weakly holomorphic modular forms for $\Gamma_0^+(2)$ and obtained a similar result [1].
In this paper, we consider in the cases of $\Gamma_0^+(3)$, $\Gamma_0^+(5)$, and $\Gamma_0^+(7)$.

2 Natural basis

A holomorphic function f on $\mathbb{H}$ is a weakly holomorphic modular form of weight $k \in 2\mathbb{Z}$ for $\Gamma_0^+(p)$ if f satisfies

- $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for any $z \in \mathbb{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^+(p)$.
- f has a $q-$expansion of the form $f(z) = \sum_{n \in \mathbb{Z}} a_f(n) q^n$

  such that $a_f(n) = 0$ for almost all $n < 0$.

Let n_f be the smallest integer such that $a_f(n_f) \neq 0$, we define f is a holomorphic modular form if $n_f \geq 0$, a cusp form if $n_f > 0$. We denote the space of weakly holomorphic modular forms of weight k for $\Gamma_0^+(p)$ by $M_k^!(\Gamma_0^+(p))$, the space of holomorphic modular forms by $M_k(\Gamma_0^+(p))$, and the space of cusp forms by $S_k(\Gamma_0^+(p))$.

For $k \geq 4$, let $E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$ be the Eisenstein series of weight k for $SL_2(\mathbb{Z})$, $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ be the Dedekind eta function. Put $\delta = \begin{cases} 12 & \text{if } p = 1,3,7 \\ 8 & \text{if } p = 2 \\ 4 & \text{if } p = 5 \end{cases}$, we define $\Delta_p^+ \in S_\delta(\Gamma_0^+(p))$, and $j_p^+ \in M_0^!(\Gamma_0^+(p))$ as follows.

- $\Delta_p^+(z) = (\eta(z)\eta(pz))^\delta$,
- $j_p^+(z) = \begin{cases} \frac{1728 E_4(z)^3}{E_4(z)^3 - E_6(z)^2} - 744 & \text{if } p = 1 \\ \left(\frac{\eta(z)}{\eta(pz)}\right)^{\frac{24}{p-1}} + \frac{24}{p-1} + p^{\frac{12}{p-1}} \left(\frac{\eta(pz)}{\eta(z)}\right)^{\frac{24}{p-1}} & \text{otherwise.} \end{cases}$

For $k \in 2\mathbb{Z}$, we write

$$k = \delta \ell_k + r_k \text{ where } \ell_k \in \mathbb{Z} \text{ and } r_k = \begin{cases} \delta + 2 & \text{if } k \equiv 2 \pmod{\delta} \\ k - [\frac{k}{\delta}]\delta & \text{otherwise} \end{cases}.$$

Put $m' = m_{p,k} = \frac{p+1}{24} \delta \ell_k + \dim S_{r_k}(\Gamma_0^+(p))$. Theorem 2.4 of [2] says that there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!(\Gamma_0^+(p))$ such that

$$f_{k,m}(z) = q^{-m} + O(q^{m'+1})$$

for each integer $m \geq -m'$.

Then, $\{f_{k,m}\}_{m\geq -m'}$ form a natural basis of $M_k^!(\Gamma_0^+(p))$.
Moreover, $f_{k,m}$ can be written explicitly by

$$f_{k,m} = (\Delta_p^+)^{\ell_k} \Delta_{p,r_k} F_{k,m+m'}(j_p^+)$$

where $\Delta_{p,r_k} = f_{r_k,-m_{p,r_k}} \in M_{r_k}(\Gamma_0^+(p))$ and $F_{k,m+m'}$ is a monic polynomial of degree $m+m'$. We can check that $F_{k,m+m'}$ has rational coefficients, since the q-coefficients of Δ_p^+ and Δ_{p,r_k} are rational. Hence, the q-coefficients of $f_{k,m}$ are also rational. The following is an integral formula of $f_{k,m}$ which play an important role in investigating the zeros of $f_{k,m}$.

Proposition 2.1. [2, p. 756] *Let* $f_k := f_{k,-m'} = (\Delta_p^+)^{\ell_k}\Delta_{p,r_k}$*, then we have*

$$f_{k,m}(z) = \frac{1}{2\pi i}\oint_C \frac{f_k(z)f_{2-k}(\tau)q'^{-m-1}}{j_3^+(\tau) - j_3^+(z)}dq',$$

where $q' := e^{2\pi i\tau}$ *and* C *is the circle centered at 0 in the* q'*-plane with a sufficiently small radius.*

Duke and Jenkins proved the above formula in the case of $SL_2(\mathbb{Z})$ independently [3]. They also showed the following theorem.

Theorem 2.1. [3, Theorem 1] *Let* $\{f_{k,m}\}_{m\geq -m'}$ *be the natural basis for* $M_k^!(SL_2(\mathbb{Z}))$*. If* $m \geq |\ell_k| - \ell_k$*, then all the zeros of* $f_{k,m}$ *in* $\mathbb{F}_1^+$ *lie on the lower boundary arc.*

In the case of $\Gamma_0^+(2)$, Choi and Im obtained a similar result.

Theorem 2.2. [1, Theorem 1.2] *Let* $\{f_{k,m}\}_{m\geq -m'}$ *be the natural basis for* $M_k^!(\Gamma_0^+(2))$*. If* $m \geq 2|\ell_k| - \ell_k + 8$*, then all the zeros of* $f_{k,m}$ *in* $\mathbb{F}_2^+$ *lie on the lower boundary arc.*

3 Results

The following theorems are our main results.

Theorem 3.1. [4, Theorem 1.1] *Let* $\{f_{k,m}\}_{m\geq -m'}$ *be the natural basis for* $M_k^!(\Gamma_0^+(3))$*. If* $m \geq 18|\ell_k| + 23$*, then all the zeros of* $f_{k,m}$ *in* $\mathbb{F}_3^+$ *lie on the lower boundary arc.*

Theorem 3.2. (Kuga) *Let* $p = 5, 7$ *and* $\{f_{k,m}\}_{m\geq -m'}$ *be the natural basis for* $M_k^!(\Gamma_0^+(p))$*.*
If m *is sufficiently large, then all the zeros of* $f_{k,m}$ *in* $\mathbb{F}_p^+$ *lie on the lower boundary arcs.*

The proof motivates real estimation. For simplicity, we prove only in the case of $p = 3$ and $r_k = 0$ here.

Lemma 3.1. *If* $f = \sum_{n\geq n_f}^{\infty} a_n q^n \in M_k^!(\Gamma_0^+(3))$ *has real Fourier coefficients, then* $e^{\frac{ik\theta}{2}} f\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)$ *is real for all* $\theta \in [\frac{\pi}{2}, \frac{5\pi}{6}]$*. In particular,* $e^{\frac{ik\theta}{2}} f_{k,m}\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)$ *is real for all* $\theta \in [\frac{\pi}{2}, \frac{5\pi}{6}]$.

Proof. For all $z \in \mathbb{H}$, we note that

$$f\left(\frac{-1}{3z}\right) = (\sqrt{3}z)^k f(z),$$

and

$$\overline{f(z)} = \overline{\sum_{n\geq n_f} a(n)e^{2\pi i n z}} = \sum_{n\geq n_f} a(n)\overline{e^{2\pi i n z}} = \sum_{n\geq n_f} a(n)e^{2\pi i n(-\overline{z})} = f(-\overline{z}).$$

Put $z = \frac{1}{\sqrt{3}}e^{i\theta}$ $(\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6})$, then $\frac{-1}{3z} = -\frac{1}{\sqrt{3}}e^{-i\theta} = -\overline{z}$. Hence

$$\overline{f\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)} = \overline{f(z)} = f(-\overline{z}) = f\left(\frac{-1}{3z}\right) = (\sqrt{3}z)^k f(z) = e^{ik\theta} f\left(\frac{1}{\sqrt{3}}e^{i\theta}\right).$$

Thus, we obtain

$$\overline{e^{\frac{ik\theta}{2}} f\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)} = e^{\frac{ik\theta}{2}} f\left(\frac{1}{\sqrt{3}}e^{i\theta}\right).$$

□

The valence formula for $\Gamma_0^+(3)$ is given as follows.

Lemma 3.2. *Let* $f \in M_k^!(\Gamma_0^+(3))$*, which is not identically zero. We have*

$$v_{i\infty}(f) + \frac{1}{2}v_{\frac{i}{\sqrt{3}}}(f) + \frac{1}{6}v_{\rho_3}(f) + \sum_{\substack{\rho\neq\frac{i}{\sqrt{3}},\rho_3 \\ \rho\in\mathbb{F}_3^+}} v_\rho(f) = \frac{k}{6},$$

where $v_\rho(f)$ *is the order of* f *at* ρ.

The following Lemma can be proved by rearranging the integral formula of $f_{k,m}$.

Lemma 3.3. *Put* $h(\theta) = e^{-2\pi m\frac{1}{\sqrt{3}}\sin\theta} e^{\frac{ik\theta}{2}} f_{k,m}\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)$ *and* $\alpha(\theta) = \frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{3}}\cos\theta$.

(a) *For all* $\theta \in [\frac{\pi}{2}, \frac{23}{10}]$*, if* $m \geq 9|\ell_k| - 2\ell_k + 18$*,*

$$|h(\theta) - 2\cos\alpha(\theta)| < 1.9674.$$

(b) *For all* $\theta \in [\frac{23}{10}, \frac{5\pi}{6} - \frac{12}{25m}]$*, if* $m \geq 18|\ell_k| + 23$*,*

$$|h(\theta) - 2\cos\alpha(\theta)| < 0.99728.$$

proof of the Theorem 3.1 (for $r_k = 0$).
Put $h(\theta) = e^{-2\pi m\frac{1}{\sqrt{3}}\sin\theta} e^{\frac{ik\theta}{2}} f_{k,m}\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)$ and $\alpha(\theta) = \frac{k\theta}{2} - 2\pi m\frac{1}{\sqrt{3}}\cos\theta$. Then Lemma 3.1 implies $h(\theta)$ is a continuous real valued function. By lemma 3.2, we suffice to show that $h(\theta)$ has at least $2\ell_k + m$ zeros in the interval $[\frac{\pi}{2}, \frac{5\pi}{6}]$. When $m \geq 18|\ell_k| + 23$, we can check that

$$\alpha\left(\left[\frac{\pi}{2}, \frac{5\pi}{6} - \frac{12}{25m}\right]\right) \supset \left[3\ell_k\pi, \left(5\ell_k + m - \frac{1}{3}\right)\pi\right].$$

By lemma 3.3, we can determine the sign of $h(\theta)$ when $\alpha(\theta)$ takes the values $3\ell_k\pi$, $(3\ell_k+1)\pi$, $(3\ell_k+2)\pi$, $\ldots$, $(5\ell_k+m-2)\pi$, $(5\ell_k+m-1)\pi$, and $(5\ell_k+m-\frac{1}{3})\pi$.

$\alpha(\theta)$	$2\cos\alpha(\theta)$	$\mathrm{sgn}(h(\theta))$
$3\ell_k\pi$	$2(-1)^{\ell_k}$	$(-1)^{\ell_k}$
$(3\ell_k+1)\pi$	$2(-1)^{\ell_k+1}$	$(-1)^{\ell_k+1}$
$(3\ell_k+2)\pi$	$2(-1)^{\ell_k}$	$(-1)^{\ell_k}$
$\vdots$	$\vdots$	$\vdots$
$(5\ell_k+m-2)\pi$	$2(-1)^{\ell_k}$	$(-1)^{\ell_k}$
$(5\ell_k+m-1)\pi$	$2(-1)^{\ell_k+1}$	$(-1)^{\ell_k+1}$
$(5\ell_k+m-\frac{1}{3})\pi$	$(-1)^{\ell_k}$	$(-1)^{\ell_k}$

By the intermediate value theorem, $h(\theta)$ has at least $2\ell_k + m$ distinct zeros in the interval $[\frac{\pi}{2}, \frac{5\pi}{6}]$.
We complete the proof. □

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