Yaglom limits for Lévy processes in the domain of attraction of a stable process

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Abstract

Let X be a real valued Lévy process that is in the domain of attraction of a stable law without centering with norming function c. We prove that X conditioned not to hit zero before t has a Yaglom limit as $t \to \infty$.

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1 Introduction and main results

Let X be a real valued Lévy process with law $\mathbb{P}$ and characteristics (a, σ, Π), and $T_0 = \inf\{t > 0 : X_t < 0\}$. We will assume throughout that under $\mathbb{P}$ neither X nor $-X$ is a subordinator; in the first case the problem we are interested in makes no sense, and in the second case a different approach is needed as our methods rely on the possibility of excursions above the minimum. We also exclude the case where X is a compound Poisson process.

Kyprianou and Palmowski [8] established the existence of quasi stationary laws for Lévy processes that satisfies some hypotheses and that are classified into the families A and B. One of the conditions to belong to either of these classes is that the Lévy processes X has some exponential moments. Kyprianou and Palmowski proved that for such Lévy processes there exists a measure μ such that for every continuous and bounded function f,

$$\mu(f) = \lim_{t\to\infty} \mathbb{E}_x \left(f(X_t) | T_0 > t \right), \qquad \forall x > 0,$$

and there exists a θ such that

$$e^{\theta t} \int_{[0,\infty)} \mu(dx) \mathbb{E}_x (f(X_t), t < T_0) = \mu(f), \qquad t \geq 0,$$

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Although this result is quite general, it can not be applied to Lévy processes such as stable or attracted to stable Lévy processes. The reason for this is simple: the existence of quasi-stationary laws implies that the right tail of τ decays exponentially fast. This does not hold α-stable process. Indeed, if X is an α-stable process, it is known, see e.g. [6], that

$$\mathbb{P}_x(T_0 > t) \sim C_{\alpha,\overline{\rho}} x^{\alpha\overline{\rho}} t^{-\overline{\rho}},$$

where $\overline{\rho} = \mathbb{P}(X_1 < 0)$, and $C_{\alpha,\overline{\rho}} > 0$ is a constant. In fact it can be verified that

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \in dy | T_0 > t) = \delta_\infty(dy), \qquad y \geq 0. \tag{1}$$

Indeed, by the self-similarity property of X, we have that

$$\begin{aligned} \mathbb{E}_x\left(f(X_t)|T_0 > t\right) &= \mathbb{E}\left(f(X_t + x)|\tau_{-x} > t\right) \\ &= \frac{\mathbb{E}\left(f\left(x + t^{1/\alpha} X_1\right), 1 < \tau_{-t^{1/\alpha}x}\right)}{\mathbb{P}_x\left(T_0 > t\right)}, \end{aligned}$$

from where it is easy to see using the estimate (1) that the the right hand term tends to $f(\infty)$. We used the notation $\tau_{-x} = \inf\{t > 0 : X_t < -x\}$. It can be verified that these assertions hold also for Lévy processes attracted to stable. So, by conditioning an attracted to stable Lévy process we force the process to growth too fast and so the limit of its conditional laws is Dirac mass at infinity. Hence, in order to obtain a non-degenerate limit, it is necessary to do a further normalization on the paths of X. That is the content of our main result.

For the asymptotic results which are the main topic of this paper, we assume that X is in the domain of attraction of a stable distribution without centering, that is there exists a deterministic function $c : (0,\infty) \to (0,\infty)$ such that

$$\frac{X_t}{c(t)} \xrightarrow{\mathcal{D}} Y_1, \quad \text{as } t \to \infty, \tag{2}$$

with Y_1 a strictly stable random variable of parameter $0 < \alpha \leq 2$, and positivity parameter $\rho = \mathbb{P}(Y_1 > 0)$. In this case we will use the notation $X \in D(\alpha, \rho)$, and put $\overline{\rho} = 1 - \rho$. Hereafter $(Y_t, t \geq 0)$ will denote an α-stable Lévy process with positivity parameter $\rho = \mathbb{P}(Y_1 > 0)$.

It is well known that in this case the function c is regularly varying at infinity with index $1/\alpha$. **Throughout this paper we will use the notation** $\eta = 1/\alpha$.

In what follows, $k, k_1, k_2, \cdots$ will denote fixed positive constants whereas C will denote a generic constant whose value can change from line to line. As previously remarked, the norming function $c(\cdot) \in RV(\eta)$,where $\eta = 1/\alpha$. More precisely we will assume, WLOG, that Y is a standard stable process, and c can be taken to be a continuous, monotone increasing inverse of the quantity $x^2/m(x)$; where $m(x) = \int_{-x}^{x} y^2 \Pi(dy)$ and necessarily $m(\cdot) \in RV(2 - \alpha)$. It follows from this that, when $\alpha < 2$, we have $t\overline{\Pi}(c(t)) \to k$ and $t\overline{\Pi}^*(c(t)) \to k^*$, with $k^* > 0$ if $\alpha\overline{\rho} < 1$, and $k^* = 0$ if $\alpha\overline{\rho} = 1$, when necessarily $k > 0$. Finally when $\alpha = 2$, we have $t(\overline{\Pi}(c(t)) + \overline{\Pi}^*(c(t))) \to 0$, so we can take $k = k^* = 0$.

Theorem 1 *Assume that $X \in D(\alpha,\rho)$, $\rho > 0$ and that X is regular upwards. We have the following weak convergence of measures*

$$\lim_{t\to\infty} \mathbb{P}_x\left(\frac{X_t}{c(t)} \in dy | T_0 > t\right) = \mathbb{P}(Z_1 \in dy)$$

where Z_1 denotes the stable meander of length 1 at time 1 based on Y. See [3] for a precise definition. We will say that Z_1 is the Yaglom limit of X conditioned to stay positive.

The proof of this result is based on some Lemmas that have been established in the paper by Doney and the author [6], from where we extracted verbatim some sentences and results, thus the notation has been preserved so to facilitate the lecture of this paper.

To estate those Lemmas and proceed to the proof of Theorem 1, we need first to introduce some notation and to recall some consequences of our assumptions.

We first recall a few customary notations in fluctuation theory. For background about fluctuation theory for Lévy processes the reader is referred to the books [1], [5], and [7].

The process $X_t - I_t = X_t - \inf_{0\leq s\leq t} X_s$, $t \geq 0$ is a strong Markov process, the point process of its excursions out of zero forms a Poisson point process with intensity or excursion measure $\underline{n}$. We will denote by $\{\epsilon_t, t > 0\}$ the generic excursion process and by ζ its lifetime. It is known that under $\underline{n}$ the excursion process is Markovian with semigroup given by $\mathbb{P}_x(X_t \in dy, t < T_0)$. We will denote by L^* the local time at 0 for $X - \underline{X}$, and we will assume WLOG that it is normalized so that

$$\mathbb{E}\left(\int_0^\infty e^{-s} dL^*_s\right) = 1. \tag{3}$$

We will denote by τ^* the right continuous inverse of the local time L^*, and refer to it as the downward ladder time process, and call $\{H^*_t = -X_{\tau^*_t}, t \geq 0\}$ the downward ladder height process. The potential measure of the bivariate process (τ^*, H^*) will be denoted by

$$W^*(dt, dx) = \int_0^\infty ds\mathbb{P}(\tau^*_s \in dt, H^*_s \in dx), \qquad t \geq 0, x \geq 0.$$

The marginal in space of W^* is the potential measure of the downward ladder height process H^*, and we will denote by U^* its associated renewal function

$$U^*(a) := W^*([0,\infty) \times [0,a]) = \int_0^\infty ds\mathbb{P}(H^*_s \leq a), \qquad a \geq 0.$$

Analogously, the function V^* will denote the renewal function of the downward ladder time process, τ^*. We will use a similar notation for the analogous objects defined in terms of X^* but we will remove the symbol $*$ from them, and the excursion measure will be denoted by $\overline{n}$.

It is known, see e.g. [4], that when $X \in D(\alpha,\rho)$, the bivariate downgoing ladder process (τ^*, H^*) is in the domain of attraction of a bivariate

$(\overline{\rho}, \alpha\overline{\rho})$ stable law, and since $\overline{\rho}(t) = \mathbb{P}(X_t < 0) \to \overline{\rho}$, it follows from Spitzer's formula that

$$\underline{n}(\zeta > \cdot) \in RV(-\overline{\rho}), \tag{4}$$

where $RV(\beta)$ denotes the class of functions which are regularly varying with index β at ∞.

An important duality relation, which we will use extensively, connects W^* and W with the characteristic measures $\underline{n}$ and $\overline{n}$: see Lemma 1 in [**?**].

Lemma 2 *Let a, a^* denote the drifts in the ladder time processes τ and τ^* : then on $[0,\infty) \times [0,\infty)$ we have the identities*

$$W(dt, dx) = a^* \delta_{\{(0,0)\}}(dt, dx) + \underline{n}(\epsilon_t \in dx, \zeta > t)dt, \tag{5}$$

$$W^*(dt, dx) = a\delta_{\{(0,0)\}}(dt, dx) + \overline{n}(\epsilon_t \in dx, \zeta > t)dt, \tag{6}$$

so that, in particular

$$U(x) = a^* + \int_0^\infty \int_0^x \underline{n}(\epsilon_t \in dy, \zeta > t)dt, \tag{7}$$

$$U^*(x) = a + \int_0^\infty \int_0^x \overline{n}(\epsilon_t \in dy, \zeta > t)dt. \tag{8}$$

Remark 3 *Note that $a = 0$ (respectively $a^* = 0$) is equivalent to X being regular downwards (respectively upwards), and since we exclude the Compound Poisson case, at most one of a and a^* is positive.*

We write $\mathbb{P}^*$ for the law of the dual Lévy process $X^* = -X$,

Throughout this paper we will make systematic use of the identities in the following Lemma 4 as well as the estimates in the Lemma 5.

In order to shorten the notation throughout the rest of the paper we will understand the following terms as equal, for $s > 0$,

$$\underline{n}_s(dy) = \underline{n}(\epsilon_s \in dy) = \underline{n}(\epsilon_s \in dy, s < \zeta), \qquad y > 0.$$

Since in any case we will be integrating over $(0,\infty)$ there will not be any risk of confusion. Analogous notation will be used under the excursion measure $\overline{n}$.

Lemma 4 *The semigroup of X killed at its first entrance into $(-\infty, 0)$ can be expressed as: for $x, y \in \mathbb{R}^+$*

$$\begin{aligned} \mathbb{P}_x(X_t \in dy, t < T_0) = \int_{s=0}^t ds \int_{z \in ((x-y)^+, x]} \overline{n}_s(dz)\underline{n}_{t-s}(dy + z - x) \\ + a\underline{n}_t(dy - x)\mathbf{1}_{\{y \geq x\}} + a^*\overline{n}_t(x - dy)\mathbf{1}_{\{y \leq x\}}. \end{aligned} \tag{9}$$

A consequence of the fact that $(X(ts)/c(t), s \geq 0)$ converges in law to $(Y(s), s \geq 0)$, is that

Lemma 5 *Assume that $X \in D(\alpha, \rho)$. Then as $t \to \infty$*

$$\underline{n}(\epsilon_t \in c(t)dx | \zeta > t) \xrightarrow{D} \mathbb{P}(Z_1 \in dx),$$

where Z_1 denotes the stable meander of length 1 at time 1 based on Y.

Proof of Theorem 1. Here we will use the well known fact, see e.g. [6], that for any $x > 0$

$$\mathbb{P}_x(t < T_0) \sim U^*(x)\underline{n}(t < \zeta), \qquad t \to \infty.$$

Let $f : \mathbb{R} \to [0, \infty)$ be a continuous and bounded function. From Lemma 4 we have that

$$\begin{aligned}
&\mathbb{E}_x\left(f\left(\frac{X_t}{c(t)}\right), t < T_0\right) \\
&= \int_{s=0}^{t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \underline{n}\left(f\left(\frac{\epsilon_{t-s} - z + x}{c(t)}\right), t - s < \zeta\right) \\
&\quad + a\underline{n}_t\left(f\left(\frac{\epsilon_t + x}{c(t)}\right), t < \zeta\right) + a^*\overline{n}_t\left(f\left(\frac{x - \epsilon_t}{c(t)}\right), t < \zeta\right).
\end{aligned}$$

Assume furthermore that f has compact support, and hence it is uniformly continuous. Since $x/c(t) \to 0$ as $t \to \infty$, one can easily check, using the uniform continuity of f, the approximation

$$\begin{aligned}
&\mathbb{E}_x\left(f\left(\frac{X_t}{c(t)}\right) | t < T_0\right) \\
&= \int_{s=0}^{t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \frac{\underline{n}\left(f\left(\frac{\epsilon_{t-s}+z-x}{c(t)}\right), t - s < \zeta\right)}{\mathbb{P}_x(t < T_0)} \\
&\quad + \frac{1}{\mathbb{P}_x(t < T_0)}\left[a\underline{n}_t\left(f\left(\frac{\epsilon_t + x}{c(t)}\right), t < \zeta\right) + a^*\overline{n}_t\left(f\left(\frac{x - \epsilon_t}{c(t)}\right), t < \zeta\right)\right] \\
&\sim \int_{s=0}^{t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \frac{\underline{n}\left(f\left(\frac{\epsilon_{t-s}}{c(t)}\right), t - s < \zeta\right)}{\mathbb{P}_x(t < T_0)} \\
&\quad + \frac{1}{\mathbb{P}_x(t < T_0)}\left[a\underline{n}_t\left(f\left(\frac{\epsilon_t}{c(t)}\right), t < \zeta\right) + a^*\overline{n}_t\left(f\left(\frac{x - \epsilon_t}{c(t)}\right), t < \zeta\right)\right].
\end{aligned}$$

Since we assumed that X is regular upward $a^* = 0$. By Lemma 5, the factor of a in the rightmost term of the above expression, converges to

$$\frac{a}{U^*(x)}\mathbb{E}(f(Z_1)).$$

Now, let $0 < \lambda < 1$, be fixed, and split the integration interval $[0, t]$ into $[0, \lambda t] \cup (\lambda t, t]$. Using the convergence in Lemma 5 and the uniform convergence property of regularly varying functions (Theorem 1.5.2 in [2]), the latter integral, over $[0, \lambda t]$, can be approximated as follows

$$\begin{aligned}
&\int_{s=0}^{\lambda t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \frac{\underline{n}\left(f\left(\frac{\epsilon_{t-s}}{c(t)}\right), t - s < \zeta\right)}{\mathbb{P}_x(t < T_0)} \\
&= \int_{s=0}^{\lambda t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \frac{\underline{n}\left(f\left(\frac{\epsilon_{t-s}}{c(t)}\right), t - s < \zeta\right)}{\underline{n}(t - s < \zeta)} \frac{\underline{n}(t - s < \zeta)}{\mathbb{P}_x(t < T_0)} \\
&\sim \int_{s=0}^{\lambda t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \frac{\underline{n}\left(f\left(\frac{\epsilon_{t-s}}{c(t)}\right), t - s < \zeta\right)}{\underline{n}(t - s < \zeta)} \frac{\underline{n}(t - s < \zeta)}{U^*(x)\underline{n}(t < \zeta)} \\
&\sim \int_{s=0}^{\lambda t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \mathbb{E}(f(Z_1)) \frac{1}{U^*(x)} \sim \frac{U^*(x) - a}{U^*(x)}\mathbb{E}(f(Z_1)),
\end{aligned}$$

where in the final estimate we used Lemma 2. Lemma 16 in [6] allow us to ensure that for a fixed x and large s we have

$$\overline{n}(\epsilon_s \in (0,x], s < \zeta) \leq \frac{k_4 x U^*(x)}{sc(s)}.$$

Using again Lemma 4 and latter upper bound, we estimate the integral over the interval $(\lambda t, t]$ as follows

$$\begin{aligned}
&\int_{\lambda t}^{t} ds \int_{z\in(0,x]} \overline{n}\left(\epsilon_s \in dz, s < \zeta\right) \frac{\underline{n}\left(f\left(\frac{\epsilon_{t-s}}{c(t)}\right), t-s < \zeta\right)}{\mathbb{P}_x(t < T_0)} \\
&\leq ||f||_\infty k_4 x U^*(x) \int_{t\lambda}^{t} ds \frac{1}{sc(s)} \underline{n}(t-s < \zeta) \frac{1}{\mathbb{P}_x(t < T_0)} \\
&\sim ||f||_\infty \frac{xU^*(x)}{t\lambda c(t\lambda)} \int_0^{t(1-\lambda)} ds \underline{n}(s < \zeta) \frac{1}{U^*(x)\underline{n}(t < \zeta)} \\
&\sim ||f||_\infty \frac{x}{t\lambda c(t\lambda)} \frac{t(1-\lambda)\underline{n}(t(1-\lambda) < \zeta)}{\rho \underline{n}(t < \zeta)} \\
&\sim ||f||_\infty \frac{x(1-\lambda)^\rho}{\rho\lambda c(t\lambda)} = o(1),
\end{aligned}$$

where in the second estimate we used Karamata's theorem for regularly varying functions and in the third the fact that $t \mapsto \underline{n}(\zeta > t)$ is regularly varying with index $\overline{\rho}$. The rightmost term in the above expression tends to zero as $t \to \infty$. Adding the three three estimates above we conclude that the measures in the main theorem of this paper converge vaguely. To get the convergence in distribution we should also verify that the mass is preserved, but this is straightforward from the fact that $\mathbb{P}_x\left(\frac{X_t}{c(t)} \in (0,\infty) | t < T_0\right) = 1 = \mathbb{P}(Z_1 \in (0,\infty))$. ■

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