A DIGEST FOR "FORMAL DEGREES AND LOCAL THETA CORRESPONDENCE: QUATERNIONIC CASE"

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Abstract. This is a digest for the author's thesis "Formal degrees and local theta correspondence: quaternionic case". We determine constant occurring in a local analogue of the Siegel-Weil formula, and describe the behavior of the formal degree under the local theta correspondence for quaternionic dual pairs of almost equal rank over a non-Archimedean local field of characteristic zero. As an application, we prove the formal degree conjecture of Hiraga-Ichino-Ikeda for the non-split inner forms of Sp_4 and GSp_4.

1. Introduction

In the introduction, we explain the backgrounds of our study and the outline of the thesis.

Let F be a non-Archimedean local field of characteristic 0, let G be a classical group, and let π be an irreducible representation of $G(F)$. Then, by the local Langlands correspondence (in general a conjecture), one can define a Langlands parameter $(\phi_\pi \eta_\pi)$ of π, where ϕ_π is an L-parameter

$$\phi_\pi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$

and η_π is an irreducible representation of the finite group $\widetilde{\mathcal{S}}_\phi(\widehat{G})$. For the definition of $\widehat{G}$, ${}^L G$, see [GI14, §14.1], and for the definition of $\widetilde{\mathcal{S}}_\phi(\widehat{G})$, see [Art06, p.209]. (In Arthur's paper, the group is denoted by $\widetilde{\mathcal{S}}_\phi$.) We remark that η_π is not necessarily one-dimensional in general, although it is one-dimensional when G is a non-quaternionic classical group.

One motivation of our study is the formal degree conjecture of Hiraga-Ichino-Ikeda [HII08], which describes the formal degree in terms of the local Langlands parameter. Here, a formal degree is an invariant defined on a square integrable representation π of a reductive group G over a local field F of characteristic 0 so that

$$\int_{G/A} (\pi(g)x_1, y_1)\overline{(\pi(g)x_2, y_2)}\, dg = \frac{1}{\deg \pi}(x_1, y_1)\overline{(x_2, y_2)}$$

for all $x_1, x_2, y_1, y_2 \in \pi$ where A is the maximal F-split torus of the center of G. Here, dg is the normalized Haar measure as in [GG99, §8]. (In [GG99], it is denoted by μ_G.) Suppose that G is connected. Then, the formal degree conjecture asserts that

$$\deg \pi = \frac{\dim \eta_\pi}{|C_{\phi_\pi}(\widehat{G})|}\gamma(0, \mathrm{Ad} \circ \phi_\pi, \psi)$$

Here C_{ϕ_π} is the centralizer of the image $\mathrm{Im}\, \phi_\pi$ of ϕ_π in $\widehat{G}$.

On the other hand, the local theta correspondence in terms of the local Langlands parameter is also a background. In the non-quaternionic case, this had been conjectured by Adams and Prasad, and proved by Gan-Ichino [GI16] and Atobe-Gan [AG17] in the (almost) equal cases. Then, Gan-Ichino [GI14] observed the behavior of the formal degree under the theta correspondence via the parameter-side, and they proved it in the analytical setting. However, in the quaternionic case, this has not been formulated yet. But the thesis shows that the formal degree behaves according to certain rules under the local theta correspondence even in the quaternionic case.

Now, we explain the outline of the thesis. In the thesis, we analyze the local analogue of the Siegel-Weil formula, and we obtain a relation between the constant in the local Siegel-Weil formula and local zetas value for enough cases. Here, the constant in the local Siegel-Weil formula is known to appear in an expression of the ratio of the formal degrees of irreducible representations corresponding to each other by the local theta correspondence ([GI14, §18]). Hence, to establish the description of the behavior of the formal degrees under local theta correspondence, we compute some local zeta values. For the sake of clarity, we set a symbol for each constant. For a quaternionic dual pair $(G(V), G(W))$ of almost equal rank, we denote by $\alpha_1(W)$ the local zeta value, by $\alpha_2(V, W)$ the constant in the local Siegel-Weil formula, and by $\alpha_3(V, W)$ the proportional constant of the formal degrees which are related to each other by the local theta correspondence. Then, the goal of the thesis is the determination of the three constant $\alpha_1(W)$, $\alpha_2(V, W)$, and $\alpha_3(V, W)$.

2. Setup

This section is a summary of §§2-7.2 and §§8-9 of the thesis.

First, we explain the setting related to the doubling. Let F be a non-Archimedean local field of characteristic 0 and let D be a division quaternion algebra over F. Fix $\epsilon \in \{\pm 1\}$. Let V be a right m-dimensional D vector space equipped with a map $(\ ,\ ) : V \times V \to D$ such that

- $(xa + yb, zc) = a^*(x, z)c + b^*(y, z)c$ for $x, y, z \in V$ and $a, b, c \in D$,
- there is an $\epsilon = \pm 1$ such that $(y, x) = \epsilon(x, y)^*$ for $x, y \in V$,

and let W be a left n-dimensional D vector space equipped with a map $\langle\ ,\ \rangle : W \times W \to D$ such that

- $\langle ax + by, cz\rangle = a\langle x, z\rangle c^* + b\langle y, z\rangle c^*$ for $x, y, z \in W$ and $a, b, c \in D$,
- there is an $\epsilon = \pm 1$ such that $\langle y, x\rangle = -\epsilon\langle x, y\rangle^*$ for $x, y \in W$.

Here we denote by $* : D \to D$ the main involution. We assume that $(\ ,\ )$ and $\langle\ ,\ \rangle$ are non-degenerate. We put $G(V)$ and $G(W)$ the isometry groups of $(V, (\ ,\ ))$ and $(W, \langle\ ,\ \rangle)$ respectively. This pair $(G(V), G(W))$ is called a quaternion dual pair. We assume that $2n - 2m - \epsilon = 1$. We consider normalized Haar measures on $G(V)$ and $G(W)$ as in [GG99, §8]. If we take a basis $\underline{e}$ for V, then we denote by $\mathfrak{d}(V)$ the modulo ${F^\times}^2$ class of the reduced norm of $R = (e_i, e_j)_{i,j} \in \mathrm{GL}_m(D)$, which is called the discriminant of V. It is known that $\mathfrak{d}(V)$ does not depend on the basis for V. We denote by χ_V the character of $F^\times$ defined by

$$\chi_V(a) = (a, \mathfrak{d}(V))_F$$

where $(\ ,\ )_F$ is the Hilbert symbol of F. Moreover, we introduce the doubled space: let $W^\square = W \times W$ and let $\langle\ ,\ \rangle^\square : W^\square \times W^\square \to D$ be the map given by

$$\langle (x_1, x_2), (y_1, y_2)\rangle^\square = \langle x_1, y_1\rangle - \langle x_2, y_2\rangle$$

for $x_1, x_2, y_1, y_2 \in W$. A notable thing is that $W^\square$ possesses the polar decomposition $W^\square = W^\triangle \oplus W^\triangledown$ where

$$W^\triangle = \{(x, x) \in W^\square \mid x \in W\}, \text{ and}$$
$$W^\triangledown = \{(x, -x) \in W^\square \mid x \in W\}.$$

We denote by $P(W^\triangle)$ by the parabolic subgroup preserving $W^\triangle$. For a character χ of $F\times$, we denote by $I(s, \chi)$ the space of smooth functions F on $G(W^\square)$ satisfying

$$F(pg) = \delta(p) \cdot \chi(\Delta(p)) \cdot F(g)$$

for $p \in P(W^{\triangle})$ and $g \in G(W^{\square})$. Here, we denote by $\Delta(p)$ the inverse of the reduced norm on $\mathrm{End}(W^{\triangle})$ of the restriction $p|_{W^{\triangle}}$. Then, we define the doubling zeta integral: for a section $F \in I(s,\chi)$ and for a matrix coefficient ξ of an irreducible representation π of $G(W)$, we define

$$Z^W(F,\xi) = \int_{G(W)} F(\iota(g))\xi(g)\,dg$$

where $\iota : G(W) \to G(W^{\square})$ is an embedding given by $\iota(g)(x_1,x_2) = (x_1 \cdot g, x_2)$. It is known that the integral convergent absolutely when $\Re s \gg 0$, and it admits a meromorphic continuation to $\mathbb{C}$. Note that the doubling γ-factor, which is expected to coincide with the standard γ-factor, appears in a local functional equation of the doubling zeta integrals. (In the quaternionic case, this is established in [Yam14], [Kak20].)

Now we explain the Weil representation which we use in the thesis. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^{\times}$. Let $\omega_{\psi}^{\square}$ be the Weil representation of the reductive dual pair $(G(V), G(W^{\square}))$. If we fix a basis $\underline{e'}^{\square} = (e'_1,\dots,e'_{2n})$ for $W^{\square}$ so that $e'_1,\dots,e'_n$ span $W^{\triangle}$ and $e'_{n+1},\dots,e'_{2n}$ span $W^{\triangledown}$, we define the Weyl element τ of $G(W^{\square})$ by

$$\begin{cases} \tau(e'_i) = e'_{n+i} & i = 1,\dots,n, \\ \tau(e'_i) = -\epsilon e'_{i-n} & i = n+1,\dots,2n. \end{cases}$$

Then, $\omega_{\psi}^{\square}$ is realized on the space $\mathcal{S}(V \otimes W^{\triangledown})$ of the Schwartz-Bruhat functions on $V \otimes W^{\triangledown}$, and the action of $G(V) \times G(W^{\square})$ on $\mathcal{S}(V \otimes W^{\triangledown})$ is computed by Kudla [Kud94, p.40] as follows:

Proposition 1. *Let $\phi \in \mathcal{S}(V \otimes W^{\triangledown})$. Then, $\omega_{\psi}^{\square}(h,g)\phi = \beta_V(g)r(g)(\phi \circ h^{-1})$. More precisely,*

- $[\omega_{\psi}^{\square}(h,1)\phi](x) = \phi(h^{-1}x)$ *for $h \in G(V)$,*
- $[\omega_{\psi}^{\square}(1,m(a))\phi](x) = \beta_V(m(a))|N(a)|^{-m}\phi(x \cdot {}^t a^{*-1})$ *for $a \in \mathrm{GL}(W^{\triangledown})$,*
- $[\omega_{\psi}^{\square}(1,b)\phi](x) = \psi(\frac{1}{4}\langle\langle x, x \cdot b\rangle\rangle)\phi(x)$ *for $b \in U(W^{\triangle})$,*
- *the action of the Weyl element τ is given by*

$$[\omega_{\psi}^{\square}(1,\tau)\phi](x) = \beta_V(\tau) \cdot \int_{V \otimes W^{\triangledown}} \psi(\frac{1}{2}\langle\langle x, y\tau\rangle\rangle)\phi(y)\,dy$$

where dy is the self-dual measure of $V \otimes W^{\triangledown}$ with respect to the pairing

$$V \otimes W^{\triangledown} \times V \otimes W^{\triangledown} \to \mathbb{C} : x, y \mapsto \psi(\frac{1}{2}\langle\langle x, y\tau\rangle\rangle).$$

Here, we denote by N the reduced norm of $\mathrm{End}_D(W^{\triangledown})$ over F, and $\beta_V(g)$ is the constant given by

- $\beta_V(m(a)) = 1, \beta_V(\tau) = (-1)^{mn}$ *in the case $\epsilon = 1$, and*
- $\beta_V(m(a)) = \chi_V(N(a)), \beta_V(\tau) = (-1, \det V)_F^n(-1)^{mn}(-1,-1)_F^{mn}$ *in the case $\epsilon = -1$.*

Finally, we explain the local theta correspondence. Put $\mathbb{W} = V \otimes_D W$. We denote by $\mathrm{Mp}_{\psi}(\mathbb{W})$ the metaplectic group, and by $\widetilde{j} : G(V) \times G(W) \to \mathrm{Mp}_{\psi}(\mathbb{W})$ a splitting of the embedding

$$j : G(W) \times G(V) \to \mathrm{Sp}(\mathbb{W}) : (g,h) \mapsto g \otimes h.$$

For an irreducible representation π of $G(W)$, we define $\Theta_{\psi}(\pi, V)$ as the largest quotient module

$$(\widetilde{j}^*\omega_{\psi} \otimes \pi^{\vee})_{G(W)}$$

of $\widetilde{j}^*\omega_{\psi} \otimes \pi^{\vee}$ on which $G(W)$ acts trivially. This is a representation of $G(V)$. We define the theta correspondence $\theta_{\psi}(\pi, V)$ of π by

$$\theta_{\psi}(\pi,V) = \begin{cases} 0 & (\Theta_{\psi}(\pi,V) = 0), \\ \text{the maximal semisimple quotient of } \Theta_{\psi}(\pi,V) & (\Theta_{\psi}(\pi,V) \neq 0). \end{cases}$$

Then, the following property called the Howe duality holds, which is completely proved in [GS17, Theorem 1.3] in the quaternionic case.

Proposition 2 (Howe duality)**.** *For irreducible representations π_1, π_2 of $G(W)$, we have*

(1) $\theta_\psi(\pi_1, V)$ is irreducible if it is non-zero,
(2) $\pi_1 \cong \pi_2$ if $\theta_\psi(\pi_1, V) \cong \theta_\psi(\pi_2, V) \neq 0$,
(3) $\theta_\psi(\pi_1, V)^\vee \cong \theta_{\overline{\psi}}(\pi_1^\vee, V)$.

3. Main results

This section is a summary of §7.3 and §§10-11 of the thesis. In this section, we define the three invariants $\alpha_1(W)$, $\alpha_2(V, W)$, and $\alpha_3(V, W)$ which we mentioned in §1, and we state their explicit formulas.

3.1. **The constant $\alpha_1(W)$.** Take a basis $\underline{e} = (e_1, \ldots, e_n)$ for W. We denote $R = (\langle e_i, e_j \rangle)_{i,j} \in \mathrm{GL}_n(D)$, and we denote by $N(R)$ its reduced norm. Moreover, we take the bases $\underline{e}'^{\square} = (e'_1, \ldots, e'_{2n})$ for $W^\square$ defined by

$$e'_i = (e_i, e_i) \ e'_{n+i} = \sum_{k=1}^{n} a_{jk}(e_i, -e_i)$$

for $i = 1, \ldots, n$, where $(a_{jk})_{j,k} = R^{-1}$. Then, we choose the maximal compact subgroup K of $G(W^\square)$ preserving the lattice $\mathcal{O}_D e'_1 \oplus \cdots \oplus \mathcal{O}_D e'_{2n}$ of $W^\square$. We denote by f°_s the unique K-invariant section of $I(s, 1)$ with $f^\circ_s(1) = 1$, and we denote by ξ° the matrix coefficient of the trivial representation with $\xi^\circ(1) = 1$. We put

$$\alpha_1(W) := Z^W(f^\circ_\rho, \xi^\circ)$$

where $\rho = n - \frac{\epsilon}{2}$. Then, we have:

Proposition 3. *(1) In the case $-\epsilon = 1$, we have*

$$\alpha_1(W) = |2|_F^{n(2n+1)} \cdot |N(R)|^{-n-\frac{1}{2}} \cdot q^{-(2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil)} \cdot \prod_{i=1}^{n} (1 + q^{-(2i-1)}).$$

(2) In the case $-\epsilon = -1$, we have

$$\alpha_1(W) = |2|_F^{n(2n-1)} \cdot |N(R)|^{-n+\frac{1}{2}} \cdot q^{-(2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor)} \cdot \prod_{i=1}^{n} (1 + q^{-(2i-1)}).$$

3.2. **The constant $\alpha_2(V, W)$.** We choose the self-dual Haar measure on $V \otimes W^\triangledown$ with respect to the pairing

$$V \otimes W^\triangledown \times V \otimes W^\triangledown \to \mathbb{C} : (x, y) \mapsto \psi(\frac{1}{2}\langle\langle x, y\tau \rangle\rangle)$$

where τ is the Weyl element in $G(W^\square)$ defined by

$$\begin{cases} \tau(e'_i) = e'_{n+i} & i = 1, \ldots, n, \\ \tau(e'_i) = -\epsilon e'_{i-n} & i = n+1, \ldots, 2n. \end{cases}$$

Then, we define the map $\mathcal{I}$ by

$$\mathcal{I}(\phi, \phi') = \int_{G(V)} (\omega^\square_\psi(h, 1)\phi, \phi') \, dh$$

for $\phi, \phi' \in \mathcal{S}(V \otimes W^\triangledown)$. Here, we denote $(\ ,\ )$ the L^2-inner product on $\mathcal{S}(V \otimes W^\triangledown)$. On the other hand, we define the map $\mathcal{E} : \mathcal{S}(V \otimes W^\triangledown)^2 \to \mathbb{C}$ as follows: for $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, we define

$F_\phi \in I(-\frac{1}{2}, \chi_V)$ by $F_\phi(g) = [\omega_\psi^\square(g)\phi](0)$, and we choose $F_\phi^\dagger \in I(\frac{1}{2}, \chi_V)$ so that $M(\frac{1}{2}, \chi_V)F^\dagger = F_\phi$ where $M(s, \chi_V) : I(s, \chi_V) \to I(-s, \chi_V)$ is the intertwining operator defined by

$$[M(s, \chi_V)F](g) = \int_{U(W^\triangle)} F(\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \tau g)\, dX$$

for $F \in I(s, \chi_V)$. Here, dX is the self-dual Haar measure on $\mathfrak{u}$ with respect to the pairing

$$\mathfrak{u} \times \mathfrak{u} \to \mathbb{C} : (X, Y) \mapsto \psi(T(XY))$$

where T is the reduced trace on $\mathrm{M}_n(D)$. Then the map $\mathcal{E}$ is defined by

$$\mathcal{E}(\phi, \phi') = \int_{G(W)} F_\phi^\dagger(\iota(g))\overline{F_{\phi'}(\iota(g))}\, dg$$

for $\phi, \phi' \in \mathcal{S}(V \otimes W^\triangledown)$. The maps $\mathcal{I}$ and $\mathcal{E}$ are both $G(V) \times G(V) \times \Delta G(W)$-equivalent functionals. On the other hand, it is known that the space of $G(V) \times G(V) \times \Delta G(W)$-equivalent functionals

$$\lambda : \omega_\psi \otimes \overline{\omega_\psi} \to \mathbb{C}$$

is one dimensional. Thus, we can define $\alpha_2(V, W)$ as the non-zero constant satisfying $\mathcal{I} = \alpha_2(V, W) \cdot \mathcal{E}$. Then our second result is the following:

Theorem 4. *Put $\rho = n - \frac{\epsilon}{2}$. Then we have*

$$\alpha_2(V, W) = |2|_F^{-2n\rho + n(n-\frac{1}{2})} \cdot |N(R)|^\rho \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(1-2i)}{\zeta_F(2i)}$$
$$\times \begin{cases} 2(-1)^n \gamma(1-n, \chi_V, \psi)^{-1} \epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ 1 & (-\epsilon = -1). \end{cases}$$

3.3. **The constant** $\alpha_3(V, W)$**.** For an irreducible representation σ of $G(V)$ and for a character χ of $F^\times$, we denote by $\gamma^V(s, \sigma \times \chi, \psi)$ the γ-factor defined by the functional equation of the doubling zeta integrals. Recall that this γ-factor is expected to coincide with the standard γ-factor. It is known that $\theta_\psi(\pi, V)$ is square integrable if π is square integrable and $\theta_\psi(\pi, V) \neq 0$. Moreover, it is proved that there is a constant $\alpha_3(V, W)$ such that

$$\frac{\deg \pi}{\deg \theta_\psi(\pi, V)} = \alpha_3(V, W) \cdot \omega_\pi(-1) \cdot \gamma^V(0, \theta_\psi(\pi, V) \times \chi_W, \psi)$$

for all square integrable irreducible representations π so that $\theta_\psi(\pi, V) \neq 0$. This follows from the existence of the proportional constant $\alpha_2(V, W)$ by a Gan-Ichino's observation using a local analogue of the proof of Rallis inner product formula [GI14, §18]. In the thesis, we determine the constant $\alpha_3(V, W)$:

Theorem 5. *We have*

$$\alpha_3(V, W) = \begin{cases} (-1)^n \chi_V(-1) \epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ \frac{1}{2} \cdot \chi_W(-1) \epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases}$$

4. Outline of the proofs

This section is a summary of §§12-19 of the thesis. First, we compute $\alpha_1(W)$ in the case where either W is anisotropic or $R \in \mathrm{GL}_n(\mathcal{O}_D)$. In the case where W is anisotropic, we compute it by using the local functional equation of the doubling zeta integral. In the case $R \in \mathrm{GL}_n(\mathcal{O}_D)$, we use the following lemma, which relates the doubling zeta integral with the intertwining operator:

Lemma 6. *For $f \in I(\rho, 1)$, we have*

$$\int_{G(W)} f((g,1))\,dg = m^\circ(\rho)^{-1} \cdot \alpha_1(W) \cdot \int_{U(W^\triangle)} f(\tau u)\,du$$

where

$$m^\circ(s) = \begin{cases} |2|_F^{n(n-\frac{1}{2})} q^{-\frac{1}{2}n(n+1)} \frac{\zeta_F(s-n+\frac{1}{2})}{\zeta_F(s+n+\frac{1}{2})} \prod_{i=0}^{n-1} \frac{\zeta_F(2s-2i)}{\zeta_F(2s+2n-4i-3)} & (-\epsilon = 1), \\ |2|_F^{n(n-\frac{1}{2})} q^{-\frac{1}{2}n(n-1)} \prod_{i=0}^{n-1} \frac{\zeta_F(2s-2i)}{\zeta_F(2s+2n-4i-1)} & (-\epsilon = -1). \end{cases}$$

Then, we compute $\alpha_2(V, W)$ in the case where either V or W is anisotropic. This part is the key ingredient of the thesis. Suppose that V is anisotropic. In this case, we can compute $\mathcal{I}(\phi, \phi')$ and $\mathcal{E}(\phi, \phi')$ explicitly when the support of ϕ is sufficiently small. More precisely, for ϕ in $\mathcal{S}(V \otimes W^\triangledown)$ and $t \in \mathbb{Z}$, we define $\phi_t \in \mathcal{S}(V \otimes W^\triangledown)$ by

$$\phi_t(x) := q^{-4mnt} \phi(x\varpi_F^t)$$

where ϖ_F is a uniformizer of F. Then, we have:

Lemma 7. *Let $\phi, \phi' \in \mathcal{S}(V \otimes W^\triangledown)$. For sufficiently large t, we have*

$$\mathcal{I}(\phi_t, \phi') = q^{-4mnt} |G(V)| F_\phi(1) \overline{F_{\phi'}(\tau)},$$

and

$$\mathcal{E}(\phi_t, \phi') = m^\circ(\rho)^{-1} \alpha_1(W) q^{-4mnt} F_\phi(1) \overline{F_{\phi'}(\tau)}.$$

Then, suppose that W is anisotropic and V is isotropic. It only occurs in the case where $\epsilon = 1, \dim V = 2, \dim W = 3$, and $\chi_W = 1$. In this case, if we take a certain lattice L of $V \otimes W^\triangledown$ and denote by 1_L the characteristic function of L, then we can compute $\mathcal{I}(1_L, 1_L)$ and $\mathcal{E}(1_L, 1_L)$ explicitly. Thus, we have the formula of $\alpha_2(V, W)$ in the case where either V or W is anisotropic.

Then, we compute $\alpha_3(V, W)$. The two constant $\alpha_2(V, W)$ and $\alpha_3(V, W)$ are related to each other by considering a local analogue of the Rallis inner product formula:

Lemma 8. *We have*

$$\begin{aligned} \alpha_3(V, W) =& \frac{1}{2} \cdot \alpha_2(V, W) \cdot e(G) \cdot |2|_F^{2n\rho - n(n-\frac{1}{2})} \cdot |N(R)|^{-\rho} \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(2i)}{\zeta_F(1-2i)} \\ & \times \begin{cases} \chi_V(-1)\gamma(1-n, \chi_V, \psi) & (-\epsilon = 1), \\ \chi_W(-1)\epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases} \end{aligned}$$

Proof. Similar to [GI14, §18]. □

Hence, we have Theorem 5 in the case where either V or W is anisotropic. To complete the proof of Theorem 5, we consider parabolic inductions.

Lemma 9. *we have*

$$\alpha_3(V', W') = \alpha_3(V, W).$$

Proof. We prove this lemma in the same line with [GI14, §§20.3-20.7]. However, we need to prepare some properties of representations of quaternionic unitary groups. Let $W' = X + W + X^*$ be another $(-\epsilon)$-Hermitian space where X, X^* are isotropic subspaces with $\dim X = \dim X^* = t$ and $X + X^*$ is the orthogonal complement of W. Let π be an irreducible supercuspidal representation of $G(W)$, let τ be an irreducible supercuspidal representation of $\mathrm{GL}(X)$. We analyze poles of Plancherel measures, and we conclude that there is a positive real number $s_0 > 0$ such that $\mathrm{Ind}_{P(X)}^{G(W')} \pi \boxtimes \tau_{s_0}$ has a square integrable irreducible subquotient π'. Here, $P(X)$ denotes the parabolic subgroup of $G(W')$ preserving X, and τ_{s_0} is the representation of $\mathrm{GL}(X)$ defined by $\tau_{s_0}(g) = |N(g)|^{s_0}\tau(g)$ where $N(g)$ is the reduced norm. Moreover, we consider another ϵ-Hermitian space $V' = Y + V + Y^*$ where Y, Y^* are isotropic subspaces with $\dim Y = \dim Y^* = t$ and $Y + Y^*$ is the orthogonal complement of V. Then, when either V or W is anisotropic, we prove that there exists an irreducible supercuspidal representation π of $G(W)$ so that $\theta_\psi(\pi, V)$ is non-zero supercuspidal. Then, one can prove that $\mathrm{Ind}_{P(Y)}^{G(V)}\theta_\psi(\pi, V)$ also has a square integrable irreducible subquotient σ'. Moreover, we have $\sigma' \cong \theta_\psi(\pi', V')$. As in [GI14], we use the Heiremann's result [Hei04] to describe $\deg \pi'$ in terms of $\deg \pi$ and $\deg \tau$. The same argument holds for σ'. Therefore, we have $\alpha_3(V', W') = \alpha_3(V, W)$. □

This completes the proof of Theorem 5. Moreover, by tracing the above argument conversely, we have

$$\text{Theorem 5} \Rightarrow \text{Theorem 4} \Rightarrow \text{Proposition 3.}$$

Thus, we obtain the complete formulas for $\alpha_1(W)$, $\alpha_2(V, W)$, and $\alpha_3(V, W)$.

5. Formal degree conjecture for the non-split inner forms of Sp_4, GSp_4

As an application of Theorem 5, we prove the formal degree conjecture for the non-split inner forms of Sp_4 and GSp_4. For these groups, the local Langlands correspondence has been established by Choiy [Cho17] and by Gan-Tantono [GT14]. Let G be the non-split inner form of Sp_4 or that of GSp_4, let π be a square integrable irreducible representation of G. Then, the Langlands parameter of π is defined via the local theta correspondence between G and the (similitude) unitary group of a skew-Hermitian space of two or three dimensional. On the other hand, by accidental isomorphisms, the formal degree conjecture for the similitude unitary groups above is available. Hence, we have:

Corollary 10.

$$\deg \pi = \frac{\dim \eta}{\# C_\phi}\gamma(0, \pi, \mathrm{Ad}, \psi)$$

where $\gamma(s, \pi, \mathrm{Ad}, \psi)$ is the Adjoint γ-factor of π.

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