FORMAL DEGREES AND LOCAL THETA CORRESPONDENCE: QUATERNIONIC CASE

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Abstract. In this paper, we determine a constant occurring in a local analogue of the Siegel-Weil formula, and describe the behavior of the formal degree under the local theta correspondence for quaternion dual pairs of almost equal rank over a non-Archimedean local field of characteristic 0. As an application, we prove the formal degree conjecture of Hiraga-Ichino-Ikeda for the non-split inner forms of Sp_4 and GSp_4.

Contents

1. Backgrounds and Introduction

1.1. **Motivations.** The theory of automorphic representations is a common generalization of that of Dirichlet characters and that of classical modular forms. This is formulated on the groups of the adelic points of reductive groups over global fields. The setting we will consider in this paper is the local aspect of it, which we explain in more detail. Let F be a local field, let G be a reductive group over F, and let $G(F)$ be the group of its F-valued points.

Suppose that F is non-Archimedean. Then, as the local aspect, we discuss smooth admissible representations of $G(F)$, which are defined to be representations on $\mathbb{C}$ vector spaces that have direct sum decompositions into irreducible representations with finite multiplicities when they are restricted to an open compact subgroup C of $G(F)$. Note that $G(F)$ possesses an open compact subgroup (for example, Iwahori subgroups (§5.3)) since F is non-Archimedean. On the other hand, when F is Archimedean, we need to discuss moderate growth representations or $(\mathfrak{g}, K)$-modules (for example, see [Cas89]), but we do not explain details of them because our interest in this paper is totally non-Archimedean cases. It is conjectured that the representations above are parametrized by "Langlands parameters" (§1.3). Note that in §1.3 we omit the Archimedean case. This conjecture gives a lot of motivations in representation theory. Actually, the main result in this paper has a background of the formal degree conjecture (§1.5), which describes the formal degree in terms of the Langlands parameter (§1.3). Here, the formal degree is an invariant associated with a square integrable irreducible smooth admissible representation which is defined to have a non-zero square integrable matrix coefficient (see §1.5). More precisely, we approach the formal degree conjecture by using a correspondence between irreducible representations of two different classical groups (see §1.2), which is called the theta correspondence (see §1.6). The main result describes the behavior of the formal degree under the theta correspondence between "quaternionic dual pair". This extends a result of Gan-Ichino (see §1.7), which will be explained more precisely in §1.8.

Finally, we note that "a smooth admissible representation over $\mathbb{C}$" of $G(F)$ will be abbreviated as "a representation" of $G(F)$ in this paper.

1.2. **Classical groups.** Now, we introduce the classical groups. Advantages to consider this class of groups are that they have explicit constructions, a simple classification, and particular methods in the representation theory (for example, the doubling method (§1.4) and the theta correspondence (§1.6)). Let F be a local field of characteristic 0, and let E be either F itself, a 2-dimensional semisimple F-algebra, or a quaternion algebra over F. Although our interest is primarily when E is division, we allow E to be split (i.e. $E = F \times F$ or $E = \mathrm{M}_2(F)$) since they appear as a localization of global division algebras. We denote by $* : E \to E$ the main involution over F, and by Cent.E the center of E. Let W be a free right E module of rank n equipped with a map $\langle\ ,\ \rangle : W \times W \to E$ such that

- $\langle\ ,\ \rangle$ is either 0 or non-degenerate,
- $\langle ax + by, cz \rangle = a\langle x, z\rangle c^* + b\langle y, z\rangle c^*$ for $x, y, z \in W$ and $a, b, c \in E$,
- there is an $\epsilon = \pm 1$ such that $\langle y, x\rangle = -\epsilon \langle x, y\rangle^*$ for $x, y \in W$.

Such W is called $(-\epsilon)$-Hermitian spaces. We denote by $G(W)$ the algebraic group

$$\{g \in \mathrm{GL}(W) \mid \langle gx, gy\rangle = \langle x, y\rangle \text{ for all } x, y \in W\}.$$

The groups $G(W)$ are called **classical groups**. Let $\underline{e} = (e_1, \ldots, e_n)$ be a basis for W over E. We define $R = (\langle e_i, e_j\rangle)_{i,j} \in \mathrm{GL}_n(E)$, and we define

$$\mathfrak{d}(W) = N(R) \times \begin{cases} (-1)^{\frac{1}{2}n(n-1)}2^{-n} & (E = F), \\ (-1)^{\frac{1}{2}n(n-1)} & ([E : F] = 2), \\ (-1)^n & ([E : F] = 4). \end{cases}$$

Here, N denotes the reduced norm of $\mathrm{M}_n(E)$ over the center Cent.E of E. The modulo $(F^\times)^2$ class of $\mathfrak{d}(W)$ does not depend on the choice of the basis, and it is called the discriminant of W. Note that the isometry class of W are determined by the dimension, the discriminant, the Hasse invariant, and the signature (for details, see Scharlau's book [Sch85]). We denote by χ_W the character of $(\mathrm{Cent}.E)^\times$ given by

- $\chi_W(a) = (a, \mathfrak{d}(W))_F$ for $a \in F$ where $(\ , \ )_F$ is the Hilbert symbol of F when either $E = F$ or $[E : F] = 4$,
- a fixed character so that $\chi_W|_{F^\times} = \chi_E^n$ where χ_E is the quadratic character of $F^\times$ associated with E via the local class field theory when $[E : F] = 2$.

Then, we will denote by

- $\mathrm{Sp}(W)$ instead of $G(W)$ when $E = F$ and $-\epsilon = -1$,
- $\mathrm{O}(W)$ instead of $G(W)$ when $E = F$ and $-\epsilon = 1$,
- $\mathrm{U}(W)$ instead of $G(W)$ when E is a quadratic extension field of F and $-\epsilon = 1$.

Moreover, we will consider a double cover $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$, which is called the **metaplectic group** of W (for definition, see e.g. [GS12], [RR93, Theorem 4.1]). Although the metaplectic group $\mathrm{Mp}(W)$ does not have a structure of an algebraic group, we may include it in the classical groups. Since we consider the theory of inner forms (see §1.3.2), it is useful to introduce particular notations for quasi-split groups;

- for a positive integer t, we denote $\mathrm{Sp}(2t) = \mathrm{Sp}(W)$ for W with $E = F$, $-\epsilon = -1$ and $\dim W = 2t$;
- for a positive integer t and a quadratic character χ, we denote $\mathrm{O}(2t, \chi) = \mathrm{O}(W)$ for W with $E = F$, $-\epsilon = 1$, $\chi_W = \chi$, and W has a t dimensional isotropic subspace;
- for a non-negative integer t, we denote $\mathrm{O}(2t + 1) = \mathrm{O}(W)$ for W with $E = F$, $-\epsilon = 1$, and W has a t-dimensional isotropic subspace;
- we denote $\mathrm{U}(n) = \mathrm{U}(W)$ for W with $[E : F] = 2$, and W has a $\lfloor n/2 \rfloor$ dimensional isotropic subspace.

Let W be a non-degenerate n-dimensional $(-\epsilon)$-Hermitian space. Then, it is known that $G(W)$ is an inner form of one of $\mathrm{Sp}(2t)$, $\mathrm{O}(2t, \chi)$, $\mathrm{O}(2t + 1)$, $\mathrm{U}(n)$. Here, we put $t = \lfloor n/2 \rfloor$ (more precisely, see §1.3). As the local Langlands correspondence which we will see in §1.3 indicates, the representation theory of $G(W)$ is expected to have some similarities to that of the quasi-split inner form. However, in general, the representation theory for quaternionic unitary groups might be more complicated and is less developed than that for non-quaternionic classical groups. Because our result is an extension work to quaternionic unitary groups, we will pay more attention to the difference between them in the later subsections.

1.3. **Local Langlands correspondence.**

Now, we explain the local Langlands correspondence. Roughly speaking, the local Langlands correspondence (in general a conjecture) is a classification theory of the irreducible representations of reductive groups over a local field, which is a far extension of the highest weight theory for compact Lie groups. Many invariants of irreducible representations are expected to be interpreted in terms of Langlands parameters (for example, see §1.4, §1.5). The main result of this paper has a background of the "Langlands functoriality": it compares the formal degrees of two representations whose Langlands parameters would be related to each other.

1.3.1. *L-parameters.*

Let F be a non-Archimedean local field, and let G be a connected reductive group over F. We omit the Archimedean theory for simplicity. We denote by F^s the separate closure of F. For a Galois extension E/F, and by $\Gamma_{E/F}$ its Galois group.

We define the Weil group by

$$W_F = \langle I, \mathrm{Fr} \rangle$$

where I is the inertia subgroup of $\Gamma_{F^s/F}$ and Fr is a Frobenius element of F. The structure of the topological group of W_F is defined so that a fundamental neighborhood system of $1 \in W_F$ consists of the open subgroups of I. Moreover, we define the Langlands group by

$$L_F = W_F \times \mathrm{SL}_2(\mathbb{C}).$$

Let $\widehat{G}$ be the Langlands dual group of G, and let ${}^L G$ be the L-group of G. We do not recall here the definition, but we give $\widehat{G}$ and ${}^L G$ explicitly for quasi-split classical groups listed in §1.2;

- if $G = \mathrm{GL}_n$, then $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$, and ${}^L G$ is the direct product $\mathrm{GL}_n(\mathbb{C}) \times W_F$;
- if $G = \mathrm{Sp}(2n)$, then $\widehat{G} = \mathrm{SO}(2n+1, \mathbb{C})$ and ${}^L G$ is the direct product $\mathrm{SO}(2n+1, \mathbb{C}) \times W_F$;
- if $G = \mathrm{Mp}(2n)$, then $\widehat{G} = \mathrm{Sp}(2n, \mathbb{C})$ and ${}^L G$ is the direct product $\mathrm{Sp}(2n, \mathbb{C}) \times W_F$;
- if $G = \mathrm{O}(2n, \chi)$, then $\widehat{G} = \mathrm{O}(2n, \mathbb{C})$ and ${}^L G$ is the subgroup
$$\{(g, w) \in \mathrm{O}(2n, \mathbb{C}) \times W_F \mid \det(g) = \chi(w)\},$$
and ${}^L G \to W_F$ is the natural projection;
- if $G = \mathrm{O}(2n+1)$, then, $\widehat{G} = (\mathrm{Sp}(2n, \mathbb{C}) \times \{\pm 1\})$, and ${}^L G$ is the subset
$$\{(g, \chi_W(w), w) \mid g \in \mathrm{Sp}(2n, \mathbb{C}), w \in W_F\}$$
of the direct product $(\mathrm{Sp}(2n, \mathbb{C}) \times \{\pm 1\}) \times W_F$;
- if $G = \mathrm{U}(n)$, then ${}^L G$ is the semi-direct product $\mathrm{GL}_n(\mathbb{C}) \rtimes W_F$. Here the action of W_F on $\mathrm{GL}_n(\mathbb{C})$ is given by
$$w \cdot g = \begin{cases} g & (w \in W_F), \\ \Phi_n {}^t g^{-1} \Phi_n^{-1} & (w \in W_F \setminus W_E) \end{cases}$$
where Φ_n is a matrix whose (i, j)-component is written by the Kronecker's delta $(-1)^{i-1} \delta_{i+j, n+1}$ for each i, j.

Note here that, although they are not an algebraic group (resp. not connected), we listed the metaplectic groups (resp. the orthogonal groups) because they are necessary when we consider the theta correspondence (§1.6 below). An L-parameter for G is a homomorphism
$$\phi : L_F \to {}^L G$$
so that $\phi|_{W_F}$ is continuous, $\phi(\mathrm{Fr})$ is semisimple where Fr is the Frobenius element in W_F, $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is an algebraic homomorphism, the image $\mathrm{Im}(\phi)$ of ϕ is not contained in any non-relevant Levi subgroup of ${}^L G$ (see [Bor79]), and the following diagram is commutative:
$$\begin{array}{ccc} L_F & \xrightarrow{\phi} & {}^L G \\ \downarrow & & \downarrow \\ W_F & = & W_F \end{array}$$
where the vertical maps are natural projections.

1.3.2. *Inner forms.* Let G_1 and G_2 be two reductive algebraic groups over F. We denote by $\mathrm{Inn}(G)$ the algebraic group consisting of the inner automorphisms of G, which is isomorphic to G/Z where Z is the center of G. Then G_2 is said to be an inner form of G_1 if there is an isomorphism $\Psi : G_1 \to G_2$ over F^s and a 1-cocycle $c \in H^1(\Gamma_{F^s/F}, \mathrm{Inn}(G_1))$ such that the action of $\Gamma_{F^s/F}$ on $G_2(F^s)$ is given by
$$\Gamma_{F^s/F} \times G_2(F^s) \to G_2(F^s) : (\gamma, g) \mapsto \Psi(c_\gamma(\gamma \cdot \Psi^{-1}(g)))$$
where $\gamma \cdot g$ is the action of $\Gamma_{F^s/F}$ on G_1. For classical groups, it is known that;

- the inner forms of GL_n are all $\mathrm{GL}_{n/d'}(D')$ so that D' is a central division quaternion algebra over F of $[D' : F] = d'^2$ for $d' | n$;
- in the case where $E = F$, $-\epsilon = 1$, and $\dim W = 2t$, then $\mathrm{O}(W)$ is an inner form of $\mathrm{O}(2t, \chi_W)$;
- in the case where $E = F$, $-\epsilon = 1$, and $\dim W = 2t+1$, then $\mathrm{O}(W)$ is an inner form of $\mathrm{O}(2t+1)$;

- in the case where E is a quadratic extension field of F and $\dim W = n$, then $\mathrm{U}(W)$ is an inner form of $\mathrm{U}(n)$;
- in the case where E is a division quaternion algebra over F, $-\epsilon = 1$, and $\dim W = n$, then $G(W)$ is an inner form of $\mathrm{Sp}(2n)$;
- in the case where E is a division quaternion algebra over F, $-\epsilon = -1$, and $\dim W = n$, then $G(W)$ is an inner form of $\mathrm{O}(2n, \chi_W)$.

Now we explain some basic properties of the inner forms. Let G be a connected quasi-split reductive group over F, and let G' be an inner form of G. First, we have $\Phi(G') \subset \Phi(G)$. Second, there is an isomorphism

$$H^1(\Gamma_{F^s/F}, \mathrm{Inn}(G)) \cong \mathrm{Hom}(Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_{F^s/F}}, \mathbb{C}^\times)$$

(see [Kot84, Proposition 6.4]). Here, we denote by $Z(\widehat{G}_{\mathrm{sc}})$ the center of the simply connected cover of the adjoint group $\widehat{G}_{\mathrm{ad}}$ of $\widehat{G}$. We denote by $\zeta_{G'}$ the character of $Z(\widehat{G}_{\mathrm{sc}})^{\Gamma_{F^s/F}}$ which is associated with the inner form G' of G by the above correspondence. We also consider the case $G = \mathrm{O}(2t, \chi)$. Then, for an inner form G' of G, we denote by $\zeta_{G'}$ the character $\zeta_{G'^\circ}$ of the group $Z(\widehat{G}^\circ_{\mathrm{sc}})^{\Gamma_{F_s/F}}$. Here, G'° denotes the Zariski connected component of G'.

1.3.3. *Local Langlands correspondence.* The local Langlands correspondence is usually formulated on a connected reductive group. But as explained in §1.3.1, we also need to consider a bit different types of groups. Hence, for a while, we assume that G is a classical group. We denote by $\Phi_F(G)$ the set of $\widehat{G}$-conjugacy classes of L-parameters for G. By the local Langlands conjecture, we expect a finite to one map

$$\Pi(G(F)) \to \Phi_F(G) : \pi \mapsto \phi_\pi$$

where we denote by $\Pi(G(F))$ the set of equivalent classes of the irreducible representations of $G(F)$ (see [Bor79, Chapter III]). For the connection between the maps for even orthogonal groups and those for even special orthogonal groups, see [AG17, §3.6]. Note that, for an odd orthogonal group, the definition of the L-parameter in Atobe-Gan [AG17] differs from that of [GI14]. In this paper, we use the latter one. For a metaplectic group, we consider the set of the genuine irreducible representations instead of $\Pi(G(F))$, and the above map is defined by using the local theta correspondence (see [GS12]). In any case, for $\phi \in \Phi_F(G)$, we denote by

$$\Pi_\phi(G(F)) = \{\pi \in \Pi(G(F)) \mid \phi_\pi \sim \phi\},$$

and we call it the L-packet for ϕ. Here, "$\sim$" denotes the conjugacy equivalence by $\widehat{G}$. We are then interested in the internal structure of $\Pi_\phi(G(F))$. We explain here an expectation based on a conjecture of Arthur [Art06]. Note that Arthur discussed only tempered L-packets, but we can extend the discussion to non-tempered ones for at least classical groups (see [SZ18]). Denote by $C_\phi(\widehat{G})$ the centralizer of $\mathrm{Im}\,\phi$ in $\widehat{G}$, by $S_\phi(\widehat{G})$ the image of $C_\phi(\widehat{G})$ in $\widehat{G}/Z(\widehat{G})$, by $\widetilde{S}_\phi(\widehat{G})$ the preimage of $S_\phi(\widehat{G})$ in the simply connected cover $\widehat{G}_{\mathrm{sc}}$ of $\widehat{G}/Z(\widehat{G})$, and by $\widetilde{\mathcal{S}}_\phi(\widehat{G})$ the component group $\pi_0(\widetilde{S}_\phi(\widehat{G}))$. If we take a character ζ'_G of $Z(\widehat{G^\circ}_{\mathrm{sc}})$ so that its restriction to $Z(\widehat{G^\circ}_{\mathrm{sc}})^{\Gamma_{F^s/F}}$ is ζ_G, then we have a bijection

$$\Pi_\phi(G(F)) \to \mathrm{I}(\widetilde{\mathcal{S}}_\phi(\widehat{G}), \zeta'_G) \tag{1.1}$$

(see [Kal18, §4.6]). Here, we denote by $\mathrm{I}(\widetilde{\mathcal{S}}_\phi(\widehat{G}), \zeta'_G)$ the set of the irreducible constituents of $\mathrm{Ind}_{\widetilde{\mathcal{S}}_\phi(\widehat{G^\circ})}^{\widetilde{\mathcal{S}}_\phi(\widehat{G})} \rho$ for all irreducible representations ρ of $\widetilde{\mathcal{S}}_\phi(\widehat{G^\circ})$ with $\mathrm{Hom}_{Z(\widehat{G^\circ}_{\mathrm{sc}})}(\zeta'_G, \rho) \neq 0$. Note that the map may not be canonical, and that is an obstacle to formulate the "endoscopic character relation". Note that Kaletha discussed how to remove this ambiguity (see [Kal16, §5]). However, it has no effect on the formulation of the formal degree conjecture (see §1.5). Therefore, in this paper, for an irreducible representation π of G, we define the **Langlands parameter** (ϕ, η) of π

to be a pair consisting of the L-parameter $\phi = \phi_\pi$ and an irreducible representation η associated with π by the map (1.1).

The local Langlands correspondence has been established for

- the general linear group GL_n by Harris-Taylor [HT01], by Henniart [Hen00], and by Scholze [Sch13],
- the quasi-split special orthogonal groups $\mathrm{SO}(2n+1)$, $\mathrm{SO}(2n,\chi)$ by Arthur [Art13],
- the symplectic group $\mathrm{Sp}(2n)$ by Arthur [Art13],
- quasi-split unitary groups $\mathrm{U}(n)$ by Mok [Mok15],
- (non-quasi-split) unitary groups $\mathrm{U}(W)$ by Kaletha-Minguez-Shin-White [KMSW14].

For quaternionic unitary groups, the Langlands correspodence has not been established yet in general. However, thanks to accidental isomorphisms, it is available for inner forms of $\mathrm{O}(2)$, $\mathrm{O}(4)$, $\mathrm{Sp}(4)$ (see [Cho17]).

1.4. **Local factors.** L-functions have contributed to the development of number theory. The local L-factor of an irreducible representation of $G(F)$ is a far generalization of the p-factor of the Eular product of Dirichlet L-function, which is an invariant associated with a Dirichlet character. By using the Galois side (Langlands parameter side), we can define various local L-factors in a unified manner: let ρ be a finite-dimensional representation of the local Langlands group L_F. Then, ρ decomposes as

$$\bigoplus_{k\geq 0} S_k \otimes \mathrm{Sym}^k$$

where S_k is a representation of W_F and Sym^k is the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension $k+1$. Then we define the L-factor of ρ by

$$L(s,\rho) = \prod_{k\geq 0} \det(1 - q^{-\frac{k}{2}-s}\phi(\mathrm{Fr})|S_k^I)$$

where Fr is a Frobenius element in W_F and $I \subset W_F$ is the inertia subgroup. Now, let G be a reductive group over a local field F, let π be an irreducible representation of $G(F)$, and let r be a finite-dimensional representation of ${}^L G$. Then, we define a local L-factor

$$L(s,\pi,r) := L(s, r\circ\phi)$$

where ϕ is the L-parameter associated with π.

Then, we define the local ϵ-factor, which appears in the global functional equation of L-functions as a factor of an Euler product. Fix a non-trivial additive character ψ of F. Let π be an irreducible representation of $G(F)$, and let ϕ be its L-parameter. Then, for a finite-dimensional representation r of ${}^L G$, we define the ϵ-factor by

$$\epsilon(s,\pi,r,\psi) := \epsilon(s, r\circ\phi, \psi)$$

where the right-hand side is the ϵ-factor defined in [GR10, §2.2].

Finally, we define the local γ-factor $\gamma(s,r,\pi,\psi)$ by

$$\gamma(s,r,\pi,\psi) = \epsilon(s,r,\pi,\psi)\cdot\frac{L(1-s,r,\pi^\vee)}{L(s,r,\pi)}$$

where $\pi^\vee$ is the contragredient representation of π.

An advantage to consider the γ-factor is the "multiplicativity" for some r. More precisely, if π is an irreducible subquotient representation of $\mathrm{Ind}_P^G\sigma$ for some parabolic subgroup P of G and some irreducible representation σ of the Levi subgroup M of P, then the γ-factor of π is expected to be described by γ-factors of σ. (At least when r is the standard representation std (see §1.4.2) or r is an irreducible representation r_i of [Sha90, p. 278] for $i = 1, 2, \ldots$, the multiplicativity is expect to be satisfied). Moreover, the global functional equation can be interpreted as "$\prod_v \gamma_v(s,r,\pi,\psi) = 1$" (although the left-hand side does not converge).

1.4.1. *Adjoint γ-factor.* We denote by Z the center of G, by $\widehat{\mathfrak{g}}$ the Lie algebra of ${}^L G$, and by $\widehat{\mathfrak{g}}_0$ the Lie algebra of ${}^L(G/Z)$. Then we define a finite-dimensional representation

$$\mathrm{Ad} : {}^L G \to \mathrm{GL}(\widehat{\mathfrak{g}}_0)$$

by $\mathrm{Ad}(g)X = gXg^{-1}$ for $g \in {}^L G$ and $X \in \widehat{\mathfrak{g}}_0$. This representation is called the adjoint representation, and the γ-factor $\gamma(s,\pi,\mathrm{Ad},\psi)$ is called the adjoint γ-factor. This γ-factor appears in the formulation of the formal degree conjecture (§1.5).

1.4.2. *Standard γ-factor.* We define a finite dimensional representation std case by case:

- $\mathrm{std} : \mathrm{GL}_n(\mathbb{C}) \times W_F \to \mathrm{GL}_n(\mathbb{C}) : (g,w) \mapsto g$,
- $\mathrm{std} : \mathrm{GL}_n(\mathbb{C}) \rtimes W_F \to \mathrm{GL}_{2n}(\mathbb{C}) : (g,w) \mapsto \begin{pmatrix} g & 0 \\ 0 & \Phi_n {}^t g^{-1} \Phi_n^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}^{a_{\chi_E}(w)}$ where Φ_n is a matrix whose (i,j)-component is written by the Kronecker's delta $(-1)^{i-1}\delta_{i+j,n+1}$ for each i,j,
- $\mathrm{std} : \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \rtimes W_F \to \mathrm{GL}_2(\mathbb{C}) : (z_1,z_2,w) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{a_{\chi_E}(w)}$,
- $\mathrm{std} : \{(g,\chi_W(w),w) \mid g \in \mathrm{Sp}_{2t}(\mathbb{C}), w \in W_F\} \to \mathrm{GL}_{2t}(\mathbb{C}) : (g,\chi_W(w),w) \mapsto g$,
- $\mathrm{std} : \mathrm{SO}_{2t+1} \times W_F \to \mathrm{GL}_{2t+1}(\mathbb{C}) : (g,w) \mapsto g$,
- $\mathrm{std} : \mathrm{SO}_{2t} \rtimes W_F \to \mathrm{GL}_{2t}(\mathbb{C}) : (g,w) \mapsto g w_1^{a_\chi(w)}$

where χ_E is the character of $F^\times$ (or W_F) associated with E/F if it is a quadratic field extension, and

$$a_\chi(w) = \begin{cases} 0 & (\chi(w) = 1), \\ 1 & (\chi(w) = -1) \end{cases}$$

and w_1 is an element of $\mathrm{O}(2t)$ so that $\det(w_1) = -1$.

Let G be either

$$\begin{cases} \mathrm{Mp}(W) \times \mathrm{GL}_1 & \text{if } E = F \text{ and } (-\epsilon) = -1, \\ G(W) \times \mathrm{Res}_{E/F}\mathrm{GL}_1 & \text{if } [E:F] = 2, \text{ or} \\ G(W) \times \mathrm{GL}_1 & \text{otherwise.} \end{cases}$$

Then we denote by std the representation $\mathrm{std} \boxtimes \mathrm{std}$ of ${}^L G$. Now we consider the γ-factor $\gamma(s,\pi \boxtimes \omega,\mathrm{std},\psi)$ for G. We call it the standard γ-factor, and we will denote it by $\gamma(s,\pi\boxtimes\omega,\psi)$ abbreviating "std".

1.4.3. *Doubling γ-factor.* The standard γ-factor has an analytic definition by using the doubling method of Piatetski-Shapiro and Rallis [GPSR87, PSR86]. An advantage of the analytic definition is that one can relate directly the representation theory with the local factors. Actually, the standard local factors have an application to the non-vanishing problem for the theta correspondence. (See [HKS96], [GS12], [GI14, Proposition 11.2]. See also [Yam14] for a global application.)

Let $W^\square$ be an $(-\epsilon)$-Hermitian space $W \times W$ equipped with a $(-\epsilon)$-Hermitian form

$$\langle (x_1,y_1),(x_2,y_2)\rangle^\square = \langle x_1,x_2\rangle - \langle y_1,y_2\rangle$$

for $x_1,x_2,y_1,y_2 \in W$. We denote by $W^\triangle$ the diagonal subset of $W^\square$, and by $P(W^\triangle)$ the parabolic subgroup preserving $W^\triangle$. For a character ω of $F^\times$, we denote by $I(s,\omega)$ the representation of $G(W^\square)$ induced by the character $\omega_s \circ \Delta$ of $P(W^\triangle)$, which is given by $\omega_s(\Delta(p)) =$

$\omega(N(p|_{W^\triangle})^{-1}|N(p|_{W^\triangle})|^{-s}$ for $p \in P(W^\triangle)$. (Here, we denote by $N(x)$ the reduced norm of $x \in \mathrm{End}(W^\triangle)$.) Consider an intertwining operator

$$M(s,\chi) : I(s,\omega) \to I(-s,\theta(\omega)^{-1}).$$

Besides, for an irreducible representation π of $G(W)$, we define the doubling zeta integral

$$Z(f_s,\xi) = \int_{G(W)} f_s(g,1)\xi(g)\,dg$$

for $f_s \in I(s,\omega)$ and a matrix coefficient of π. Then, they satisfy the following functional equation:

$$Z(M(s,\omega)f_s,\xi) = \Gamma(s,\pi,\chi)Z(f_s,\xi).$$

Then, by using appropriate normalization factors $c(s,\omega,A,\psi)$ and $R(s,\omega,A,\psi)$, we have an analytic definition of the standard γ-factor:

$$\gamma(s,\pi\times\omega,\psi) := \pi(-1)c(s,\omega,A,\psi)^{-1}\Gamma(s,\pi,\omega)R(s,\omega,A,\psi).$$

More precisely, there are expected properties of the local γ-factor, which characterizes itself, and we can prove that the function $\gamma^W(s,-,\psi)$ satisfies them. (This is proved by Lapid-Rallis [LR05] for non-quaternionic classical groups over local fields of characteristic 0, by Gan [Gan12] for metaplectic groups over local fields of characteristic 0, by Yamana [Yam11] and the author [Kak20b] for quaternionic unitary groups over local fields of characteristic 0, and the author [Kak20a] for classical groups over function fields.) Note that we can retrieve the local standard L- and ϵ-factors from the standard γ-factors. Thus, we also have analytic definitions of L- and ϵ-factors. We also note that the local standard L-factor satisfies the "g.c.d property" which characterizes itself in terms of the doubling zeta integrals directly [Yam14].

1.5. **Formal degree conjecture.** Now we state a motivating problem of our study. Let G be a connected reductive group over a local field F, and let A be the maximal F-split torus of the center of G. An irreducible representation π of $G(F)$ is said to be square integrable if π is a unitary representation and the integral

$$\int_{G/A} |(\pi(g)x,y)|^2\,dg$$

converges for all $x,y \in \pi$. Here, $(\ ,\ )$ is a $G(F)$-invariant non-zero Hermitian pairing of π. For a square integrable irreducible representation π of $G(F)$, we define the formal degree $\deg\pi$ of π as the positive real number satisfying

$$\int_{G/A} (\pi(g)x_1,x_2)\cdot\overline{(\pi(g)y_1,y_2)}\,dg = \frac{1}{\deg\pi}(x_1,y_1)\overline{(x_2,y_2)} \tag{1.2}$$

for $x_1,x_2,y_1,y_2 \in \pi$. Here, dg is a canonical Haar measure defined by using the motive of G (see §6). Note that $\deg\pi$ does not depend on the $G(F)$-invariant non-zero Hermitian pairing but the group $G(F)$ and the non-trivial additive character ψ. For example, if G is anisotropic, then it is known that

$$\deg\pi = |G(F)|^{-1}\cdot\dim\pi.$$

Hiraga-Ichino-Ikeda [HII08] conjectured that the formal degree is described explicitly in terms of the Langlands parameter, and it was refined by Gross-Reeder [GR10]:

Conjecture 1.1. *(*[HII08, Conjecture 1.4] *and* [GR10, Conjecture 6.1]*). Let π be a square integrable irreducible representation of $G(F)$. Then,*

$$\deg\pi = \zeta_\pi\frac{\dim\eta}{\#C'_\phi(\widehat{G})}\gamma(0,\mathrm{Ad},\pi,\psi)$$

where (ϕ, η) is the Langlands parameter of π, and

$$\zeta_\pi = \frac{\epsilon(\frac{1}{2}, \mathrm{Ad}, \pi, \psi)}{\epsilon(\frac{1}{2}, \mathrm{Ad}, \mathrm{St}, \psi)} \in \{\pm 1\}.$$

Here we denote by $C'_\phi(\widehat{G})$ the finite group $C_\phi(\widehat{G}) \cap \widehat{G/A}$, and by St *the Steinberg representation of $G(F)$.*

Note that the ambiguity in §1.3.3 of the irreducible representation η is due to the twisting by a character of $Z(\widehat{G}_{\mathrm{sc}})$ and does not affect the dimension $\dim \eta$ of η. Since the Weil group is close to the Galois group, the adjoint γ-factor can be regarded as a number theoretic invariant. Then, Conjecture 1.1 asserts that a number theoretic invariant appears in the analytic equation (1.2). Note that this type of phenomenon is interesting in the representation theory of reductive groups over local fields (for examples, an analytic definition of the standard γ-factor for classical groups (see §1.4.3), and an expression of Plancherel measures in terms of γ-factors (see [Sha90])). The formal degree conjecture has been already proved for inner forms of GL_n, inner forms of SL_n [HII08], SO_{2n+1}, Mp_{2n} [ILM17], and unitary groups [BP18]. Moreover, Gan-Ichino [GI14] observed the behavior of the formal degrees under the Langlands functorial lifting coming from the local theta correspondences for non-quaternionic dual pairs and proved the formal degree conjecture for Sp_4, GSp_4 and U_3. (We will explain the Gan-Ichino's work more precisely in §1.7.) And in this paper, we will prove it for the non-split inner forms of Sp_4 and GSp_4 (§20).

1.6. **Local theta correspondence.** Then, we discuss an approach to the formal degree conjecture since it seems to be difficult to prove it directly. A key tool is the local theta correspondence, which is a correspondence between irreducible representations of a certain classical group and those of another classical group. In this subsection, we explain the definition, a property related to the see-saw diagram, and an expression in terms of Langlands parameters.

1.6.1. *Definition.* Let V be an m-dimensional right ϵ-Hermitian space over E equipped with an ϵ-Hermitian form $(\ ,\ )$, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space over E equipped with a $(-\epsilon)$-Hermitian form $\langle\ ,\ \rangle$. We put

$$l = l_{V,W} = \begin{cases} n - m + \epsilon & (E = F), \\ n - m & ([E : F] = 2), \\ 2n - 2m - \epsilon & ([E : F] = 4). \end{cases}$$

Then, $(G(V), G(W))$ forms a reductive dual pair, that is, $G(V)$ and $G(W)$ are reductive groups over F such that there is an embedding $j : G(V) \times G(W) \to \mathrm{Sp}(\mathbb{W})$ for some symplectic space $\mathbb{W}$ and they satisfy $G(V) = Z_{\mathrm{Sp}(\mathbb{W})}(G(W))$ and $G(W) = Z_{\mathrm{Sp}(\mathbb{W})}(G(V))$. Fix a non-trivial additive character ψ of F, and fix a pair $\kappa = (\chi_V, \chi_W)$ of characters as in §1.2. Then, there is a diagram

$$\begin{array}{ccc} G(V) \times G(W) & \xrightarrow{\quad j \quad} & \mathrm{Sp}(\mathbb{W}) \\ \uparrow & & \uparrow \\ \widetilde{G}(V) \times \widetilde{G}(W) & \xrightarrow{\quad \widetilde{j}_{\kappa,\psi} \quad} & \mathrm{Mp}(\mathbb{W}) \end{array}$$

where

$$\widetilde{G}(V) = \begin{cases} \mathrm{Mp}(V) & E = F,\ \epsilon = -1, \text{ and } \dim W \text{ is odd}, \\ G(V) & \text{otherwise} \end{cases}$$

and

$$\widetilde{G}(W) = \begin{cases} \mathrm{Mp}(W) & E = F,\ -\epsilon = -1, \text{ and } \dim V \text{ is odd}, \\ G(W) & \text{otherwise,} \end{cases}$$

the vertical maps are natural projections, and the upper horizontal map is the natural embedding. Note that $\widetilde{j}_{\kappa,\psi}$ depends on the choice of κ, ψ. Let ω_ψ be the Weil representation of $\mathrm{Mp}(\mathbb{W})$. Then, for an irreducible representation of $\widetilde{G}(W)$, we define

$$\Theta_{\kappa,\psi}(\pi, V) := (\widetilde{j}^*_{\kappa,\psi}\omega_\psi \otimes \pi^\vee)_{\widetilde{G}(W)}$$

the co-invariant space of $\widetilde{G}(W)$, and we define

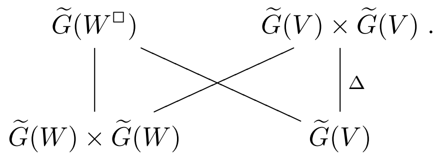
$$\theta_{\kappa,\psi}(\pi, V) = \begin{cases} 0 & (\Theta_{\kappa,\psi}(\pi, V) = 0), \\ \text{the maximal semisimple quotient of } \Theta_{\kappa,\psi}(\pi, V) & (\Theta_{\kappa,\psi}(\pi, V) \neq 0). \end{cases}$$

Then, the Howe duality, a fundamental theorem in the theory of theta correspondence guarantees that $\theta_{\kappa,\psi}(\pi, V)$ is irreducible if it is non-zero. Moreover, it also asserts that if π_1 and π_2 are different irreducible representations of $\widetilde{G}(W)$, and both $\theta_{\kappa,\psi}(\pi_1, V)$ and $\theta_{\kappa,\psi}(\pi_2, V)$ are non-zero, then $\theta_{\kappa,\psi}(\pi_1, V) \not\cong \theta_{\kappa,\psi}(\pi_2, V)$. The Howe duality was proved in [Wal90] in the case where the residual characteristic is not equal to 2, and was proved in [GT16, GS17] in the remaining cases. We call $\theta_{\kappa,\psi}(\pi, V)$ the theta correspondence of π to $\widetilde{G}(V)$.

1.6.2. *See-saw diagram.* Then, we explain an important property which is called the doubling see-saw. Let $W^\square$ be the doubled space of W. Then, the natural action of $G(W) \times G(W)$ on $W^\square = W \times W$ induces the inclusion $G(W) \times G(W) \hookrightarrow G(W^\square)$ and the natural map $\widetilde{G}(W) \times \widetilde{G}(W) \to G(W^\square)$. On the other hand, we consider the diagonal map $\Delta : \widetilde{G}(V) \to \widetilde{G}(V) \times \widetilde{G}(V)$. Then, for irreducible representations π_1, π_2 of $\widetilde{G}(W)$ and for an irreducible representation σ of $\widetilde{G}(V)$, we have

$$\begin{aligned} &\mathrm{Hom}_{\widetilde{G}(V)}(\Delta^*(\Theta_{\kappa,\psi}(\pi_1, V) \otimes \Theta_{\kappa,\psi}(\pi_2, V)), \sigma) \\ &= \mathrm{Hom}_{\widetilde{G}(W)\times\widetilde{G}(W)}(\Theta_{\kappa^\square,\psi}(\sigma, W^\square), \pi_1 \boxtimes \pi_2). \end{aligned}$$

Here, we denote by $\kappa^\square$ a pair $(\chi_{W^\square}, \chi_V)$ of characters as in §1.2. Note here that we denote by $\omega^\square_{\kappa^\square,\psi}$ the Weil representation associated with the reductive dual pair $(G(W^\square), G(V))$. This equation is called the see-saw identity. The setting and the equation are exhibited by the following diagram: the diagonal lines indicate that we consider the theta correspondence.



The doubling see-saw allows us using the doubling method to analyze the theta correspondence as follows: the image of the $\widetilde{G}(W^\square)$-invariant map

$$\omega^\square_{\kappa^\square,\psi} \to I(-\frac{l}{2}, \chi_V) : \phi \mapsto F_\phi$$

given by $F_\phi(g) = [\omega_{\kappa,\psi}(g)\phi](0)$ is isomorphic to $\Theta_{\kappa,\psi}(1_V, W^\square)$. Moreover, there is a $\widetilde{G}(W) \times \widetilde{G}(W)$-invariant bijection

$$\delta : \omega_{\kappa,\psi} \otimes \overline{\omega_{\kappa,\psi}} \to \omega^\square_{\kappa^\square,\psi}$$

so that $(\omega_{\kappa,\psi}(g)\phi_1, \phi_2) = F_{\delta(\phi_1\otimes\phi_2)}((g, 1))$. If π is square integrable, the doubling zeta integral $Z(F_{\delta(\phi_1\otimes\phi_2)}, \xi_\pi)$ converges absolutely, and the map

$$\overline{\pi} \otimes \pi \otimes \omega_{\kappa,\psi} \otimes \overline{\omega_{\kappa,\psi}} \to \mathbb{C} : (x, y, \phi_1, \phi_2) \mapsto Z(F_{\delta(\phi_1\otimes\phi_2)}, \xi_\pi)$$

factors through the canonical projection

$$\overline{\pi} \otimes \pi \otimes \omega_{\kappa,\psi} \otimes \overline{\omega_{\kappa,\psi}} \to \sigma \otimes \overline{\sigma}$$

where $\sigma = \theta_{\kappa,\psi}(\pi, V)$. Here ξ_π is the coefficient of π given by $\xi_\pi(g) = (\pi(g)x, y)$. Thus, we obtain a $\widetilde{G}(V)$-invariant non-zero Hermitian pairing on σ from the $\widetilde{G}(W)$-invariant non-zero Hermitian pairing on π.

1.6.3. *Langlands functoriality.* For the non-quaternionic dual pair, there is an expression of the theta correspondence in terms of Langlands parameters. Suppose $l = 1$ for simplicity. Moreover, we exclude the $(\mathrm{O}, \mathrm{Mp})$ pair because the local Langlands correspondence for Mp is defined via the local theta correspondence (see [GS12]).

Let π be an irreducible representation of $G(W)$, let $\sigma = \theta_{\kappa,\psi}(\pi, V)$ be its theta correspondence, and let ϕ_π and ϕ_σ be the L-parameters associated with π and σ respectively. Assume that σ is non-zero. Then, by Adams' conjecture, we can guess that

$$\phi_\pi = (\phi_\sigma \otimes \chi_V^{-1}\chi_W) \oplus \chi_W \tag{1.3}$$

(see [GI14, §15]). This is proved for unitary cases [GI16]. Now we explain the Prasad conjecture [Pra93, Pra00], which describes the behavior of the characters of the component group under the local theta correspondence. Since $G(W)$ is a non-quaternionic classical group, the character η_π of the component group factors through the projection

$$\widetilde{S}_\phi(\widehat{G(W)}) \to C_\phi(\widehat{G(W)}).$$

On the other hand, the map ${}^L G(V) \to {}^L G(W)$ of Adams' conjecture induces an embedding

$$C_{\phi_\sigma}(G(V)) \to C_{\phi_\pi}(G(W)).$$

Then, the Prasad conjecture asserts that η_π is the composition of η_σ and above embedding. Note that the conjecture is proved by Gan-Ichino [GI16] for unitary dual pairs, and by Atobe-Gan [AG17] for symplectic-even orthogonal dual pairs.

Note that the Prasad conjecture is not formulated yet for the quaternionic dual pairs since the character η does not factor through the projection $\widetilde{S}_\phi(\widehat{G(W)}) \to C_\phi(\widehat{G(W)})$. Thus, the behavior of $\dim \eta$ under the theta correspondence is more complicated than that for non-quaternionic dual pairs.

1.7. **Gan-Ichino's result.** In this subsection, we explain an approach of Gan-Ichino [GI14] to the formal degree conjecture. Note that our result of this paper extends their results, and it will be explained in §1.8.

1.7.1. *Observations.* Suppose that the residue characteristic is not 2 and $[E : F] \leq 2$ (i.e. non-quaternionic case). Moreover, we exclude the $(\mathrm{O}, \mathrm{Mp})$ pair, and suppose that $l = 1$ so that we can refer to the result of §1.6.3. Let π be a square integrable irreducible representation of $G(W)$, and let σ be an irreducible representation of $G(V)$ associated with π via the theta correspondence. By the equation (1.3), we have

$$\frac{\# C_{\phi_\pi}(\widehat{G(W)})}{\# C_{\phi_\sigma}(\widehat{G(V)})} = \begin{cases} 2 & ([E:F] = 2), \\ 1 & (E = F,\ \epsilon = 1), \\ 4 & (E = F,\ \epsilon = -1) \end{cases}$$

(see [GI14, p.581]), and

$$\frac{\gamma(s, \mathrm{Ad}, \pi \times \chi_V, \psi)}{\gamma(s, \mathrm{Ad}, \sigma \times \chi_W, \psi)} = \gamma^V(s, \sigma \times \chi_W, \psi).$$

Moreover, since both η_π and η_σ are one dimensional, we have

$$\frac{\dim \eta_\pi}{\dim \eta_\sigma} = 1.$$

Thus, they guessed the ratio $\deg(\pi)/\deg(\sigma)$, and prove that

Theorem 1.2. *(*[GI14, Theorem 15.1]*) We have*

$$\frac{\deg \pi}{\deg \sigma} = C(V,W) \cdot \gamma(0, \sigma \times \chi_W, \psi) \tag{1.4}$$

where

$$C(V,W) = \begin{cases} 2^{-1} \cdot \chi_W(\lambda)^{-m} \epsilon(\frac{1}{2}, \chi_E, \psi)^{-1} \gamma(0, \chi_E, \psi)^{-1} & ([E:F]=2), \\ \epsilon(\frac{1}{2}, \chi_V, \psi)^{-1} & (E=F, \epsilon=1), \\ 2^{-2} \cdot \chi_W(-1)^{m/2} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi)^{-1} & (E=F, \epsilon=-1), \end{cases}$$

and $\lambda \in E^\times$ is a fixed element so that $\lambda^* = -\lambda$.

1.7.2. *Outline of the proof.* We sketch the proof of Theorem 1.2 to explain what are the obstacles in the case of quaternionic dual pairs. First, they consider a local analogue of the Siegel-Weil formula as follows: we define the map

$$\mathcal{I} : \omega_\psi^\square \otimes \overline{\omega_\psi^\square} \otimes \overline{\chi_W} \otimes \chi_W \to \mathbb{C}$$

by

$$\mathcal{I}(\phi_1, \phi_2) = \int_{G(V)} (\omega_\psi^\square(h)\phi_1, \phi_2) \cdot \overline{\chi_W(\det h)}\, dh$$

for $\phi_1, \phi_2 \in \omega_\psi^\square$. Besides, we define the map

$$\mathcal{E} : \omega_\psi^\square \otimes \overline{\omega_\psi^\square} \otimes \overline{\chi_W} \otimes \chi_W \to \mathbb{C}$$

by

$$\mathcal{E}(\phi_1, \phi_2) = \int_{G(W)} F_{\phi_1}(g,1) \cdot \overline{F^\dagger_{\phi_2}(g,1)}\, dg$$

for $\phi_1, \phi_2 \in \omega_\psi^\square$. Here $F^\dagger_{\phi_2}$ is a certain section of $I(\frac{1}{2}, \chi_V)$ so that $M(\frac{1}{2}, \chi_V)F^\dagger_{\phi_2} = F_{\phi_2}$. Then, we can prove that there is a constant C_{SW} such that

$$\mathcal{I} = C_{\mathrm{SW}} \cdot \mathcal{E}$$

(see [GI14, §17]). Second, by a local analogue of the proof of the Rallis inner product formula, we can prove that there is a constant C' such that

$$\frac{\deg \pi}{\deg \sigma} = C_{\mathrm{SW}} \cdot C' \cdot \omega_\sigma(-1) \gamma(0, \sigma \times \chi_W, \psi)$$

for all square integrable irreducible representations π with $\sigma = \theta_{\kappa,\psi}(\pi, V) \neq 0$. Thus, we conclude that

$$\frac{\deg \pi}{\deg \sigma} \cdot \omega_\sigma(-1) \cdot \gamma(0, \sigma \times \chi_W, \psi)^{-1}$$

is a constant independent of the square integrable irreducible representation π so that $\theta_{\kappa,\psi}(\pi, V) \neq 0$. Third, they discuss an induction argument. Suppose that V has an r-dimensional isotropic subspace X, and suppose also that W has an r-dimensional isotropic subspace Y. Then, we have the decompositions

$$V = X + V' + X', \text{ and } W = Y + W' + Y'$$

where X', Y' are r-dimensional isotropic subspaces, and V' and W' are non-degenerate subspaces such that $X + X'$ is orthogonal to V' and $Y + Y'$ is orthogonal to W'. Consider a parabolic subgroup Q of $G(V)$ which preserves X, and consider irreducible supercuspidal representations

σ' of $G(V')$ and τ of $\mathrm{GL}(X)$ so that $\mathrm{Ind}_Q^{G(V)} \sigma' \boxtimes \tau|\det|^{s_0}$ has a square integrable constituent σ' for some $s_0 > 0$. Then, by using the result of Heiermann [Hei04], we can relate $\deg \sigma$ with $\deg \sigma'$. Then, we can conclude that $C(V,W) = C(V',W')$. Finally, they proved Theorem 1.2 for minimal cases of the induction argument: this step was done case by case ([GI14, §20.8]). We should remark that their proof makes full use of the properties of L-packets and local theta correspondence.

1.8. **Summary of this paper.** Now we summarize our results. In this paper, we describe the behavior of the formal degree under the local theta correspondence of almost equal rank for quaternionic dual pairs over a local field of characteristic 0. As an application, we prove the formal degree conjecture of [HII08, GR10] for the non-split inner forms of Sp_4 and GSp_4. These results extend Gan-Ichino's work [GI14] to quaternionic dual pairs. However, there are some differences as follows.

- The dimension $\dim \eta$ of an irreducible representation η of the component group may not be 1. Moreover, the behavior of $\dim \eta$ under the theta correspondence has not been formulated.
- Case-by-case discussions of [GI14] cannot be applicable to our cases (see §1.7). More precisely, it seems to be difficult to find enough examples of quaternionic dual pairs (H,G) and square integrable irreducible representations π of G such that we can know the formal degree $\deg \pi$ of π, the formal degree $\deg \sigma$ of the theta correspondence σ of π, and the standard local γ-factor $\gamma(s, \sigma \boxtimes \chi, \psi)$ with a quadratic character χ at the same time even in low-rank cases.

To avoid the second difficulty, we analyze the local analogue of the Siegel-Weil formula, and we obtain a relation between the constant in the local Siegel-Weil formula and the local zeta value for enough cases. Here, the constant in the local Siegel-Weil formula appears in an expression of the ratio of the formal degrees of irreducible representations corresponding to each other by the local theta correspondence. Hence, to establish the description of the behavior of the formal degrees under local theta correspondence, we compute some local zeta values. On the other hand, a general formula of the local zeta value is obtained by reversing the above discussion. For a quaternionic dual pair $(G(V), G(W))$ of almost equal rank, we denote by $\alpha_1(W)$ the local zeta value, by $\alpha_2(V,W)$ the constant in the local Siegel-Weil formula (it was denoted by C_{SW} in §1.7), and by $\alpha_3(V,W)$ the constant appearing in the behavior of the formal degree under the theta correspondence. Then the results in this paper are summarized as follows:

1.8.1. *The constant* $\alpha_1(W)$. Let $\epsilon = \pm 1$, let W be an n-dimensional right $(-\epsilon)$-Hermitian space equipped with the $(-\epsilon)$-Hermitian form $\langle\ ,\ \rangle$ (see §3), and let $G(W)$ be the unitary group of W. We denote by $W^\square$ the doubled space which is the vector space $W \oplus W$ equipped with an $(-\epsilon)$-Hermitian form $\langle\ ,\ \rangle^\square = \langle\ ,\rangle \oplus (-\langle\ ,\ \rangle)$, by $W^\triangle$ the diagonal subset of $W^\square$, and by $P(W^\triangle)$ the parabolic subgroup preserving $W^\triangle$. For a character ω of $F^\times$, we denote by $I(s,\omega)$ the representation of $G(W^\square)$ induced by the character $\omega_s \circ \Delta$ of $P(W^\triangle)$, which is given by $\omega_s(\Delta(p)) = \omega(N(p|_{W^\triangle})^{-1}|N(p|_{W^\triangle})|^{-s}$. (Here, we denote by $N(x)$ the reduced norm of $x \in \mathrm{End}(W^\triangle)$.) Then we define

$$\alpha_1(W) = Z^W(f_\rho^\circ, \xi^\circ).$$

Here

- $Z^W(\ ,\ )$ is the doubling zeta integral (see §7.1),
- f_s° is the $K(\underline{e}'^\square)$-invariant section of $I(s,1)$ so that $f_s^\circ(1) = 1$ where $K(\underline{e}'^\square)$ is a special maximal compact subgroup of the unitary group $G(W^\square)$ of $W^\square$, which depends on the choice of a basis $\underline{e}$ for W (see §7.1),
- ξ° is the coefficient of the trivial representation of $G(W)$ so that $\xi^\circ(1) = 1$, and

- $\rho = n - \frac{\epsilon}{2}$.

This invariant is technically important because it appears in a certain local functional equation, which relates the zeta integral with the intertwining operator (see Lemma 7.8). In this paper, we first compute $\alpha_1(W)$ directly for some W (Proposition 7.6), and finally we complete the formula for the remaining cases as a corollary of Theorem 1.4 (Proposition 19.4). We also note here that by determining $\alpha_1(W)$, we can compute the constant by which the scalar multiplication appearing in a formula of the zeta integral for a certain section (see §21) is given, which has not been computed yet.

1.8.2. *The constant* $\alpha_2(V, W)$. Let V be an m-dimensional ϵ-Hermitian space, let $(\ ,\ )$ be an ϵ-Hermitian form on V, let $\omega_\psi^\square$ be the Weil representation of $G(V) \times G(W^\square)$. It is realized on the Schwartz space $\mathcal{S}(V \otimes W^\triangledown)$ where $W^\triangledown$ is the anti-diagonal subset of $W^\square$. We suppose that $2n - 2m = 1 + \epsilon$. Then we define the local theta integral

$$\mathcal{I}(\phi, \phi') = \int_{G(V)} (\omega_\psi^\square(h, 1)\phi, \phi')\, dh$$

for $\phi, \phi' \in \mathcal{S}(V \otimes W^\triangledown)$. Here, we denote by $(\ ,\ )$ the normalized L^2-inner product on $\mathcal{S}(V \otimes W^\triangledown)$ (as in Proposition 8.3). Moreover, we define another map $\mathcal{E} : \mathcal{S}(V \otimes W^\triangledown)^2 \to \mathbb{C}$ as follows: for $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, we define $F_\phi \in I(-\frac{1}{2}, \chi_V)$ by $F_\phi(g) = [\omega_\psi^\square(g)\phi](0)$, and we choose $F_\phi^\dagger \in I(\frac{1}{2}, \chi_V)$ so that $M(\frac{1}{2}, \chi_V)F^\dagger = F_\phi$ where $M(s, \chi_V)$ is an intertwining operator (see §7.1). Then the map $\mathcal{E}$ is defined by

$$\mathcal{E}(\phi, \phi') = \int_{G(W)} F_\phi^\dagger(\iota(g, 1))\overline{F_{\phi'}(\iota(g, 1))}\, dg$$

for $\phi, \phi' \in \mathcal{S}(V \otimes W^\triangledown)$. Here, $\iota : G(W)^2 \to G(W^\square)$ is given by the natural action of $G(W)^2$ on $W^\square$. Then the constant $\alpha_2(V, W)$ is defined as a non-zero constant C_{SW} so that $\mathcal{I} = C_{\mathrm{SW}} \cdot \mathcal{E}$ (see Lemma 10.2). Then we have

Theorem 1.3. *Choose the basis* $\underline{e}$ *for* V *as in §7.1. Then,*

$$\alpha_2(V, W) = |N(R)|^\rho \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(1 - 2i)}{\zeta_F(2i)} \times \begin{cases} 2(-1)^n \gamma(1 - n, \chi_V, \psi)^{-1} \epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ 1 & (-\epsilon = -1). \end{cases}$$

Here, $R = ((e_i, e_j))_{i,j} \in \mathrm{GL}_m(D)$.

To prove this theorem, we will first prove it in the case where either V or W is non-zero anisotropic (§§12-13). Note that in this case we can express $\alpha_2(V, W)$ using $\alpha_1(W)$, and thus Theorem 1.3 follows from the formula of $\alpha_1(W)$. For the remaining cases, it will be proved as a corollary of Theorem 1.4 (§19).

1.8.3. *The constant* $\alpha_3(V, W)$. Let π be a square integrable irreducible representation of $G(W)$. We choose canonical Haar measures dh and dg on $G(V)$ and $G(W)$ respectively (see §6.1). Then, as we explained in §1.7 (more precisely, as in [GI14, p.597]), we can prove that there is a constant $\alpha_3(V, W)$ such that

$$\frac{\deg \pi}{\deg \sigma} = \alpha_3(V, W)\omega_\pi(-1)\gamma^V(0, \sigma \times \chi_W, \psi)$$

for all square integrable irreducible representation π of $G(W)$ and the irreducible square integrable representation σ of $G(V)$ associated with π by the local theta correspondence whenever $\sigma \neq 0$. Then, our main theorem is stated as follows:

Theorem 1.4. *We have*

$$\alpha_3(V,W) = \begin{cases} (-1)^n \chi_V(-1)\epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ \frac{1}{2}\chi_W(-1)\epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases}$$

When either V or W is anisotropic, we prove this theorem by expressing $\alpha_3(V,W)$ using $\alpha_2(V,W)$ as in §1.7 (more precisely, see Proposition 15.1). In general, we use an induction on $\dim W$ to compute $\alpha_3(V,W)$ (§18). Note that since the relation of Proposition 15.1 between $\alpha_2(V,W)$ and $\alpha_3(V,W)$ still holds in general, we have the general formula of $\alpha_2(V,W)$.

As an application of Theorem 1.4, we prove the formal degree conjecture for the non-split inner forms of GSp_4 and Sp_4. Note that, for these groups, the Langlands correspondence is established by Gan-Tantono [GT14] and Choiy [Cho17] respectively.

Theorem 1.5. *Let F be a local field of characteristic 0. Then the formal degree conjecture holds for the non-split inner forms of $\mathrm{Sp}_4(F)$ and $\mathrm{GSp}_4(F)$.*

1.8.4. *Structure of this paper.* Now, we explain the structure of this paper. In §§2-3, we set up the notations for fields, quaternion algebras, and $\pm\epsilon$-Hermitian spaces. In §4, we define some symbols which are referred to when we take bases for $\pm\epsilon$-Hermitian spaces. In §5, we discuss the Bruhat-Tits theory for quaternionic unitary groups, and we give a formula of the index of an Iwahori subgroup in a certain special compact subgroup (Proposition 5.6). In §6, we explain the normalization of Haar measures on reductive groups and certain nilpotent groups, and we give some volume formulas. In §7, we explain the doubling method, and we recall the definition of the doubling γ-factor. Moreover, we compute the constant $\alpha_1(W)$ for some cases. In §8, we set up and explain the doubling method and the Weil representations. In §9, we set up the theta correspondence. In §§10-11, 19-20, we state our main results. In §§12-18, we prove these results. More precisely, §§12-13 are devoted to the computation of $\alpha_2(V,W)$ when either V or W is anisotropic, §14 is a preliminary for §15 which associates $\alpha_2(V,W)$ with $\alpha_3(V,W)$, and §§16-17 are preliminaries for §18 in which we verify the commutativity of $\alpha_3(V,W)$ with the parabolic inductions. Finally, in the Appendix §21, we give a formula of doubling zeta integrals of certain sections as an application of the formula of $\alpha_1(W)$. Note that this corrects the errors in [Kak20b, Proposition 8.3].

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2. Quaternion algebras over local fields

Let F be a non-Archimedean local field of characteristic 0, let D be a quaternion algebra over F. For a while, we assume that D is division. We denote by

- $\mathrm{ord}_F : F^\times \to \mathbb{Z}$ the normalized additive valuation,
- $|\ |_F$ the normalized absolute value,
- $\mathcal{O}_F$ the valuation ring of F,
- ϖ_F a uniformizer of F,
- q the cardinality of $\mathcal{O}_F/\varpi_F$,
- $* : D \to D$ the canonical involution of D,
- $N_D : D \to F$ the reduced norm,
- $T_D : D \to F$ the reduced trace,
- $\mathrm{ord}_D = \mathrm{ord}_F \circ N_D$ the normalized additive valuation of D,
- $|\ |_D = |\ |_F \circ N_D$ the absolute value,
- $\mathcal{O}_D$ the valuation ring of D,

- α and ϖ_D two elements of D satisfying $T_D(\alpha) = T_D(\varpi_D) = 0$, $\mathrm{ord}_D\,\alpha = 0$, $\mathrm{ord}_D\,\varpi_D = 1$, and $\alpha\varpi_D + \varpi_D\alpha = 0$,
- L the subfield $F(\alpha) \subset D$, and
- $\mathcal{O}_L$ the valuation ring of L.

Moreover, we denote the set

$$\{x \in D \mid x + x^* = 0\}$$

by D_0, and $(a\mathcal{O}_D)_0 = a\mathcal{O}_D \cap D_0$ for $a \in D$. Then, one can show that $[(\mathcal{O}_D)_0 : (\varpi_D\mathcal{O}_D)_0] = q$. Fix an additive non-trivial character $\psi : F \to \mathbb{C}^\times$ whose order is 0. We note some basic properties:

Lemma 2.1. *(1) We denote by $\mathcal{O}_D^*$ the dual lattice of $\mathcal{O}_D$ with respect to the pairing*

$$D \times D \to \mathbb{C}^\times : (x, y) \mapsto \psi(T_D(xy)). \tag{2.1}$$

Then, we have $\mathcal{O}_D^* = \varpi_D^{-1}\mathcal{O}_D$.

(2) We denote by $(\mathcal{O}_D)_0^*$ the dual lattice of $(\mathcal{O}_D)_0$ with respect to the pairing

$$D_0 \times D_0 \to \mathbb{C}^\times : (x, y) \mapsto \psi(T_D(xy)). \tag{2.2}$$

Then, we have $(\mathcal{O}_D)_0^* = \frac{1}{2}\alpha\mathcal{O}_F + \varpi_D^{-1}\mathcal{O}_L$.

In particular,

Corollary 2.2. *(1) The volume $|\mathcal{O}_D|$ of $\mathcal{O}_D$ with the self-dual Haar measure with respect to the pairing (2.1) is q^{-1}.*

(2) The volume $|(\mathcal{O}_D)_0|$ of $(\mathcal{O}_D)_0$ with the self-dual Haar measure with respect to the pairing (2.2) is $|2|^{\frac{1}{2}}q^{-1}$.

3. ϵ-Hermitian spaces and their unitary groups

Let $\epsilon \in \{\pm 1\}$. Now, we consider the following:

- a pair $(W, \langle\ ,\ \rangle)$ where W is a left free D-module of rank n, and $\langle\ ,\ \rangle$ is a map $W \times W \to D$ satisfying

$$\langle ax, by\rangle = a\langle x, y\rangle b^*,\ \ \langle y, x\rangle = -\epsilon\langle x, y\rangle$$

  for $x, y \in W$ and $a, b \in D$,
- a pair $(V, (\ ,\ ))$ where V is a right free D-module of rank m, and $(\ ,\ )$ is a map $V \times V \to D$ satisfying

$$(v_1 a, v_2 b) = a^*(x, y)b,\ \ (y, x) = \epsilon(x, y)^*$$

  for $x, y \in V$ and $a, b \in D$.

We call them an n-dimensional right ϵ-Hermitian space and an m-dimensional left $(-\epsilon)$-Hermitian space respectively if they are non-degenerate. We denote by $G(W)$ by the group of the left D-linear automorphisms g of W such that

$$\langle x \cdot g, y \cdot g\rangle = \langle x, y\rangle$$

for all $x, y \in W$. We also denote by $G(V)$ by the group of the right D-linear automorphisms g of V as a right D-module such that

$$(g \cdot x, g \cdot y) = (x, y)$$

for all $x, y \in V$.

Remark 3.1. *When $-\epsilon = 1$, the unitary group $G(W)$ is an inner form of a symplectic group as an algebraic group. On the other hand, when $-\epsilon = -1$, one can regard the group of F-valued points of $G(W)$ in two ways: it is a group of the F-valued points of an inner form of a special orthogonal group, and it is also a group of the F-valued points of an inner form of an orthogonal group. This is caused by the fact that the Zariski connected component of $G(W)$ which does not contain 1 has no F-valued point.*

Put $\mathbb{W} = V \otimes_F W$ and define $\langle\langle\ ,\ \rangle\rangle$ by

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle = T((x_1, y_1)\langle x_2, y_2\rangle^*)$$

for $x_1, y_1 \in V$ and $x_2, y_2 \in W$. Then, $\langle\langle\ ,\ \rangle\rangle$ is a symplectic form on $\mathbb{W}$, and the $(G(W), G(V))$ is a reductive dual pair in $\mathrm{Sp}(\mathbb{W})$. We define

$$l = l_{V,W} = \begin{cases} 2n - 2m - 1 & (\epsilon = 1), \\ 2n - 2m + 1 & (\epsilon = -1). \end{cases}$$

We define the characters χ_V and χ_W of $F^\times$ by

$$\chi_V(a) = \begin{cases} 1 & (\epsilon = 1), \\ (a, \mathfrak{d}(V))_F & (\epsilon = -1) \end{cases} \text{ and } \chi_W(a) = \begin{cases} (a, \mathfrak{d}(V))_F & (\epsilon = -1), \\ 1 & (\epsilon = -1). \end{cases}$$

4. Bases for W and V

In this section, we discuss bases for W, which we will consider in this paper. The discussion for V goes the same line with that of W. For a basis $\underline{e} = \{e_1, \ldots, e_n\}$ for W, we define

$$R(\underline{e}) := (\langle e_i, e_j \rangle)_{ij} \in \mathrm{GL}_n(D).$$

Denote by W_0 the anisotropic kernel of W, and put $n_0 = \dim_D W_0$, $r = \frac{1}{2}(n - n_0)$. We assume that

$$W_0 = \sum_{i=r+1}^{r+n_0} e_i D,$$

both

$$X = \sum_{i=1}^{r} e_i D \text{ and } \sum_{i=r+n_0+1}^{n} e_i X^*$$

are isotropic subspaces of W, and

$$R(\underline{e}) = \begin{pmatrix} 0 & 0 & J_r \\ 0 & R_0 & 0 \\ -\epsilon J_r & 0 & 0 \end{pmatrix} \tag{4.1}$$

where

$$J_r = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

and $R_0 \in \mathrm{GL}_{n_0}(D)$. By this basis, we regard $G(W)$ as a subgroup of $\mathrm{GL}_n(D)$.

5. Bruhat-Tits theory

The main purposes of this section are to explain the explicit description of a certain Iwahori subgroup $\mathcal{B}$ of $G(W)$ (§5.3), and to give a formula of the index $[K_W : \mathcal{B}]$ where K_W is a certain special maximal compact subgroup of $G(W)$ (§5.4). Note that in §6, the normalization of Haar measures will be given by the volume of the Iwahori subgroup.

5.1. **Apartments.** Take a basis $\underline{e}$ as in §4. Put $I = \{e_1, \dots, e_r\}$, $I_0 = \{e_{r+1}, \dots, e_{n-r}\}$, and $I^* = \{e_{n-r+1}, \dots, e_n\}$. We denote by S the maximal F-split torus

$$\{\mathrm{diag}(x_1, \dots, x_r, 1, \dots, 1, x_r^{-1}, \dots, x_1^{-1}) \mid x_1, \dots, x_r \in F^\times\}$$

of $G(W)$. We denote by $Z_{G(W)}(S)$ the centralizer of S in $G(W)$, by $N_{G(W)}(S)$ the normalizer of S in $G(W)$, by $\mathcal{W} = N_{G(W)}(S)/Z_{G(W)}(S)$ the relative Weyl group with respect to S, by Φ the relative root system of $G(W)$ with respect to S, by $X^*(S)$ the group of algebraic characters of S, by $E^\vee$ the vector space $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, and by E the $\mathbb{R}$ dual space of $E^\vee$. Moreover, we define the bilinear map $\langle\ ,\ \rangle : E \times E^\vee \to \mathbb{R}$ by $\langle y, \eta \rangle = \eta(y)$ for $y \in E^\vee$ and $\eta \in E$. Then, we can define the map $\mu : Z_{G(W)}(S) \to E$ by

$$[\mu(z)](a') = -\mathrm{ord}_F(a'(z))$$

for $a' \in X^*(S)$. Then, there is a unique morphism $\nu : N_{G(W)}(S) \to \mathrm{Aff}(E)$ so that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_{G(W)}(S) & \longrightarrow & N_{G(W)}(S) & \longrightarrow & \mathcal{W} & \longrightarrow & 1\ . \\ & & \downarrow{\scriptstyle \mu} & & \vdots{\scriptstyle \nu} & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & \mathrm{Aff}(E) & \longrightarrow & \mathrm{Aut}(E) & \longrightarrow & 1 \end{array}$$

For $a \in \Phi$, we denote by X_a the root subgroup in $G(W)$. Let $u \in X_a \setminus \{1\}$. Then one can prove that $X_{-a} \cdot u \cdot X_{-a} \cap N_{G(W)}(S)$ consists of an unique element. We denote it by $m_a(u)$. We define a map $\varphi_a : X_a \setminus \{1\} \to \mathbb{R}$ by

$$m_a(u)(\eta) = \eta - (\langle a, \eta \rangle + \varphi_a(u))a^\vee$$

for all $\eta \in E$. We put Φ_{aff} the affine root system

$$\{(a, t) \mid a \in \Phi,\ t = \varphi_a(u) \text{ for some } u \in X_a \setminus \{1\}\} \subset \Phi \times \mathbb{R},$$

and by $E_{a,t}$ the subset $\{\eta \in E \mid [m_a(u)](\eta) = \eta\}$ where $u \in X_a$ so that $\varphi_a(u) = t$. We call a connected component of

$$E \setminus \bigcup_{(a,t) \in \Phi_{\mathrm{aff}}} E_{a,t}$$

a chamber of E. For $i \in I \cup I^*$, we define $a_i \in X^*(S) \subset E^\vee$ by $a_i(x) = N_D(x_i)$ for

$$x = \mathrm{diag}(x_1, \dots, x_r, 1, \dots, 1, x_r^{-1}, \dots, x_1^{-1}) \in S.$$

Note that $a_{n-i} = -a_i$ for $i \in I$ (the multiplication of E is denoted by "+").

Now we describe φ_a explicitly following [BT72, §10]. The root system of $G(W)$ with respect to S is divided into

$$\Phi = \Phi_1^+ \cup \Phi_1^- \cup \Phi_2^+ \cup \Phi_2^- \cup \Phi_3^+ \cup \Phi_3^- \cup \Phi_4^+ \cup \Phi_4^-$$

where

$$\begin{aligned} \Phi_1^+ &= \{a_i - a_j \mid 1 \le j < i \le r\}, \\ \Phi_2^+ &= \{a_i \mid i = 1, \dots, r\}, \\ \Phi_3^+ &= \{a_i + a_j \mid 1 \le j < i \le r\}, \\ \Phi_4^+ &= \{2a_i \mid i = 1, \dots, r\}, \end{aligned}$$

and $\Phi_k^- = -\Phi_k^+$ for $k = 1,2,3,4$. Let $a = a_i - a_j \in \Phi_1^+ \cup \Phi_1^-$. For $x \in D$, we define $u_a(x) \in X_a$ by

$$e_k \cdot u_a(x) = \begin{cases} e_k & (k \neq i, n-i), \\ e_i + x \cdot e_j & (k = i), \\ e_{n-i} + x^* \cdot e_{n-j} & (k = n-i). \end{cases}$$

Let $a = a_i \in \Phi_2^+$. For $c = (c_1, \ldots, c_{n_0}) \in W_0 = D^{n_0}$ and $d \in D$ with $(d^* - \epsilon d) + \langle c, c \rangle = 0$, we define $u_a(c,d) \in X_a$ by

$$e_k \cdot u_a(c,d) = \begin{cases} e_k & (k \neq i, r+1, \ldots, r+n_0), \\ e_i + \sum_{t=1}^{n_0} c_t e_{r+t} + d e_{n-i} & (k = i), \\ e_k + \alpha_{k-r} c_{k-r}^* e_{n-i} & (k = r+1, \ldots, r+n_0). \end{cases}$$

Let $a = -a_i \in \Phi_2^-$. For $c = (c_1, \ldots, c_{n_0}) \in W_0 = D^{n_0}$ and $d \in D$ with $(d - \epsilon d^*) + \langle c, c \rangle = 0$, we define $u_a(c,d) \in X_a$ by

$$e_k \cdot u_a(c,d) = \begin{cases} e_k & (k \neq r+1, \ldots, r+n_0, n-i), \\ e_k - \alpha_{k-r} c_{k-r}^* e_i & (k = r+1, \ldots, r+n_0), \\ d e_i + \sum_{t=1}^{n_0} c_t e_{r+t} + e_{n-i} & (k = n-i). \end{cases}$$

Let $a = (a_i + a_j) \in \Phi_3^+$. For $x \in D$, we define $u_a(x) \in X_a$ by

$$e_k \cdot u_a(x) = \begin{cases} e_k & (k \neq i, j), \\ e_i + x \cdot e_{n-i} & (k = i), \\ e_j + \epsilon x^* e_{n-j} & (k = j). \end{cases}$$

Let $a \in \Phi_3^-$. For $x \in D$, we define $u_a(x) := {}^t u_{-a}(x)^* \in X_a$. Finally, let $a = \pm 2a_i \in \Phi_4^\pm$. For $d \in D$ with $d^* - \epsilon d = 0$, we define $u_a(d) := u_{\pm a_i}(0, d) \in X_{2a}$.

Lemma 5.1. *For $a \in \Phi$, we have*

- $\varphi_a(u_a(x)) = \mathrm{ord}_D(x)$ *for $x \in D$ if $a \in \Phi_1^+ \cup \Phi_1^- \cup \Phi_3^+ \cup \Phi_3^-$,*
- $\varphi_a(u_a(c,d)) = \frac{1}{2}\mathrm{ord}_D(d)$ *for $c \in D^{n_0}$ and $d \in D$ with $(d^* - \epsilon d) \pm \langle c, c \rangle = 0$ if $a \in \Phi_2^\pm$,*
- $\varphi_a(u_a(d)) = \mathrm{ord}_D(d)$ *for $d \in D$ with $d^* - \epsilon d = 0$ if $a \in \Phi_4^+ \cup \Phi_4^-$.*

5.2. **Lattice functions.** To know the action of $G(W)$ on its building, it is useful to consider lattice functions. Let Y be a left vector space over D. A lattice of Y is a free $\mathcal{O}_D$-submodule $\mathcal{Y}$ of Y so that $D \cdot \mathcal{Y} = Y$. For a lattice $\mathcal{L}$ of W, we denote by $\mathcal{L}^\vee$ the dual lattice of $\mathcal{L}$ defined by

$$\mathcal{L}^\vee = \{x \in W \mid \langle x, y \rangle \in \varpi_D \mathcal{O}_D \text{ for all } y \in \mathcal{L}\}.$$

Definition 5.2. *A mapping Λ from a real number s to a lattice $\Lambda(s)$ of W is called a lattice function if*

(1) $\Lambda(s) \supset \Lambda(t)$ *when* $s < t$,
(2) $\Lambda(s+1) = \varpi_D \Lambda(s)$,
(3) $\Lambda(s) = \cap_{t<s} \Lambda(t)$.

Let Λ be a lattice function. For $s \in \mathbb{R}$, we denote by $\Lambda(s)^+$ the lattice $\cup_{t<s}\Lambda(t)$. We define dual lattice function $\Lambda^\vee$ by

$$\Lambda^\vee(s) := (\Lambda(-s)^+)^\vee,$$

and we say that Λ is self-dual if $\Lambda = \Lambda^\vee$. For $p \in E$, we define the self-dual lattice function Λ_p by

$$\Lambda_p(s) = \left(\bigoplus_{i \in I} \varpi_D^{\lceil s + a_i(p) \rceil} \mathcal{O}_D \cdot e_i \right) \oplus \mathcal{X}_0(s) \oplus \left(\bigoplus_{i \in I^*} \varpi_D^{\lceil s + a_i(p) \rceil} \mathcal{O}_D \cdot e_i \right)$$

where $\mathcal{X}_0(s)$ be the lattice of W_0 defined by

$$\mathcal{X}_0(s) = \{x \in W_0 \mid \frac{1}{2}\langle x, x\rangle \in \varpi_D^{\lceil 2s \rceil}\mathcal{O}_D\}.$$

Then, $\mathrm{Stab}_{G(W)}(p) = \mathrm{Stab}_{G(W)}\Lambda_p$.

5.3. **Iwahori subgroups.** Before stating the definition of the Iwahori subgroup, we explain a map of Kottwitz. Let F^{ur} be the maximal unramified extension of F, let F^s be the separable closure of F, let $I = \mathrm{Gal}(F^s/F^{\mathrm{ur}})$ be the inertia group of F, and let Fr be a Frobenius element. Then, Kottwitz defined a surjective map

$$\kappa_W : G(W) \to \mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)^{\mathrm{Fr}}$$

(see [Kot97, §7.4]). Here, we denote by $\widehat{G(W)}$ the Langlands dual group of $G(W)$, by $Z(\widehat{G(W)})^I$ the I-invariant subgroup of the center of $\widehat{G(W)}$, and by $\mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)^{\mathrm{Fr}}$ the Fr-invariant subgroup of $\mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)$. Then, an Iwahori subgroup of $G(W)$ is defined to be a subgroup consisting of the elements g of $G(W)$ which preserves each point of a chamber of the building and $\kappa_W(g) = 1$. Now we describe an Iwahori subgroup of $G(W)$. Let $\mathcal{C}$ be a chamber in E so that

- for any root $a \in \Phi(S, G(W))$ with $X_a \subset B$, $\langle a, \mathcal{C}\rangle \subset \mathbb{R}_{>0}$,
- the closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ contains the origin $0 \in E$.

Then, the Iwahori subgroup associated with the chamber $\mathcal{C}$ is given by

$$\mathcal{B} := \{g \in G(W) \mid \kappa_W(g) = 1 \text{ and } g \cdot p = p \text{ for all } p \in \mathcal{C}\}.$$

By the construction of the map κ_W, the following diagram is commutative:

$$\begin{array}{ccc} Z_{G(W)}(S) & \xrightarrow{\kappa_{Z_{G(W)}(S)}} & \mathrm{Hom}(Z(\widehat{Z_{G(W)}(S)})^I, \mathbb{C}^\times)^{\mathrm{Fr}} \\ \downarrow & & \downarrow \\ G(W) & \xrightarrow{\kappa_W} & \mathrm{Hom}(Z(\widehat{G(W)})^I, \mathbb{C}^\times)^{\mathrm{Fr}} \end{array}$$

where the vertical maps are (induced from) the natural embeddings. Hence, we have:

Lemma 5.3.

$$\mathcal{B} = Z_{G(W)}(S)_1 \cdot \prod_{a \in \Phi^+} X_{a,0} \cdot \prod_{a \in \Phi^-} X_{a,\frac{1}{2}}$$

where $Z_{G(W)}(S)_1$ is the set of matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & g_0 & 0 \\ 0 & 0 & a^{*-1} \end{pmatrix} \quad (a = \mathrm{diag}(a_1, \ldots, a_r), g_0 \in G(W_0))$$

such that $a_i \in \mathcal{O}_D^\times$ for $i = 1, \ldots, r$, and $\kappa_{W_0}(g_0) = 1$. Here, we denote by $X_{a,t}$ the subset

$$\{u \in X_a \mid \varphi_a(u) \geq t\}$$

of X_a for $t \in \mathbb{R}$.

5.4. **Special maximal compact subgroups.** We denote by K_W the special maximal compact subgroup of $G(W)$ fixing the origin $0 \in E$. Then, $\mathcal{B} \subset K_W$. In this subsection, we compute the index $[K_W : \mathcal{B}]$. To do this, we first note that the "$\mathcal{B}N\mathcal{B}$ decomposition" of Buruhat-Tits [BT72], which was completely proved by Haines-Rapoport [PR08, Appendix].

Proposition 5.4. *We have the decomposition* $G(W) = \mathcal{B} \cdot N_{G(W)}(S) \cdot \mathcal{B}$.

Hence we have:

Corollary 5.5.

$$K_W \cap (\ker \kappa_W) = \bigsqcup_{w \in \mathcal{W}'} \mathcal{B} w \mathcal{B}$$

where $\mathcal{W}' = (N_{G(W)}(S) \cap K_W \cap \ker \kappa_W)/Z_{G(W)}(S)_1$.

Now we obtain the formula of the index. Note that this formula is not necessary for the proof of the main theorem. However, it is useful to note it here since that enables us to deduce the formula of $\alpha_1(W)$ by more direct computation in some cases (see Proposition 7.6).

Proposition 5.6. *We have*

$$[K_W : \mathcal{B}] = \prod_{i=1}^{r} (1 + q^{2(n_0+i)-1}) \cdot \frac{\prod_{i=1}^{r}(q^{2i} - 1)}{(q^2 - 1)^r} \times \begin{cases} 1 & n_0 = 0, 1, \text{ and } \chi_W \text{ is unramified}, \\ 2 & \text{otherwise}. \end{cases}$$

In the rest of this section, we will prove the proposition. At first, denoting by $\mathfrak{S}_r$ the r-th permutation group, we have a natural isomorphism

$$\mathcal{W}' \cong \mathfrak{S}_r \ltimes (\mathbb{Z}/2\mathbb{Z})^r$$

by the actions $s \cdot a_i = a_{s(i)}$ for $i \in I, s \in \mathfrak{S}_r$ and $u \cdot a_i = (-1)^{u_i} a_i$ for $i \in I, u = (u_1, \ldots, u_r) \in (\mathbb{Z}/2\mathbb{Z})^r$. Then, we have:

Lemma 5.7. *Let* $w \in \mathcal{W}'$ *and suppose that* w *corresponds to* $(w_0, u) \in \mathfrak{S}_r \ltimes (\mathbb{Z}/2\mathbb{Z})^r$. *Then there is an element* s_u *of* $\mathfrak{S}_r$ *independent of* w_0, *such that*

$$[\mathcal{B} w \mathcal{B} : \mathcal{B}] = q^{2 \cdot l(w_0 s_u)} \cdot \prod_{i : u_i = 1} q^{2(n_0 + r - i) + 1}$$

where $l(w_0 s_u)$ *is the length of* $w_0 s_u$ *in the relative Weyl group of* $\mathrm{GL}_r(D)$ *with respect to the positive system* $\{a_j - a_{j+1} \mid j = 1, \ldots, r-1\}$, *and* $u = (u_1, \ldots, u_r) \in (\mathbb{Z}/2\mathbb{Z})^r$.

Proof. Let X be an isotropic subspace of W spanned by $e_1, \ldots, e_r$, let P be a maximal parabolic subgroup of $G(W)$ preserving X. We identify $\mathrm{GL}(X)$ with $\mathrm{GL}_r(D)$ by the basis $e_1, \ldots, e_r$. Then we denote by M the Levi subgroup $\mathrm{GL}_r(D) \times G(W_0)$ of P, and by U the unipotent radical of P. If we put $s_i \in \mathfrak{S}_r$ by

$$s_i(j) = \begin{cases} j & (j < i), \\ j + 1 & (i \le j < r), \\ i & (j = r) \end{cases}$$

and if we put

$$s_u := s_r^{u_r} \cdot \cdots \cdot s_2^{u_2} \cdot s_1^{u_1} \in \mathfrak{S}_r,$$

then we have $(\mathcal{B} \cap M)^{(s_u^{-1}, u)} = \mathcal{B} \cap M$. Hence, we have

$$\begin{aligned} [\mathcal{B} w \mathcal{B} : \mathcal{B}] &= [\mathcal{B} : \mathcal{B} \cap w \mathcal{B} w^{-1}] \\ &= [\mathcal{B} \cap M : (\mathcal{B} \cap M) \cap (\mathcal{B} \cap M)^{w_0 s_u}][\mathcal{B} \cap U : (\mathcal{B} \cap U) \cap (\mathcal{B} \cap U)^u]. \end{aligned}$$

Here,

$$[\mathcal{B}\cap M : (\mathcal{B}\cap M)\cap(\mathcal{B}\cap M)^{w_0 s_u}] = q^{2l(w_0 s_u)}$$

and

$$[\mathcal{B}\cap U : (\mathcal{B}\cap U)\cap(\mathcal{B}\cap U)^{u}] = \prod_{i:u_i=1} q^{2(n_0+r-i)+1}.$$

Thus we have the lemma. □

Now we prove Proposition 5.6. By the above lemmas, we have

$$\begin{aligned}[K_W\cap(\ker\kappa_W):\mathcal{B}] &= \sum_{w\in\mathcal{W}'}[\mathcal{B}w\mathcal{B}:\mathcal{B}] \\ &= \left(\sum_{w_0\in\mathfrak{S}_r} q^{2\cdot l(w_0)}\right)\cdot\left(\sum_{u\in(\mathbb{Z}/2\mathbb{Z})^r}\prod_{i:u_i=1} q^{2(n_0+i)-1}\right).\end{aligned}$$

The summation

$$\sum_{w_0\in\mathfrak{S}_r} q^{2\cdot l(w_0)}$$

is equal to $[\mathrm{GL}_r(\mathbb{F}_{q^2}) : B(\mathbb{F}_{q^2})]$ where B is the Borel subgroup of GL_r. It is known that

$$|\mathrm{GL}_r(\mathbb{F}_{q^2})| = \prod_{i=0}^{r-1}(q^{2r^2}-q^{2ri}),\ \text{and}\ |B(\mathbb{F}_{q^2})| = (q^2-1)^r\cdot q^{r(r-1)}.$$

Moreover, we have

$$\sum_{u\in(\mathbb{Z}/2\mathbb{Z})^r}\prod_{i:u_i=1} q^{2(n_0+r-i)+1} = \prod_{j=1}^{r}(1+q^{2(n_0+j)-1}).$$

Finally, consider the lattice model (§5.2). In the case κ_W is not trivial, κ_{W_0} is also non-trivial. Thus, $\kappa_W|_{K_W}$ is non-trivial if κ_W is non-trivial since $G(W_0)$ preserves the lattice function Λ_0 where $0\in E$ is the origin. Hence, we have

$$[K_W : K_W\cap(\ker\kappa_W)] = \begin{cases}1 & n_0 = 0,1,\ \text{and } \chi_W \text{ is unramified,}\\ 2 & \text{otherwise.}\end{cases}$$

Hence we have Proposition 5.6.

6. Haar measures

In this section, we explain how we choose a Haar measure in this paper for reductive groups and unipotent groups. Let $\psi : F\to\mathbb{C}^\times$ be a non-trivial additive character of F. For a reductive group, Gan-Gross constructed a Haar measure dg depending only on the group G and the non-trivial additive character ψ [GG99, §8]. (In [GG99], it is denoted by μ_G.) On the other hand, for a unipotent group, it is useful to consider the "self-dual measures" du with respect to ψ. Note that, in both cases, we denote by $|X|$ the volume of X for a measurable set X.

6.1. **Measures on reductive groups.** Let G be a connected reductive group, and let G' be the quasi-split inner form of G. Moreover, let S' be a maximal F-split torus of G', let T' be the centralizer of S' in G' (it becomes a torus over F^s), and let $\mathcal{W}(T', G')$ be the Weyl group of G' with respect to T'. Put $E' := X^*(T') \otimes \mathbb{Q}$. Then the space E' can be regarded as a graded $\mathbb{Q}[\Gamma]$-module

$$E' = \oplus_{d \geq 1} E'_d$$

as follows: consider a $\mathcal{W}(T', G')$-invariant subalgebra $R = \mathrm{Sym}^\bullet(E')^{\mathcal{W}(T',G')}$ of symmetric algebra $\mathrm{Sym}^\bullet(E')$. We denote by R_+ the ideal consisting of the elements of positive degrees. Then, there is a $\mathbb{Q}[\Gamma]$-isomorphism $E' \cong R_+/R_+^2$. Then, the grading of E' is the one deduced from the natural grading of R_+/R_+^2.

Let $\Psi : G' \to G$ be an inner isomorphism defined over F^{ur}, and let w_G be the element of the Weyl group of G' given by $w_G = \Psi^{-1} \circ \mathrm{Fr}(\Psi)$. We denote by $\mathfrak{M}$ the motive

$$\oplus_{d \geq 1} E'_d(d-1)$$

of G (see [Gro97]), and by $a(\mathfrak{M})$ the Artin invariant

$$\sum_{d \geq 1} (2d-1) \cdot a(E'_d)$$

of $\mathfrak{M}$ (see [GG99]). Then, the normalized Haar measure dg is given by the volume of the Iwahori subgroup $\mathcal{B}$ of G over F:

$$|\mathcal{B}| = q^{-\mathfrak{N} - \frac{1}{2} a(\mathfrak{M})} \cdot \det(1 - \mathrm{Fr} \circ w_G; E'(1)^I).$$

Here, we put

$$\mathfrak{N} = \sum_{d \geq 1} (d-1) \dim_{\mathbb{Q}} {E'_d}^I.$$

Now, consider the case $G = G(W)$ where W is an n-dimensional $(-\epsilon)$-Hermitian space over D. Then, E', $a(\mathfrak{M})$, and $\mathfrak{N}$ are given by the following:

Lemma 6.1. *(1) In the case $-\epsilon = 1$, we have*

$$E' \cong \mathbb{Q}X^2 + \mathbb{Q}X^4 + \cdots + \mathbb{Q}X^{2n} \subset \mathbb{Q}[X]$$

as graded $\mathbb{Q}[\Gamma]$-modules. Here, the grading and the action of Γ on $\mathbb{Q}[X]$ are given by

$$\deg X^k = k \ (k = 0, 1, \ldots), \text{ and } \Gamma \text{ acts on } \mathbb{Q}[X] \text{ trivially}.$$

Moreover, we have

$$\mathfrak{N} = n^2 \text{ and } a(\mathfrak{M}) = 0.$$

(2) In the case $-\epsilon = -1$, we have

$$E' \cong \mathbb{Q}X^2 + \mathbb{Q}X^4 + \cdots + \mathbb{Q}X^{2n-2} + \mathbb{Q}Y \subset \mathbb{Q}[X, Y]$$

as graded $\mathbb{Q}[\Gamma]$-modules. Here, the grading and the action of Γ on $\mathbb{Q}[X, Y]$ are given by

$$\deg X^k = k, \ \deg Y^l = nl \ (k, l = 0, 1, \ldots), \text{ and}$$
$$\sigma \cdot f(X, Y) = f(X, \eta_W(\sigma)Y) \text{ for } f(X, Y) \in \mathbb{Q}[X, Y], \sigma \in \Gamma.$$

Moreover, we have

$$\mathfrak{N} = \begin{cases} n^2 - n & \chi_W \text{ is unramified,} \\ n^2 - 2n + 1 & \chi_W \text{ is ramified,} \end{cases}$$

and

$$a(\mathfrak{M}) = \begin{cases} 0 & \chi_W \text{ is unramified,} \\ 2n - 1 & \chi_W \text{ is ramified.} \end{cases}$$

Then, the normalization of the Haar measure dg is given by the following:

Proposition 6.2. *(1) Suppose that $-\epsilon = 1$. Then, we have*

$$|\mathcal{B}| = (1-q^{-1})^{\lfloor \frac{n}{2} \rfloor} \cdot (1+q^{-1})^{\lceil \frac{n}{2} \rceil} \cdot q^{-n^2}$$

where $\mathcal{B}$ is an Iwahori subgroup of $G(W)$.

(2) Suppose that $-\epsilon = -1$. Then, we have

$$|\mathcal{B}| = \begin{cases} (1-q^{-2})^{\frac{n}{2}} \cdot q^{-n^2+n} & n_0 = 0, \\ (1-q^{-2})^{\frac{n-1}{2}} \cdot q^{-n^2+n-\frac{1}{2}} & n_0 = 1, \chi_W : \textit{ramified}, \\ (1-q^{-2})^{\frac{n-1}{2}} \cdot (1+q^{-1}) \cdot q^{-n^2+n} & n_0 = 1, \chi_W : \textit{unramified}, \\ (1-q^{-2})^{\frac{n-2}{2}} \cdot (1+q^{-1}) \cdot q^{-n^2+n-\frac{1}{2}} & n_0 = 2, \chi_W : \textit{ramified}, \\ (1-q^{-2})^{\frac{n-2}{2}} \cdot (1+q^{-2}) \cdot q^{-n^2+n} & n_0 = 2, \chi_W : \textit{unramified}, \\ (1-q^{-2})^{\frac{n-3}{2}} \cdot (1+q^{-1}+q^{-2}+q^{-3}) \cdot q^{-n^2+n} & n_0 = 3. \end{cases}$$

If $G(W)$ is anisotropic, then $\mathcal{B} = \ker \kappa_W$ (see §5.3). Hence, its total volume is given by the following corollary:

Corollary 6.3. *(1) Suppose that $-\epsilon = 1$ and $n = 1$. Then we have $|G(W)| = q^{-1}(1+q^{-1})$.*

(2) Suppose that $-\epsilon = -1$. Then we have

$$|G(W)| = \begin{cases} 1+q^{-1} & n=1,\ \chi_W:\ \textit{unramified}, \\ 2q^{-\frac{1}{2}} & n=1,\ \chi_W:\ \textit{ramified}, \\ 2q^{-2} \cdot (1+q^{-2}) & n=2,\ 1 \neq \chi_W:\ \textit{unramified}, \\ 2q^{-\frac{5}{2}} \cdot (1+q^{-1}) & n=2,\ \chi_W:\ \textit{ramified}, \\ 2q^{-6} \cdot (1+q^{-1})(1+q^{-2}) & n=3,\ \chi_W = 1. \end{cases}$$

Proof. We have

$$[G(W):\mathcal{B}] = \#(X^*(Z(\widehat{G})^I)^{\mathrm{Fr}}) = \begin{cases} 1 & n=1,\ \chi_W:\ \text{unramified}, \\ 2 & \text{otherwise} \end{cases}$$

where I is the inertia group of F, and Fr is a Frobenius element of F. Hence we have the claim. □

6.2. **Measures on unipotent groups.** Take a basis $\underline{e}$ and regard $G(W)$ as a subgroup of $\mathrm{GL}_n(D)$ as in §4. Let

$$\mathfrak{f} : 0 = X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = X$$

be a flag consisting of isotropic subspaces. We put $r_i = \dim_D X_i/X_{i-1}$ for $i = 1, \ldots, k$. Moreover, we put

$$\mathfrak{u}_{r'} = \{z \in \mathrm{M}_{r'}(D) \mid {}^t z^* - \epsilon z = 0\}$$

for a positive integer r'. We denote by P the parabolic subgroup of all $p \in G(W)$ satisfying $X_i \cdot p \subset X_i$ for $i = 0, \ldots, k$, and by $U(P)$ the unipotent radical of P. Moreover, we denote by $U_i(P)$ the subgroup

$$\{u \in U(P) \mid X \cdot (u-1) \subset X_i\}$$

for $i = 1, \ldots, k$. Then, for $i = 1, \ldots, k$, we have the exact sequence

$$1 \to U_{i-1}(P) \to U_i(P) \to \prod_{j=(i+2)/2}^{i} \mathrm{M}_{r_j, r_{i+1-j}}(D) \to 0 \tag{6.1}$$

if i is even, and the exact sequence

$$1 \to U_{i-1}(P) \to U_i(P) \to \mathfrak{u}_{r_{(i+1)/2}} \times \prod_{j=(i+3)/2}^{i} \mathrm{M}_{r_{i+1-j}, r_j}(D) \to 0 \tag{6.2}$$

if i is odd. Here, the first maps are the inclusions and the second maps are given by

$$u = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ z_1 & 0 & 1 & & & \\ * & z_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & 0 & 1 & \\ * & \cdots & * & z_i & 0 & 1 \end{pmatrix} \mapsto (z_{\lceil i/2 \rceil} J_{r_{\lceil i/2 \rceil}}, \dots, z_i J_{r_i}).$$

for $u \in U_i(P)$. We define a measure dz on $\mathfrak{u}_{r'}$ to be the self-dual Haar measure with respect to a pairing

$$\mathfrak{u}_r \times \mathfrak{u}_r \to \mathbb{C} : (z, z') \mapsto \psi(T(z \cdot {}^t z'^*)),$$

and we define a measure dx on $\mathrm{M}_{r',r''}(D)$ to be the self-dual Haar measure with respect to a pairing

$$\mathrm{M}_{r',r''}(D) \times \mathrm{M}_{r',r''}(D) \to \mathbb{C}^\times : (x, x') \mapsto \psi(T(x \cdot {}^t x'^*)).$$

Then, the Haar measure du on $U_i(P)$ is defined inductively by the exact sequences (6.1) and (6.2) for $i = 1, \dots, k$.

7. Doubling method and local γ-factors

In this section, we explain the doubling method, and we recall the analytic definition of the local standard γ-factor (§7.2). Note that the doubling method also appears in the formulation of the local Siegel-Weil formula (§10 below) and the local Rallis inner product formula (§15 below). Let W be a $(-\epsilon)$-Hermitian space. In this section, we also define the local zeta value $\alpha_1(W)$, which depends on W and its basis $\underline{e}$. In §7.3, we compute $\alpha_1(W)$ for a $(-\epsilon)$-Hermitian space and for a basis $\underline{e}$ for W under some assumptions. As explained in the introduction, this computation of the constant $\alpha_1(W)$ will play an important role in the computation of the constant in the local Siegel-Weil formula (§10).

7.1. **Doubling method.** Let $(W^\square, \langle\ ,\ \rangle^\square)$ be the pair where $W^\square = W \oplus W$ and $\langle\ ,\ \rangle^\square$ is the map $W^\square \times W^\square \to D$ defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle^\square = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle$$

for $x_1, x_2, y_1, y_2 \in W$. Let $G(W^\square)$ be the isometric group of $W^\square$. Then, the natural action

$$G(W) \times G(W) \curvearrowright W \oplus W : (x_1, x_2) \cdot (g_1, g_2) = (x_1 \cdot g_1, x_2 \cdot g_2)$$

induces an embedding $\iota : G(W) \times G(W) \to G(W^\square)$. Consider maximal totally isotropic subspaces

$$W^\triangle = \{(x, x) \in W^\square \mid x \in W\}, \text{ and}$$
$$W^\triangledown = \{(x, -x) \in W^\square \mid x \in W\}.$$

Note that there is a polar decomposition $W^\square = W^\triangle \oplus W^\triangledown$. We denote by $P(W^\triangle)$ the maximal parabolic subgroup of $G(W^\square)$ which preserves $W^\triangle$. Then, a Levi subgroup of $P(W^\triangle)$ is isomorphic to $\mathrm{GL}(W^\triangle)$. We denote by Δ the character of $P(W^\triangle)$ given by

$$\Delta(x) = N_{W^\triangle}(x)^{-1}.$$

Here $N_{W^\triangle}(x)$ is the reduced norm of the image of x in $\mathrm{End}_D(W^\triangle)$. Let $\omega : F^\times \to \mathbb{C}^\times$ be a character. For $s \in \mathbb{C}$, put $\omega_s = \omega \cdot |-|^s$. Let $\underline{e}$ be a basis for W. Then we define a basis $\underline{e}'^\square = (e'_1, \dots, e'_{2n})$ for $W^\square$ by

$$e'_i = (e_i, e_i),\ e'_{n+i} = \sum_{k=1}^{n} a_{jk}(e_i, -e_i)$$

for $i = 1, \dots, n$, where $(a_{jk})_{j,k} = R(\underline{e})^{-1}$. Note that

$$(\langle e'_i, e'_j \rangle)_{i,j} = \begin{pmatrix} 0 & 2 \cdot I_n \\ -2\epsilon \cdot I_n & 0 \end{pmatrix}.$$

We choose a maximal compact subgroup $K(\underline{e}'^\square)$ of $G(W^\square)$ which preserves the lattice

$$\mathcal{O}_{W^\square} = \sum_{i=1}^{2n} \mathcal{O}_D e'_i$$

of $W^\square$. Then, we have $P(W^\triangle)K(\underline{e}'^\square) = G(W^\square)$. Denote by $I(s, \omega)$ the degenerate principal series representation

$$\mathrm{Ind}_{P(W^\triangle)}^{G(W^\square)}(\omega_s \circ \Delta)$$

consisting of the smooth right $K(\underline{e}'^\square)$-finite functions $f : G(W^\square) \to \mathbb{C}$ satisfying

$$f(pg) = \delta_{P(W^\triangle)}^{\frac{1}{2}}(p) \cdot \omega_s(\Delta(p)) \cdot f(g)$$

for $p \in P(W^\triangle)$ and $g \in G(W^\square)$, where $\delta_{P(W^\triangle)}$ is the modular function of $P(W^\triangle)$. We may extend $|\Delta|$ to a right $K(\underline{e}'^\square)$-invariant function on $G(W^\square)$ uniquely. For $f \in I(0, \omega)$, put $f_s = f \cdot |\Delta|^s \in I(s, \omega)$. Then, we define an intertwining operator $M(s, \omega) : I(s, \omega) \to I(-s, \omega^{-1})$ by

$$[M(s, \omega)f_s](g) = \int_{U(W^\triangle)} f_s(\tau u g)\, du$$

where τ is the Weyl element of $G(W^\square)$ given by

$$\begin{cases} \tau(e'_i) = e'_{n+i} & i = 1, \dots, n, \\ \tau(e'_i) = -\epsilon e'_{i-n} & i = n+1, \dots, 2n. \end{cases}$$

This integral converges absolutely for $\Re s > 0$ and admits a meromorphic continuation to $\mathbb{C}$. Let π be a representation of $G(W)$ of finite length. For a matrix coefficient ξ of π, and for $f \in I(0, \omega)$, we define the doubling zeta integral by

$$Z^W(f_s, \xi) = \int_{G(W)} f_s(\iota(g, 1))\xi(g)\, dg.$$

Then the zeta integral satisfies the following properties, which is stated in [Yam14, Theorem 4.1]. This gives a generalization of [LR05, Theorem 3].

Proposition 7.1. *(1) The integral $Z^W(f_s, \xi)$ converges absolutely for $\Re s \geq n - \epsilon$ and has an analytic continuation to a rational function of q^{-s}.*

(2) There is a meromorphic function $\Gamma^W(s, \pi, \omega)$ such that

$$Z^W(M(s, \omega)f_s, \xi) = \Gamma^W(s, \pi, \omega) Z^W(f_s, \xi)$$

for all matrix coefficient ξ of π and $f_s \in I(s, \omega)$.

7.2. **Local γ-factor.** We regard $\mathfrak{u}_n$ as a subspace of $\mathrm{End}_D(W^\square)$ and we denote by $\mathfrak{u}_{\mathrm{reg}}$ the set of $A \in \mathfrak{u}$ of rank n. Fix a non-trivial additive character $\psi : F \to \mathbb{C}^\times$ and $A \in \mathfrak{u}_{\mathrm{reg}}$. We define a Haar measure du on $U(W^\triangle)$ by the identification $U(W^\triangle) \cong \mathfrak{u}_n$ by the basis $\underline{e}'$ (see §6.2). We define

$$\psi_A : U(W^\triangledown) \to \mathbb{C}^\times : u \mapsto \psi(\mathrm{T}_{W^\square}(uA))$$

where $\mathrm{T}_{W^\square}$ denotes the reduced trace of $\mathrm{End}_D(V^\square)$. For $f \in I(0,\omega)$ we define

$$l_{\psi_A}(f_s) = \int_{U(W^\triangledown)} f_s(u)\psi_A(u)\,du.$$

Then, this integral defining l_{ψ_A} converges for $\Re s \gg 0$ and admits a holomorphic continuation to $\mathbb{C}$ ([Kar79, §3.2]). Let $A_0 \in \mathrm{GL}_n(D)$ the matrix representation of the linear map $A : W^\triangledown \to W^\triangle$ with respect to the bases $e'_{n+1}, \ldots, e'_{2n}$ for $W^\triangledown$ and $e'_1, \ldots, e'_n$ for $W^\triangle$. We denote by $e(G(W))$ the Kottwitz sign of $G(W)$, which is given by

$$e(G(W)) = \begin{cases} (-1)^{\frac{1}{2}n(n+1)} & (-\epsilon = 1), \\ (-1)^{\frac{1}{2}n(n-1)} & (-\epsilon = -1). \end{cases}$$

Proposition 7.2. *We have*

$$l_{\psi_A} \circ M(s,\omega) = c(s,\omega,A,\psi) \cdot l_{\psi_A},$$

where $c(s,\omega,A,\psi)$ is the meromorphic function of s given by

$$c(s,\omega,A,\psi) = e(G(W)) \cdot \omega_s(N(A_0))^{-1} \cdot |2|^{-2ns+n(n-\frac{1}{2})} \cdot \omega^{-1}(4) \cdot \gamma(s-n+\frac{1}{2},\omega,\psi)^{-1}$$
$$\times \prod_{i=0}^{n-1} \gamma(2s-2i,\omega^2,\psi)^{-1} \cdot \gamma(s+\frac{1}{2},\omega\chi_{A_0},\psi) \cdot \epsilon(\frac{1}{2},\chi_{A_0},\psi)^{-1}$$

in the case $-\epsilon = 1$, and

$$c(s,\omega,A,\psi) = e(G(W)) \cdot \omega_s(N(A_0))^{-1} \cdot |2|^{-2ns+n(n-\frac{1}{2})} \cdot \omega^{-1}(4) \cdot \prod_{i=0}^{n-1} \gamma(2s-2i,\omega^2,\psi)^{-1}$$

in the case $-\epsilon = -1$.

Remark 7.3. *These formulas differ from those in* [Kak20b, Proposition 4.2]*. This is caused by a typo where $\omega_{n\pm\frac{1}{2}}(N(R))$ should be replaced by $\omega_{n\pm\frac{1}{2}}(N(R))^{-1}$ in* [Kak20b, Proposition 4.2].

Now we define the doubling γ-factor as in [Kak20b]. Note that the above error has no effects on the definition in [Kak20b].

Definition 7.4. *Let π be an irreducible representation of $G(W)$, let ω be a character of $F^\times$, let ψ be a non-trivial character of F. Then we define the γ-factor by*

$$\gamma^W(s+\frac{1}{2},\pi\times\omega,\psi) = c(s,\omega,A,\psi)^{-1} \cdot \Gamma^W(s,\pi,\omega) \cdot c_\pi(-1) \cdot R(s,\omega,A,\psi).$$

where c_π be the central character of π, and

$$R(s,\omega,A,\psi) = \begin{cases} \omega_s(N(R(\underline{e})A_0)^{-1}\gamma(s+\frac{1}{2},\omega\chi_{\mathfrak{d}(A)},\psi)\epsilon(\frac{1}{2},\chi_{\mathfrak{d}(A)},\psi)^{-1} & \text{in the case } -\epsilon = 1, \\ \omega_s(N(R(\underline{e})A_0)^{-1}\epsilon(\frac{1}{2},\chi_{\mathfrak{d}(W)},\psi) & \text{in the case } -\epsilon = -1. \end{cases}$$

The doubling γ-factor $\gamma^W(s+\frac{1}{2},\pi\boxtimes\omega,\psi)$ is expected to coincide with the standard γ-factor $\gamma(s+\frac{1}{2},\pi\boxtimes\omega,\mathrm{std},\psi)$ where std is the standard embedding of ${}^L(G(W)\times\mathrm{GL}_1)$. Another notable property is the commutativity with parabolic inductions, which is useful in the computation. For example, the doubling γ-factor of the trivial representation is given by the following lemma, which we use in the computation of the doubling zeta integral (§7.3 and §21 below).

Lemma 7.5. *Denote by 1_W the trivial representation of $G(W)$. Then we have*

$$\gamma^W(s+\frac{1}{2}, 1_W \times 1, \psi) = \begin{cases} \prod_{i=-n}^{n} \gamma_F(s+\frac{1}{2}+i, 1, \psi) & -\epsilon = 1, \\ \gamma_F(s+\frac{1}{2}, \chi_W, \psi) \prod_{i=-n+1}^{n-1} \gamma_F(s+\frac{1}{2}+i, 1, \psi) & -\epsilon = -1. \end{cases}$$

7.3. **Local zeta value.** We use the same setting and notation of §7.1. Let $f_s^\circ \in I(s,1)$ be the unique $K(\underline{e}'^\square)$-fixed section with $f_s^\circ(1) = 1$, and let ξ° be the matrix coefficient of the trivial representation of $G(W)$ with $\xi^\circ(1) = 1$. Then, we define

$$\alpha_1(W) := Z^W(f_\rho^\circ, \xi^\circ),$$

which is the first constant we are interested in. The purpose of this subsection is to obtain a formula of $\alpha_1(W)$ in the case where either $R(\underline{e}) \in \mathrm{GL}_n(\mathcal{O}_D)$ or W is anisotropic. Note that the general formula of $\alpha_1(W)$ will be obtained in §19.

Proposition 7.6. *(1) In the case $-\epsilon = 1$ and $R(\underline{e}) \in \mathrm{GL}_n(\mathcal{O}_D)$, we have*

$$\alpha_1(W) = |2|^{n(2n+1)} \cdot q^{-n_0^2-(2n_0+1)r-2r^2} \cdot \prod_{i=1}^{n}(1+q^{-(2i-1)}).$$

(2) In the case $-\epsilon = -1$ and $R(\underline{e}) \in \mathrm{GL}_n(\mathcal{O}_D)$, we have

$$\alpha_1(W) = |2|^{n(2n-1)} \cdot q^{-2rn_0-2r^2+r} \cdot \prod_{i=1}^{n}(1+q^{-(2i-1)}).$$

(3) In the case $-\epsilon = -1$ and W is anisotropic, we have

$$\alpha_1(W) = |N(R(\underline{e}))|^{-n+\frac{1}{2}} \times \begin{cases} |2|_F \cdot (1+q^{-1}) & n=1, \\ |2|_F^6 \cdot q^{-1} \cdot (1+q^{-1})(1+q^{-3}) & n=2, \\ |2|_F^{15} \cdot q^{-3} \cdot (1+q^{-1})(1+q^{-3})(1+q^{-5}) & n=3. \end{cases}$$

Note that in the case $2 \not| q$, the assertions (1) and (2) are conclusions of [Kak20b, Proposition 8.3] and the volume formula of K_W (Proposition 5.6). However, to contain the case $2|q$, we prove them in another way. Before proving this lemma, we observe following two important lemmas:

Lemma 7.7.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G(W)\times G(W)}(I(\rho,1), \mathbb{C}) = 1.$$

Proof. First, the map

$$Z : I(\rho,1) \to \mathbb{C} : f \mapsto \int_{G(W)} f((g,1))\, dg$$

is contained in $\mathrm{Hom}_{G(W)\times G(W)}(I(\rho,1), \mathbb{C})$. To prove the lemma, it suffices to show that $\ker Z$ is spanned by the set

$$\{h - R(g)h \mid h \in I(\rho,1), g \in G(W)\times G(W)\}.$$

Here, we denote by $R(g)$ the right translation by g. Let $f \in \ker Z$. Take a compact open subgroup K' of $K(\underline{e}'^\square)$, complex numbers $a_i \in \mathbb{C}$ and elements $g_i \in G(W)\times G(W)$ for $i = 1, \dots, t$ so that

$$f = \sum_{i=1}^{t} a_i R(g_i)\mathfrak{c}$$

where $\mathfrak{c} \in I(\rho,1)$ is the section defined by

$$\mathfrak{c}(g) := \begin{cases} \delta_{P(W^\triangle)}(p) & g = pk' \ (p \in P(W^\triangle), k' \in K'), \\ 0 & g \notin P(W^\triangle)K'. \end{cases}$$

Then, we have

$$a_1 + \cdots + a_t = \frac{Z(f)}{Z(\mathfrak{c})} = 0$$

and we have

$$\sum_{i=1}^{t-1} b_i(R(g_i)\mathfrak{c} - R(g_{i+1})\mathfrak{c}) = f$$

where $b_i := a_1 + \cdots + a_i$ for $i = 1, \ldots, t-1$. Hence we have the lemma. □

Lemma 7.8. *For $f \in I(\rho, 1)$, we have*

$$\int_{G(W)} f((g,1))\, dg = m^\circ(\rho)^{-1} \cdot \alpha_1(W) \cdot \int_{U(W^\triangle)} f(\tau u)\, du$$

where

$$m^\circ(s) = \begin{cases} |2|_F^{n(n-\frac{1}{2})} q^{-\frac{1}{2}n(n+1)} \frac{\zeta_F(s-n+\frac{1}{2})}{\zeta_F(s+n+\frac{1}{2})} \prod_{i=0}^{n-1} \frac{\zeta_F(2s-2i)}{\zeta_F(2s+2n-4i-3)} & (-\epsilon = 1), \\ |2|_F^{n(n-\frac{1}{2})} q^{-\frac{1}{2}n(n-1)} \prod_{i=0}^{n-1} \frac{\zeta_F(2s-2i)}{\zeta_F(2s+2n-4i-1)} & (-\epsilon = -1). \end{cases}$$

Proof. Define a map $\mathfrak{A} : \mathcal{S}(G(W^\square)) \to I(\rho, 1)$ by

$$[\mathfrak{A}\varphi](g) = \int_{P(W^\triangle)} \delta_{P(W^\triangle)}(p)^{-1} \varphi(pg)\, dp.$$

Then $\mathfrak{A}$ is surjective. Moreover, we have

$$\begin{aligned} \int_{U(W^\triangle)} [\mathfrak{A}\varphi](\tau u)\, du &= \int_{U(W^\triangle)} \int_{M(W^\triangle)} \int_{U(W^\triangle)} \delta^{-1}_{P(W^\triangle)}(m) \varphi(xm\tau y)\, dydmdx \\ &= \gamma(G(W^\square)/P(W^\triangle)) \int_{G(W^\square)} \varphi(g)\, dg. \end{aligned}$$

Here, $\gamma(G(W^\square)/P(W^\triangle))$ is the constant defined by

$$\gamma(G(W^\square)/P(W^\triangle)) = \int_{U(W^\triangle)} f^\circ(\tau u)\, du$$

where $f^\circ \in I(\rho, 1)$ is an unique $K(\underline{e}'^\square)$-invariant section with $f^\circ(1) = 1$. Hence we conclude that the map

$$I(\rho, 1) \to \mathbb{C} : f \mapsto \int_{U(W^\triangle)} f(\tau u)\, du$$

is $G(W^\square)$-invariant, in particular, it is $G(W) \times G(W)$-invariant. Hence, by Lemma 7.7, we conclude that there is a constant $\alpha' \in \mathbb{C}$ such that

$$\int_{G(W)} f((g,1))\, dg = \alpha' \int_{U(W^\triangle)} f(\tau u)\, du$$

for all $f \in I(\rho, 1)$. To determine the constant α', we use f° as a test function. By Gindikin-Karperevich formula ([Cas80, Theorem 3.1]) or Shimura's computation ([Shi99, Proposition 3.5]), we have

$$\int_{U(W^\triangle)} f^\circ(\tau u)\, du = m^\circ(\rho).$$

Moreover, comparing this to Lemma 21.2, we have the claim. □

Now we prove Proposition 7.6. As a consequence of Lemma 7.8, we use another section $f(s, 1_{\varpi_F \mathcal{O}_\mathfrak{u}}, -) \in I(s, 1)$ to compute the ratio $\alpha_1(W) m^\circ(\rho)^{-1}$. Here, we denote the set $\mathfrak{u} \cap \mathrm{M}_n(\mathcal{O})$ by $\mathcal{O}_\mathfrak{u}$, and we define a section $f(s, \Phi, -) \in I(s, \omega)$ by

$$f(s, \Phi, g) := \begin{cases} 0 & g \notin P(W^\triangle)\tau U(W^\triangle), \\ \omega_s(\Delta(p))\Phi(X) & g = p\tau \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \ (p \in P(W^\triangle),\ X \in \mathfrak{u}) \end{cases}$$

for a character ω of $F^\times$ and $\Phi \in \mathcal{S}(\mathfrak{u})$. Let $g \in G(W)$ with $\iota(g) \in P(W^\triangle)\tau U(W^\triangle)$. Then,

$$\begin{pmatrix} \frac{g+1}{2} & \frac{g-1}{2} R(\underline{e})^{-1} \\ R(\underline{e})\frac{g-1}{2} & R(\underline{e})\frac{g+1}{2}R(\underline{e})^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & {}^t a^{*-1} \end{pmatrix} \tau \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}$$

for some $a \in \mathrm{GL}_n(D), b \in \mathrm{M}_n(D)$ and $X \in \mathfrak{u}$. If $X \in \varpi_F \mathcal{O}_\mathfrak{u}$, then a, g are given by

$$a = (X - R(\underline{e}))^{-1},\ g = a(X + R(\underline{e})) = 2aX - 1,$$

and thus $a \in \mathrm{GL}_n(\mathcal{O}_D)$ and $g \in -K^+_{2\varpi_F}$. Here we denote the set

$$\{g \in G(W) \cap \mathrm{GL}_n(\mathcal{O}_D) \mid g - 1 \in 2\varpi_F \mathrm{M}_n(\mathcal{O}_D)\}$$

by $K^+_{2\varpi_F}$. Conversely, if $g \in -K^+_{2\varpi_F}$, then a, X are given by

$$aI_n = \frac{g-1}{2} R(\underline{e})^{-1},\ aX = \frac{g+1}{2},$$

and thus $a \in \mathrm{GL}_n(\mathcal{O}_D)$ and $X \in \varpi_F \mathcal{O}_\mathfrak{u}$. Summarizing above discussions, we have

$$f(s, 1_{\varpi_F \mathfrak{u}}, -) \circ \iota = 1_{-K^+_{2\varpi_F}}$$

on $G(W)$. Put

$$m'(s) := \int_{U(W^\triangle)} f(s, 1_{2\varpi_F \mathcal{O}_\mathfrak{u}}, \tau u)\, du.$$

Then, we have

$$\begin{aligned} \frac{\alpha_1(W)}{m^\circ(\rho)} &= \frac{Z(f(\rho, 1_{2\varpi_F \mathcal{O}_\mathfrak{u}}, -))}{m'(\rho)} \\ &= \frac{|K^+_{2\varpi_F}|}{|\varpi_F \mathcal{O}_\mathfrak{u}|} \\ &= |2|_F^{2n\rho - n(n-\frac{1}{2})} q^{\frac{1}{2}n(n-\epsilon)} q^{n(2n-\epsilon)} |K^+_{\varpi_F}|. \end{aligned}$$

Since

$$[\mathcal{B}^+ : K^+_{\varpi_F}] = q^{6(n_0 r + r(r-1)) + 5r + n_0 - (2r + n_0)\epsilon}$$

and

$$|\mathcal{B}^+| = \begin{cases} -n^2 - n & (-\epsilon = 1), \\ -n^2 & (-\epsilon = -1), \end{cases}$$

we have

$$\begin{aligned}
&\log_q(q^{\frac{1}{2}n(n-\epsilon)}q^{n(2n-\epsilon)}|K^+_{\varpi_F}|)\\
&= \frac{1}{2}n(n-\epsilon) + n(2n-\epsilon) - 6(n_0 r + r(r-1)) - 5r - n_0 + (2r+n_0)\epsilon\\
&\quad - \begin{cases} -n^2 - n & (-\epsilon = 1),\\ -n^2 & (-\epsilon = -1)\end{cases}\\
&= \frac{1}{2}n(n-\epsilon) - \begin{cases} 2r^2 + (2n_0+1)r + n_0^2 & (-\epsilon = 1),\\ 2r^2 + 2n_0 r - r & (-\epsilon = -1).\end{cases}
\end{aligned}$$

Hence we have

$$\begin{aligned}
\alpha_1(W) &= m^\circ(\rho) \cdot \frac{\alpha_1(W)}{m^\circ(\rho)}\\
&= \begin{cases} |2|^{n(2n+1)} \cdot q^{-n_0^2-(2n_0+1)r-2r^2} \cdot \prod_{i=1}^n (1+q^{-(2i-1)}) & (-\epsilon = 1),\\ |2|^{n(2n-1)} \cdot q^{-2rn_0 - 2r^2 + r} \cdot \prod_{i=1}^n (1+q^{-(2i-1)}) & (-\epsilon = -1).\end{cases}
\end{aligned}$$

This proves (1) and (2) of Proposition 7.6.

Finally, we prove (3). By the definition of the γ-factor, we have the following (local) functional equation of the zeta integral:

$$\begin{aligned}
\frac{Z^W(f_s^\circ, \xi^\circ)}{m^\circ(s)} =& e(G) \frac{Z^W(f_{-s}^\circ, \xi^\circ)}{\gamma^W(s+\frac{1}{2}, 1_W \times 1, \psi)} \prod_{i=0}^{n-1} \gamma(2s-2i, 1, \psi)\\
&\times |2|_F^{2ns - n(n-\frac{1}{2})} |N(R(\underline{e}))|_F^{-s} \cdot \epsilon(\frac{1}{2}, \chi_W, \psi).
\end{aligned}$$

Note that $f_{-\rho}^\circ$ is a constant function on $G(W^\square)$, and $Z^W(f_{-\rho}^\circ, \xi^\circ) = |G(W)|$. Hence, by Lemma 7.5, we have

$$\frac{Z^W(f_{-\rho}^\circ, \xi^\circ)}{m^\circ(\rho)} = |N(R(\underline{e}))|^{-n+\frac{1}{2}} \times \begin{cases} |2|_F^{\frac{1}{2}} \cdot e(G) & n=1,\\ -|2|_F^3 \cdot e(G) & n=2,\\ -|2|_F^{\frac{15}{2}} \cdot e(G) & n=3.\end{cases}$$

Therefore, we have

$$\alpha_1(W) = |N(R(\underline{e}))|^{-n+\frac{1}{2}} \times \begin{cases} |2|_F \cdot (1+q^{-1}) & n=1,\\ |2|_F^6 \cdot q^{-1} \cdot (1+q^{-1})(1+q^{-3}) & n=2,\\ |2|_F^{15} \cdot q^{-3} \cdot (1+q^{-1})(1+q^{-3})(1+q^{-5}) & n=3.\end{cases}$$

Thus, we complete the proof of Proposition 7.6.

8. Local Weil representations

In this paper, we consider the two reductive dual pairs: $(G(V), G(W^\square))$ and $(G(V), G(W))$. For the first case, we can describe the restriction of the Weil representation to $G(V) \times G(W^\square)$. We discussed this in §8.1. The second case is discussed in §8.2.

8.1. **An explicit description.** In this subsection, we discuss an explicit description of Weil representation for the reductive dual pair $(G(V), G(W^\square))$, which is computed in [Kud94].

We fix a basis $\underline{e}$ for W. We take a basis $\underline{e}'^\square$ of $W^\square$ as in §7.1. In this subsection, we identify $G(W)$ (resp. $G(W^\square)$) with a subgroup of $\mathrm{GL}_n(D)$ (resp. $\mathrm{GL}_{2n}(D)$) by the basis $\underline{e}$ (resp.

$\underline{e}'^{\square}$). Moreover, we identify $G(V)$ with a subgroup of $\mathrm{GL}_m(D)$ by some fixed basis of V. Let $\mathbb{W}^{\square} = V \otimes_D W^{\square}$, and let $\langle\langle\ ,\ \rangle\rangle^{\square}$ be the pairing on $\mathbb{W}^{\square}$ defined by

$$\langle\langle x \otimes (y_1, y_2), x' \otimes (y_1', y_2') \rangle\rangle^{\square} = T_D((x, x') \cdot (\langle y_1, y_1' \rangle^* - \langle y_2, y_2' \rangle^*))$$

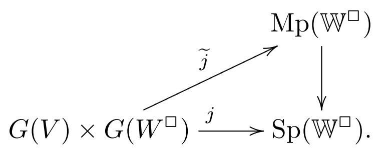
for $x_1, x_2 \in V$ and $y_1, y_2, y_1', y_2' \in W$. Then, $(G(V), G(W^{\square}))$ is a reductive dual pair in $\mathrm{Sp}(\mathbb{W}^{\square})$. We fix a non-trivial additive character $\psi : F \to \mathbb{C}^1$. Then, we consider the metaplectic $\mathbb{C}^1$-cover

$$1 \to \mathbb{C}^1 \to \mathrm{Mp}_{\psi}(\mathbb{W}^{\square}) \to \mathrm{Sp}(\mathbb{W}^{\square}) \to 1$$

of $\mathrm{Sp}(\mathbb{W}^{\square})$. We denote by $\omega_{\psi}^{\square}$ the Weil representation of $\mathrm{Mp}_{\psi}(\mathbb{W}^{\square})$. The canonical embedding

$$j : G(V) \times G(W^{\square}) \to \mathrm{Sp}(\mathbb{W}^{\square}) : (h, g) \mapsto h \otimes g$$

lifts to an embedding $\widetilde{j}$ into $\mathrm{Mp}_{\psi}(\mathbb{W}^{\square})$:



We consider a polar decomposition $\mathbb{W}^{\square} = (V \otimes W^{\triangledown}) \oplus (V \otimes W^{\triangle})$ (note that the order of $\triangle$ and $\triangledown$ is reversed compared to the decomposition of $W^{\square}$). Then, the restriction of $\omega_{\psi}^{\square}$ to $G(V) \times G(W^{\square})$ can be described explicitly on the space $\mathcal{S}(V \otimes W^{\triangledown})$ of Schwartz-Bruhat functions on $V \otimes W^{\triangledown}$. For $\phi \in \mathcal{S}(V \otimes W^{\triangledown})$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(\mathbb{W}^{\square})$, we define $r(g)\phi \in \mathcal{S}(V \otimes W^{\triangledown})$ by

$$[r(g)\phi](x) = \int_{\mathcal{Y}_c} \psi(\frac{1}{4}\langle\langle xa, xb \rangle\rangle + \frac{1}{2}\langle\langle yc, xb \rangle\rangle + \frac{1}{4}\langle\langle yc, yd \rangle\rangle)\phi(xa + yc)\, d\mu_g(y)$$

where $\mathcal{Y}_c = \ker(c) \cap V \otimes W^{\triangle} \backslash V \otimes W^{\triangle}$. Moreover, we may re-define the Haar measure $\mu_g(y)$ so that $r(g)$ keep the L^2-norm of $\mathcal{S}(V \otimes W^{\triangledown})$. For $a \in \mathrm{GL}(W^{\triangle})$, we denote by $m(a)$ the unique element of $G(W^{\square})$ such that $m(a)|_{W^{\triangle}} = a$.

Theorem 8.1. *Let $\phi \in \mathcal{S}(V \otimes W^{\triangledown})$. Then, $\omega_{\psi}^{\square}(h, g)\phi = \beta_V(g) r(g)(\phi \circ h^{-1})$. More precisely,*

- $[\omega_{\psi}^{\square}(h, 1)\phi](x) = \phi(h^{-1}x)$ *for* $h \in G(V)$,
- $[\omega_{\psi}^{\square}(1, m(a))\phi](x) = \beta_V(m(a))|N(a)|^{-m}\phi(x \cdot {}^t a^{*-1})$ *for* $a \in \mathrm{GL}(W^{\triangledown})$,
- $[\omega_{\psi}^{\square}(1, b)\phi](x) = \psi(\frac{1}{4}\langle\langle x, x \cdot b \rangle\rangle)\phi(x)$ *for* $b \in U(W^{\triangle})$,
- *the action of the Weyl element τ (for the definition, see §7.1) is given by*

$$[\omega_{\psi}^{\square}(1, \tau)\phi](x) = \beta_V(\tau) \cdot \int_{V \otimes W^{\triangledown}} \psi(\frac{1}{2}\langle\langle x, y\tau \rangle\rangle)\phi(y)\, dy$$

where dy is the self-dual measure of $V \otimes W^{\triangledown}$ with respect to the pairing

$$V \otimes W^{\triangledown} \times V \otimes W^{\triangledown} \to \mathbb{C} : x, y \mapsto \psi(\frac{1}{2}\langle\langle x, y\tau \rangle\rangle).$$

Here, we denote by N the reduced norm of $\mathrm{End}_D(W^{\triangledown})$ over F.

Proof. [Kud94, p.40]. □

Remark 8.2. *According to* [Kud94, p18]*, we can compute $\beta_V(g)$ as follows:*

- $\beta_V(m(a)) = 1, \beta_V(\tau) = (-1)^{mn}$ *in the case* $\epsilon = 1$, *and*
- $\beta_V(m(a)) = \chi_V(N(a)), \beta_V(\tau) = (-1, \det V)_F^n (-1)^{mn} (-1, -1)_F^{mn}$ *in the case* $\epsilon = -1$.

8.2. **Compatibility of Doubling and Weil representations.** Now we consider the dual pair $(G(V), G(W))$. Let $\mathbb{W} = V \otimes_D W$, and let $\langle\langle\ ,\ \rangle\rangle$ be the pairing on $\mathbb{W}$ defined by

$$\langle\langle x \otimes y, x' \otimes y' \rangle\rangle = T_D((x, x') \cdot \langle y, y' \rangle^*)$$

for $x, x' \in V$ and $y, y' \in W$. We denote by $\mathrm{Mp}_\psi(\mathbb{W})$ the metaplectic $\mathbb{C}^1$-cover of $\mathrm{Sp}(\mathbb{W})$, and by ω_ψ the Weil representation of $\mathrm{Mp}_\psi(\mathbb{W})$. Note that there is a polar decomposition $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ where $\mathbb{X}$ and $\mathbb{Y}$ are certain totally isotropic subspaces, and ω_ψ can be realized on the space $\mathcal{S}(\mathbb{X})$ of Schwartz-Bruhat functions on $\mathbb{X}$. We fix Haar measures dx and dy of $\mathbb{X}$ and $\mathbb{Y}$ so that they are the dual each other with respect to the pairing

$$\mathbb{X} \times \mathbb{Y} \to \mathbb{C}^\times : (x, y) \mapsto \psi(\langle\langle x, y \rangle\rangle).$$

Moreover, we define

$$\mathbb{X}^\triangle = (\mathbb{X} \oplus \mathbb{X}) \cap W^\triangle,\ \mathbb{X}^\triangledown = (\mathbb{X} \oplus \mathbb{X}) \cap W^\triangledown$$

and

$$\mathbb{Y}^\triangle = (\mathbb{Y} \oplus \mathbb{Y}) \cap W^\triangle,\ \mathbb{Y}^\triangledown = (\mathbb{Y} \oplus \mathbb{Y}) \cap W^\triangledown.$$

We define the Haar measure $dx^\triangle$ on $\mathbb{X}^\triangle$ by the push out measure $p_*(dx)$ where $p : \mathbb{X}^\triangle \to \mathbb{X}$ is the first projection. We define the Haar measures $dx^\triangledown$, $dy^\triangle$, $dy^\triangledown$ in the same way. The map

$$\delta : \mathcal{S}(\mathbb{X}) \otimes \overline{\mathcal{S}(\mathbb{X})} = \mathcal{S}(\mathbb{X} \oplus \mathbb{X}) \to \mathcal{S}(V \otimes W^\triangledown)$$

given by the partial Fourier transform

$$[\delta(\phi_1 \otimes \overline{\phi_2})](x) = \int_{\mathbb{X}^\triangle} (\phi_1 \otimes \overline{\phi_2})(y) \cdot \psi(\langle\langle x, y \rangle\rangle)\, dy^\triangle$$

is known to be compatible with the embedding $\iota : G(W) \times G(W) \to G(W^\square)$. Hence, we have

$$F_{\delta(\phi_1 \otimes \overline{\phi_2})}(\iota(g, 1)) = (\omega_\psi(g)\phi_1, \phi_2)_\mathbb{X}$$

for $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{X})$ where $(\ ,\ )_\mathbb{X}$ is the L^2-inner product on $\mathbb{X}$ defined by the measure dx. Moreover, we have:

Proposition 8.3. *Let dz be the self-dual Haar measure on $V \otimes W^\triangle$ with respect to the pairing*

$$V \otimes W^\triangledown \times V \otimes W^\triangledown \to \mathbb{C}^\times : (x, y) \mapsto \psi(\frac{1}{2}\langle\langle x, y\tau \rangle\rangle)$$

and let $(\ ,\ )$ be the L^2-inner product on $V \otimes W^\triangledown$ defined by dz. Then, we have

$$(\delta(\phi_1 \otimes \overline{\phi_2}), \delta(\phi_3 \otimes \overline{\phi_4})) = |2|_F^{-2mn} \cdot |N(R(\underline{e}))|^m \cdot (\phi_1, \phi_3)_\mathbb{X} \cdot \overline{(\phi_2, \phi_4)_\mathbb{X}}$$

for $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathcal{S}(\mathbb{X})$.

Proof. Note first that the vector space $V \otimes W^\triangledown$ decomposes into the direct sum

$$\mathbb{X}^\triangledown \oplus \mathbb{Y}^\triangledown.$$

For $z \in V \otimes W^{\bigtriangledown}$, we denote by z_x (resp. z_y) the $\mathbb{X}^{\bigtriangledown}$-component (resp. $\mathbb{Y}^{\bigtriangledown}$-component) of z. Then, one can prove that $dz = |N(R(\underline{e}))|^m \cdot dz_x^{\bigtriangledown} \otimes dz_y^{\bigtriangledown}$. Hence, we have

$$\begin{aligned}
&(\delta(\phi_1 \otimes \overline{\phi_2}), \delta(\phi_3 \otimes \overline{\phi_4})) \\
&= \int_{V\otimes W^{\bigtriangledown}} \delta(\phi_1 \otimes \overline{\phi_2})(z) \cdot \overline{\delta(\phi_3 \otimes \overline{\phi_4})(z)}\, dz \\
&= |N(R(\underline{e}))|^m \int_{\mathbb{X}^{\bigtriangledown}} \int_{\mathbb{Y}^{\bigtriangledown}} \delta(\phi_1 \otimes \overline{\phi_2})(z_x+z_y) \cdot \overline{\delta(\phi_3 \otimes \overline{\phi_4})(z_x+z_y)}\, dz_x^{\bigtriangledown} dz_y^{\bigtriangledown} \\
&= |N(R(\underline{e}))|^m \int_{\mathbb{X}^{\bigtriangledown}} \int_{\mathbb{Y}^{\bigtriangledown}} \int_{\mathbb{X}^{\bigtriangleup}} (\phi_1 \otimes \overline{\phi_2})(z_x + x^{\bigtriangleup}) \psi(\langle\langle x^{\bigtriangleup}, z_y\rangle\rangle) \\
&\quad \cdot \overline{\delta(\phi_3 \otimes \overline{\phi_4})(z_x+z_y)}\, dx^{\bigtriangledown} dz_x^{\bigtriangledown} dz_y^{\bigtriangledown} \\
&= |2|^{-2mn} \cdot |N(R(\underline{e}))|^m \int_{\mathbb{X}^{\bigtriangledown}} \int_{\mathbb{X}^{\bigtriangleup}} (\phi_1 \otimes \overline{\phi_2})(z_x + x^{\bigtriangleup}) \cdot \overline{(\phi_3 \otimes \overline{\phi_4})(z_x + x^{\bigtriangleup})}\, dx^{\bigtriangleup} dz_x^{\bigtriangledown} \\
&= |2|^{-2mn} \cdot |N(R(\underline{e}))|^m \int_{\mathbb{X}} \int_{\mathbb{X}} (\phi_1 \otimes \overline{\phi_2})(x, x') \cdot \overline{(\phi_3 \otimes \overline{\phi_4})(x,x')}\, dx dx' \\
&= |2|^{-2mn} \cdot |N(R(\underline{e}))|^m \cdot (\phi_1, \phi_3)_{\mathbb{X}} \cdot \overline{(\phi_2, \phi_4)_{\mathbb{X}}}.
\end{aligned}$$

Thus, we have the proposition. □

9. Local theta correspondence

In this section, we explain the notations and properties of local theta correspondence for quaternionic dual pairs.

9.1. **Definition.** Fix a non-trivial additive character ψ of F. We denote by $\mathrm{Mp}_\psi(\mathbb{W})$ the metaplectic group, and by $\widetilde{j} : G(V) \times G(W) \to \mathrm{Mp}_\psi(\mathbb{W})$ a splitting of the embedding

$$j : G(W) \times G(V) \to \mathrm{Sp}(\mathbb{W}) : (g, h) \mapsto g \otimes h.$$

For an irreducible representation π of $G(W)$, we define $\Theta_\psi(\pi, V)$ as the largest quotient module

$$(\widetilde{j}^* \omega_\psi \otimes \pi^\vee)_{G(W)}$$

of $\widetilde{j}^*\omega_\psi \otimes \pi^\vee$ on which $G(W)$ acts trivially. This is a representation of $G(V)$. We define the theta correspondence $\theta_\psi(\pi, V)$ of π by

$$\theta_\psi(\pi, V) = \begin{cases} 0 & (\Theta_\psi(\pi, V) = 0), \\ \text{the maximal semisimple quotient of } \Theta_\psi(\pi, V) & (\Theta_\psi(\pi, V) \neq 0). \end{cases}$$

Theorem 9.1 (Howe duality)**.** *For irreducible representations π_1, π_2 of $G(W)$, we have*

(1) $\theta_\psi(\pi_1, V)$ is irreducible if it is non-zero,
(2) $\pi_1 \cong \pi_2$ if $\theta_\psi(\pi_1, V) \cong \theta_\psi(\pi_2, V) \neq 0$,
(3) $\theta_\psi(\pi_1, V)^\vee \cong \theta_{\overline{\psi}}(\pi_1^\vee, V)$.

Proof. [GS17, Theorem 1.3]. □

9.2. **Square integrability.** In this subsection, we explain the preservation of the square integrability under the theta correspondence, which is necessary for the setup of the main result. Let π be an irreducible square integrable representation of $G(W)$, and let $\sigma := \theta_\psi(\pi, V)$. In this subsection, we assume that $l = 1$ and $\sigma \neq 0$. We denote by θ the $G(V)$-equivalent and $G(W)$-invariant natural quotient map

$$\omega_\psi \otimes \pi \to \sigma.$$

Let $(\ ,\ )_\pi : \pi \times \pi \to \mathbb{C}$ be a non-zero $G(W)$-invariant Hermitian pairing on π. We define a non-zero $G(V)$-invariant Hermitian pairing $(\ ,\ )_\sigma : \sigma \times \sigma \to \mathbb{C}$ by

$$(\theta(\phi_1, v_1), \theta(\phi_2, v_2))_\sigma := \int_{G(W)} (\omega_\psi(g)\phi_1, \phi_2) \cdot \overline{(\pi(g)v_1, v_2)_\pi}\, dg.$$

Lemma 9.2. *The definition of $(\ ,\ )_\sigma$ does not depend on the choice of ϕ_1, ϕ_2, v_1, v_2. Moreover, σ is a square integrable representation.*

Proof. Similar to [GI14, Appendix D]. □

9.3. **Tower properties.** In this subsection, we discuss some properties related to Witt towers. Let V_0 be a right anisotropic ϵ-Hermitian space. Put $m_0 := \dim_D V_0$. For a non-negative integer t, we define

$$V_t = X_t \oplus V_0 \oplus X_t^*$$

where X_t and X_t^* are t-dimensional right D-vector spaces. Fix a basis $\lambda_1, \ldots, \lambda_t$ for X_t and fix a basis $\lambda_{-1}, \ldots, \lambda_{-t}$ for X_t^*. Then we define an ϵ-Hermitian pairing $(\ ,\ )_t$ on V_t by

$$(\lambda_i, \lambda_{-j})_t = \delta_{ij},\ (\lambda_i, x_0)_t = (x_0, \lambda_{-j})_t = 0,\ (x_0, x_0')_t = (x_0, x_0')_0$$

for $i, j = 1, \ldots, t$ and $x_0, x_0' \in V_0$. Here $(\ ,\ )_0$ is the pairing associated with V_0.

First, we state the conservation relation of Sun-Zhu [SZ15]. Let $V_0^\dagger$ be a right anisotropic ϵ-Hermitian space such that $\chi_{V_0^\dagger} = \chi_{V_0}$ and $V_0^\dagger \not\cong V_0$. Such $V_0^\dagger$ is determined uniquely. Take $\{V_t^\dagger\}_{t\geq 0}$ as the Witt tower containing $V_0^\dagger$. Let π be an irreducible representation of $G(W)$. There is a non-negative integer $r(\pi)$ such that $\Theta_\psi(\pi, V_{r(\pi)}) \neq 0$ and $\Theta(\pi, V_t) = 0$ for $t < r(\pi)$. It is known that $\theta_\psi(\pi, V_{r(\pi)})$ is supercuspidal if π is supercuspidal. We call $r(\pi)$ the first occurrence index for the theta correspondence from π to the Witt tower $\{V_t\}_{t\geq 0}$. Denote by $r^\dagger(\pi)$ the first occurrence index for the theta correspondence from π to $\{V_t^\dagger\}_{t\geq 0}$.

Proposition 9.3. *Let π be an irreducible representation of $G(W)$. Then we have*

$$m(\pi) + m^\dagger(\pi) = 2n + 2 + \epsilon$$

where $m(\pi) = 2r(\pi) + \dim_D V_0$, and $m^\dagger(\pi) = 2r^\dagger(\pi) + \dim_D V_0^\dagger$.

Proof. [SZ15]. □

Then, we explain the behavior of theta correspondence when we change indexes of Witt towers. However, before stating them, we note here the analogue of the Gelfand-Kazhdan Theorem ([BZ76, Theorem 7.3]) for $\mathrm{GL}_r(D)$, which we use in the proof of Proposition 9.5:

Lemma 9.4. *Let τ be an irreducible representation of $\mathrm{GL}_r(D)$, and let τ^θ be the irreducible representation of $\mathrm{GL}_r(D)$ defined by $\tau^\theta(g) = \tau({}^t g^{*-1})$ for $g \in \mathrm{GL}_r(D)$. Then, τ^θ is equivalent to the contragredient representation $\pi^\vee$ of π.*

Proof. See [Rag02, Theorem 3.1]. □

Proposition 9.5. *Let $\{W_t\}_{t\geq 0}$ be a Witt tower of right $(-\epsilon)$-Hermitian spaces.*

(1) Let π be an irreducible representation of $G(W_i)$, and let $\sigma = \theta_\psi(\pi, V_j)$. Suppose that $j \geq r(\pi)$, and we denote by $\sigma_{j'}$ the representation $\theta_\psi(\pi, V_{j'})$ for $r(\pi) \leq j' \leq j$. Then, σ is a subquotient of an induced representation

$$\mathrm{Ind}_{Q_{j',j}}^{G(V_i)} \sigma_{j'} \boxtimes \chi_W |N_{X_{j',j}}|^{l_{i,j}+j-r(\pi)}.$$

Here, $l_{i,j} = 2\dim W_i - 2\dim V_j - \epsilon$, $X_{j',j}$ is a subspace of $X_{j'}$ spanned by $\lambda_{j'+1}, \ldots, \lambda_j$, $N_{X_{j',j}}$ is the reduced norm of $\mathrm{End}(X_{j',j})$, and $Q_{j',j}$ is the parabolic subgroup preserving $X_{j',j}$.

(2) *Let π be an irreducible representation of $G(W_{i'})$, let $\sigma = \theta_\psi(\pi, V_{j'})$, let τ be a non-trivial supercuspidal irreducible representation of $\mathrm{GL}_r(D)$, let s be a complex number, and let π' be an irreducible subquotient of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)}(\pi \boxtimes \tau_s\chi_V)$ where $i = i' + r$ and $P_{i',i}$ is the parabolic subgroup preserving an r-dimensional isotropic subspace of $W_{i'}$. Suppose that $\sigma \neq 0$. Then, we have that $\theta_\psi(\pi', V_j)$ is a subquotient of $\mathrm{Ind}_{Q_{j',j}}^{G(V_j)} \sigma \boxtimes \tau_s\chi_W$. Here, $j = j' + r$, and $\tau_s\chi_W$ is the representation of $\mathrm{GL}_r(D)$ defined by $\tau_s\chi_W(g) = \tau(g)\chi_W(N(g))|N(g)|^s$ for $g \in \mathrm{GL}_r(D)$, where N denotes the reduced norm.*

Proof. These properties are proved by analyzing the Jacquet module of Weil representations: it goes similar line with [Mui06], however we explain for the readers (see also [Han11]). For a while, we denote by $\omega_\psi[j,i]$ the Weil representation associated with the reductive dual pair $(G(V_j), G(W_i))$. Moreover, for a representation ρ of $G(V_j) \times G(W_i)$, for $0 \leq i' \leq i$, and for $0 \leq j' \leq j$, we denote by $J_{j',i'}\rho$ the Jacquet module of ρ with respect to the parabolic subgroup $Q_{j',j} \times P_{i',i}$. Then, by [MVW87], we have a $G(V_{j'}) \times \mathrm{GL}_{j-j'}(D) \times G(W_i)$ equivalent filtration:

$$J_{j',i}(\omega_\psi[j,i]) = R_0 \supset R_1 \supset \cdots \supset R_t \supset R_{t+1} = 0.$$

Here,

$$\begin{aligned} &t = \min\{j - j', i\},\\ &R_0/R_1 = \chi_W |N_{X_{j',j}}|^{l_{i,j}+j-j'} \boxtimes \omega_\psi[j', i],\\ &R_k/R_{k+1} = \mathrm{Ind}_{P_{i-k,i}}^{G(W_i)} \rho_k \text{ for some representation } \rho_k \ (k = 1, \ldots, t-1), \end{aligned}$$

and moreover if $j - j' \leq i$, we have

$$R_t = \mathrm{Ind}_{P_{i',i}}^{G(W_i)} \mathcal{S}(\mathrm{GL}_{j-j'}(D)) \boxtimes \omega_\psi[j', i']$$

where $i' = i - (j - j')$, and the action of $\mathrm{GL}_{j-j'}(D) \times \mathrm{GL}_{i-i'}(D)$ on $\mathcal{S}(\mathrm{GL}_{j-j'}(D))$ is given by

$$[(g_1, g_2) \cdot \varphi](g) = \chi_W(N(g_1))\chi_V(N(g_2))\varphi(g_1^{-1} g g_2)$$

for $g_1 \in \mathrm{GL}_{j-j'}(D), g \in \mathrm{GL}_{j-j'}(D)$, and $g_2 \in \mathrm{GL}_{i-i'}(D)$, where N denotes the reduced norm. Now we prove (1). Composing $J_{j',i}(\omega_\psi[j,i]) \to R_0/R_1$ with the $G(V_{j'}) \times G(W_i)$-equivalent surjection

$$\omega_\psi[j', i] \to \sigma \boxtimes \pi,$$

we have a non-zero morphism

$$J_{j',i}(\omega_\psi[j,i]) \to \chi_W |N_{X_{j',j}}|^{l+j-j'} \boxtimes \sigma \boxtimes \pi.$$

Hence we have (1). Then we prove (2). Let π' be an irreducible component of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi \boxtimes \tau_s\chi_V$. First, we have

$$\mathrm{Hom}(R_k/R_{k+1}, \pi') = \mathrm{Hom}(\rho_k, J_{i-k}\pi').$$

Here, we denote by $J_{i-k}\pi'$ the Jacquet module with respect to the parabolic subgroup $P_{i',i}$. However, since τ is supercuspidal, one can prove $J_{i-k}\,\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi \boxtimes \tau_s\chi_V = 0$ for $k = 1, 2, \ldots, t-1$ by considering the filtration of Bernstein-Zelevinsky ([BZ77, Theorem 5.2]), and thus the right hand side is 0. Hence, we have

$$R_1 \otimes \pi'^\vee \cong R_t \otimes \pi'^\vee.$$

Moreover, since $\tau_s\chi_W \not\cong \chi_W |N_{j',j}|^{l_{i,j}+j-j'}$, we have

$$R_0 \otimes (\tau_s\chi_W)^\vee \cong R_1 \otimes (\tau_s\chi_W)^\vee.$$

On th other hand, the nonzero $\mathrm{GL}_r(D)\times\mathrm{GL}_r(D)$-equivalent map

$$\mathcal{S}(\mathrm{GL}_{j-j'}(D))\otimes((\tau_s\chi_V)^\vee\boxtimes\tau_s\chi_W)\to\mathbb{C}:(\varphi,x,x')\mapsto\int_{\mathrm{GL}_r(D)}\varphi(g)\langle\tau_s(g)x,x'\rangle\chi_W\chi_V^{-1}(N_{j-j'}(g))dg$$

gives a nonzero $\mathrm{GL}_r(D)\times\mathrm{GL}_r(D)$-equivalent map

$$\mathcal{S}(\mathrm{GL}_{j-j'}(D))\otimes(\tau_s\chi_V)^\vee\to(\tau_s\chi_W)^\vee.$$

By combining the above arguments, and by Lemma 9.4, we have a nonzero $G(V_{j'})\times\mathrm{GL}_{j-j'}(D)\times G(W_i)$-equivalent map

$$\begin{aligned}
&J_{j',i}(\omega_\psi[j,i])\otimes(\sigma\boxtimes\tau_s\chi_W)^\vee\otimes(\pi')^\vee\\
&=R_t\otimes(\sigma\boxtimes\tau_s\chi_W)^\vee\boxtimes(\pi')^\vee\\
&=(\mathrm{Ind}_{P_{i',i}}^{G(W_i)}\mathcal{S}(\mathrm{GL}_{j-j'}(D))\boxtimes\omega_\psi[i',j'])\otimes(\sigma\boxtimes\tau_s\chi_W)^\vee\boxtimes(\pi')^\vee\\
&\to(\mathrm{Ind}_{P_{i',i}}^{G(W_i)}(\tau_s\chi_V)^\vee\boxtimes\pi)\otimes(\pi')^\vee\\
&\cong(\mathrm{Ind}_{P_{i',i}}^{G(W_i)}(\tau^\theta\chi_V)_{-s}\boxtimes\pi)\otimes(\pi')^\vee\\
&\cong(\mathrm{Ind}_{P_{i',i}}^{G(W_i)}(\tau_s\chi_V)\boxtimes\pi)\otimes(\pi')^\vee\\
&\to\mathbb{C}.
\end{aligned}$$

Hence we have (2). □

By the proof of Proposition 9.5, we also have a slightly different property:

Corollary 9.6. *Let $\{W_t\}_{t\geq 0}$ be a Witt tower of right $(-\epsilon)$-Hermitian spaces, let i,j,j' be non-negative integers so that $j-j'>0$, let π be an irreducible representation of $G(W_i)$, let $\sigma=\theta_\psi(\pi,V_j)$. Suppose that $\sigma\neq 0$, and σ is a subrepresention of an induced representation $\mathrm{Ind}_{Q_{j',j}}^{G(V_j)}\sigma'\boxtimes\tau_s\chi_W$ where σ' is an irreducible representation of $G(V_{j'})$, τ is an irreducible supercuspidal representation of $\mathrm{GL}_{j-j'}(D)$, and $s\in\mathbb{C}$. Moreover, we suppose that $\theta_\psi(\pi,V_{j'})=0$. Then, we have $i\geq j-j'$, and there exists an irreducible representation π' of $G(W_{i'})$ such that $\theta_\psi(\pi',V_{j'})\cong\sigma'$. Here we put $i'=i-(j-j')$. Moreover, π is an irreducible subquotient of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)}\pi'\boxtimes\tau_s\chi_W$.*

Proof. We use the notation of the proof of Proposition 9.5. Since there is a non-zero $G(V_j)\times G(W_i)$-equivalent map

$$\omega_\psi[j,i]\to\sigma\boxtimes\pi,$$

by the Frobenius reciprocity, we have a non-zero $G(V_{j'})\times\mathrm{GL}_{j-j'}(D)\times G(W_i)$-equivalent map

$$(\tau_s\chi_W)^\vee\boxtimes\pi^\vee\otimes J_{j',i}\omega_\psi[j,i]\to\sigma'. \tag{9.1}$$

Then, the assumption $\theta_\psi(\pi,V_{j'})=0$ implies that

$$\pi^\vee\otimes R_0/R_1=0.$$

Moreover, as in the proof of Proposition 9.5 (2), we have

$$(\tau_s\chi_W)^\vee\boxtimes\pi^\vee\otimes J_{j',i}\omega_\psi[j,i]=(\tau_s\chi_W)^\vee\boxtimes\pi^\vee\otimes R_{j-j'}.$$

(Here, we put $R_k=0$ for $k>t$.) Thus, $R_{i-i'}$ is forced not to be zero, and we have $i\geq j-j'$. By using the Frobenius reciprocity again, we have a nonzero $G(V_{j'})\times\mathrm{GL}_{j-j'}(D)\times G(W_{i'})\times\mathrm{GL}_{i-i'}(D)$-equivalent map

$$((\tau_s\chi_W)^\vee\boxtimes(J_{i',j}\pi)^\vee)\otimes(\mathcal{S}(\mathrm{GL}_{i-i'}(D))\boxtimes\omega_\psi[j',i'])\to\sigma'.$$

Thus, $\sigma'^{\vee} \otimes \omega_\psi[j', i'] \neq 0$. Put $\pi' := \theta_\psi(\sigma', W_{i'})$. Then, $\theta_\psi(\sigma, W_i)$ is nonzero, and it is an irreducible subquotient π'' of $\mathrm{Ind}_{P_{i',i}}^{G(W_i)} \pi' \boxtimes \tau_s\chi_V$. However, by the Howe duality (Theorem 9.1), π'' coincides with π. Thus we have the corollary. □

10. The local Siegel-Weil formula

In this section, we state the local Siegel-Weil formula, which is a local analogue of the (bounded and first term) Siegel-Weil formula. **We assume $l = 1$ and $l > 0$ in this section**.

10.1. **The map $\mathcal{I}$.** We define the $\Delta G(W^\square) \times G(V) \times G(V)$-invariant map

$$\mathcal{I} : \omega_\psi^\square \otimes \overline{\omega_\psi^\square} \to \mathbb{C}$$

by

$$\mathcal{I}(\phi, \phi') = \int_{G(V)} (\omega^\square(h)\phi, \phi')\, dh$$

for $\phi, \phi' \in \omega_\psi^\square$ where $(\ ,\ )$ is the L^2-norm of $\mathcal{S}(V \otimes W^\triangledown)$ as in Proposition 8.3. Note that the integral defining $\mathcal{I}(\ ,\ )$ converges absolutely by [Li89, Theorem 3.2].

10.2. **The map $\mathcal{E}$.** We denote by $V^\flat$ the opposite space of V, that is, an ϵ-Hermitian space such that $\dim_D V^\flat = 2n - m - \epsilon$ and $\chi_{V^\flat} = \chi_V$. (It exists because of the assumption in this section.) Consider the $G(W^\square)$-invariant map

$$\mathcal{S}(V \otimes W^\triangledown) \to I(-\frac{1}{2}, \chi_V) : \phi \mapsto F_\phi$$

defined by $F_\phi(g) = [\omega_\psi^\square(1, g)\phi](0)$ for $\phi \in$ and $g \in G(W^\square)$. Similarly, there is a $G(W^\square)$-invariant map $\mathcal{S}(V^\flat \otimes W^\triangledown) \to I(\frac{1}{2}, \chi_V)$. We denote by $R^W(V)$ and $R^W(V^\flat)$ the images of the above maps respectively. Then we have the following exact sequence:

$$0 \longrightarrow R^W(V^\flat) \longrightarrow I(\tfrac{1}{2}, \chi_V) \xrightarrow{M(\frac{1}{2}, \omega)} R^W(V) \longrightarrow 0$$

([Yam11, Proposition 7.6]). For $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, we denote by $F_\phi^\dagger \in I(-\frac{1}{2}, \chi_V)$ a section such that $M(\frac{1}{2}, \chi_V)F_\phi^\dagger = F_\phi$. Then, we define the map $\mathcal{E}$ by

$$\mathcal{E}(\phi, \phi') = \int_{G(W)} F_\phi^\dagger(\iota(g, 1)) \cdot \overline{F_{\phi'}(\iota(g, 1))}\, dg.$$

The integral defining $\mathcal{E}$ converges absolutely, and the definition of $\mathcal{E}(\phi, \phi')$ does not depend on the choice of $F_\phi^\dagger$ by the following lemma:

Lemma 10.1. *If $f \in R^W(V^\flat)$ and $h \in R^W(V)$, then we have*

$$\int_{G(W)} f(\iota(g, 1)) \cdot \overline{h(\iota(g, 1))}\, dg = 0.$$

Proof. By the proof of Lemma 7.8, we have

$$\mathrm{Hom}_{G(V)\times G(V)}(I(\rho, 1), \mathbb{C}) = \mathrm{Hom}_{G(V^\square)}(I(\rho, 1), \mathbb{C}) = Z \cdot \mathbb{C}$$

where

$$Z(F) = \int_{G(V)} F(\iota(g, 1))\, dg$$

for $F \in I(\rho, 1)$. Thus, if there are $f \in R^W(V^\flat), h \in R^W(V)$ so that $Z(f \cdot \overline{h}) \neq 0$, we would have $R^W(V^\flat) \cong \overline{R^W(V)}^\vee$. Since $\overline{I(-\frac{1}{2}, \chi_V)} \cong I(-\frac{1}{2}, \chi_V)$, we have $\overline{R^W(V)} \cong R^W(V)$. Put $\sigma := R^W(V^\flat)$. Then, we have

$$\Theta(\sigma, V^\flat) = 1_{V^\flat},\ \Theta(\sigma, V) = 1_V.$$

However, according to the conservation relation (Proposition 9.3), either of them must vanish since $\dim V + \dim V^\flat = 2n - \epsilon$. This is a contradiction, and we have the lemma. □

10.2.1. *Local Siegel-Weil formula.* The following lemma gives the definition of $\alpha_2(V, W)$, which is the second constant we are interested in.

Lemma 10.2. *There is a non-zero constant $\alpha_2(V, W)$ such that $\mathcal{I} = \alpha_2(V, W) \cdot \mathcal{E}$.*

Proof. The two maps $\mathcal{I}$, $\mathcal{E}$ are $\Delta G(W) \times G(V) \times G(V)$-invariant map. On the other hand, we have

$$\dim \mathrm{Hom}_{\Delta G(W) \times G(V) \times G(V)}(\omega_\psi^\square \otimes \overline{\omega_\psi^\square}, \mathbb{C}) = \dim \mathrm{Hom}_{\Delta G(W)}(R^W(V) \otimes \overline{R^W(V^\flat)}, \mathbb{C}) = 1.$$

Moreover, we have $\mathcal{I}(\phi, \phi) > 0$ for positive functions $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, and $\mathcal{E} \neq 0$. Hence, we have the proposition. □

We will determine the constant $\alpha_2(V, W)$ completely in §19. However we calculate $\alpha_2(V, W)$ directly when either V or W is anisotropic, which will be proved in §§12-13:

Proposition 10.3. *(1) Suppose that $-\epsilon = 1$ and V is anisotropic, then we have*

$$\alpha_2(V, W) = |N(R(\underline{e}))|^{n+\frac{1}{2}} \times \begin{cases} |2|_F^{-\frac{5}{2}} \cdot (1 + q^{-1}) & (m = 1,\ \chi_V:\ \textit{unramified}), \\ |2|_F^{-\frac{5}{2}} \cdot q^{-\frac{1}{2}} & (m = 1,\ \chi_V:\ \textit{ramified}), \\ |2|_F^{-7} \cdot q^{-2}(1 + q^{-2}) & (m = 2,\ \chi_V:\ \textit{unramified}), \\ |2|_F^{-7} \cdot q^{-\frac{5}{2}} \cdot (1 + q^{-1}) & (m = 2,\ \chi_V:\ \textit{ramified}), \\ |2|_F^{-\frac{27}{2}} \cdot q^{-6}(1 + q^{-1})(1 + q^{-2}) & (m = 3,\ \chi_V = 1). \end{cases}$$

(2) Suppose that $-\epsilon = -1$ and either V or W is anisotropic, then we have

$$\alpha_2(V, W) = |N(R(\underline{e}))|^{n-\frac{1}{2}} \times \begin{cases} |2|_F^{-3} \cdot q^{-1} \cdot (1 + q^{-1}) & (n = 2,\ \chi_W = 1), \\ |2|_F^{-3} \cdot q^{-1} \cdot (1 + q^{-1}) & (n = 2,\ \chi_W:\ \textit{ramified}), \\ |2|_F^{-3} \cdot q^{-1} \cdot (1 + q^{-1}) & (n = 2,\ \chi_W \neq 1:\ \textit{unramified}), \\ -|2|_F^{-\frac{15}{2}} q^{-4} \cdot \frac{(1+q^{-1})(1-q^{-4})}{1-q^{-3}} & (n = 3,\ \chi_W = 1). \end{cases}$$

11. Formal degrees and local theta correspondence

In this section, we state the behavior of the formal degree under the local theta correspondence, which extends the result of Gan-Ichino [GI14]. Let G be a connected reductive group over F, and let π be a square integrable irreducible representation of G. Then, the formal degree is a number $\deg \pi$ satisfying

$$\int_{G/A_G} (\pi(g)v_1, v_2) \cdot \overline{(\pi(g)v_3, v_4)}\, dg = \frac{1}{\deg \pi}(v_1, v_3) \cdot \overline{(v_2, v_4)}$$

for $v_1, \ldots, v_4 \in \pi$, where A_G is the maximal F-split torus of the center of G.

Again, we consider a right m-dimensional ϵ-Hermitian space and a left n-dimensional $(-\epsilon)$-Hermitian space. In this section, **we assume that** $l = 1$. The purpose of this section is to describe the behavior of the formal degree under the theta correspondence for the quaternionic dual pair $(G(V), G(W))$. Now we state our main theorem:

Theorem 11.1. *Let π be an irreducible square integrable representation of $G(W)$, and let $\sigma = \theta_\psi(\pi, V)$. Assume that $\sigma \neq 0$. Then, we recall that σ is also square integrable. Now, we define $\alpha_3(V, W)$ as the constant satisfying*

$$\frac{\deg \pi}{\deg \sigma} = \alpha_3(V, W)\omega_\pi(-1)\gamma^V(0, \sigma \times \chi_W, \psi).$$

Then, $\alpha_3(V, W)$ depends only on V, W and ψ. Moreover, we have

$$\alpha_3(V, W) = \begin{cases} (-1)^n\chi_V(-1)\epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ \frac{1}{2} \cdot \chi_W(-1)\epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases}$$

We prove Theorem 11.1 in later sections. In this section, we see an example:

Example 11.2. *Consider the case where $\epsilon = 1$, $m = 1$, $n = 2$, and $\chi_W = 1$. We denote by* St *the Steinberg representation of $G(W)$. Then, it is known that $\theta_\psi(\mathrm{St}, V)$ is the trivial representation 1_V of $G(V)$. The local Langlands correspondence for $G(W)$ has been established (see* [Cho17, §5]*) and the L-parameter of* St *is the principal parameter of $\widehat{G}$ (see e.g.* [GR10, §3.3]*). Then, as representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$, we have*

$$\mathrm{Ad} \circ \phi_0 = (1 \otimes r_3) \oplus (1 \otimes r_3)$$

where 1 is the trivial representation of W_F, and r_3 is the unique three dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Thus, we have

$$\gamma(s + \frac{1}{2}, \mathrm{St}, \mathrm{Ad}, \psi) = q^{-4s} \cdot \frac{\zeta_F(-s + \frac{3}{2})^2}{\zeta_F(s + \frac{3}{2})^2}.$$

Moreover, the centralizer $C_{\phi_0}(\widehat{G})$ of $\mathrm{Im}\,\phi_0$ in $\widehat{G}$ is $\{\pm 1\} \subset \widehat{G}$, and the component group $\widetilde{S}_{\phi_0}(\widehat{G})$ is abelian. Since the formal degree conjecture for $G(W)$ is available (see §20 below), we have

$$\deg \mathrm{St} = \frac{1}{2} \cdot \frac{q^2}{(1 + q^{-1})^2}.$$

On the other hand, we have

$$\deg 1_V = |G(V)|^{-1} = \frac{q}{1 + q^{-1}}.$$

(Recall that the volume $|G(V)|$ of $G(V)$ is given by Corollary 6.3.) Therefore, by Lemma 7.5, we have

$$\frac{\deg \mathrm{St}}{\deg 1_V} = \frac{1}{2} \cdot \gamma(0, 1_V \boxtimes 1, \psi)$$

which agrees with Theorem 11.1.

We note here the strategy of the proof of the theorem. At first, we consider the case where either W or V is anisotropic (i.e. the minimal cases in the sense of the parabolic induction). In theses cases, we can express $\alpha_2(V, W)$ with $\alpha_1(W)$ which is already determined in §7.3. And hence we obtain Proposition 10.3 (§§12-13). Second, we relate $\alpha_3(V, W)$ with $\alpha_2(W)$ (§§14-15). Then we have Theorem 11.1 in the minimal cases. And finally, we prove that the constant $\alpha_3(V, W)$ is compatible with parabolic inductions (§§16-18), which completes the proof of Theorem 11.1. We also note here that once $\alpha_3(V, W)$ is determined, the above processes can be reversed to obtain the general formula for $\alpha_1(W)$ and $\alpha_2(W)$ (§19).

Remark 11.3. *We remark on the local γ-factors. As written in* [Kak20b, §5.3]*, the definition of the doubling γ-factor of Lapid-Rallis* [LR05] *should be modified by a constant multiple. Thus, it is natural to ask whether the statement of the main theorem of* [GI14] *might change. However,* [GI14, Theorem 15.1] *is still true. This is because that their proof uses the doubling γ-factor not to determine the "constant $\mathcal{C}$" (see* [GI14, §20.2]*) but to show the existence of the constant $\mathcal{C}$. Hence, the difference of constant multiples is offset at the time of calculation of $\mathcal{C}$.*

12. Minimal cases (I)

In this section, we determine the constant $\alpha_2(V,W)$ and prove Proposition 10.3 (1).

Suppose that $\epsilon = 1$, $V_0 = 0$ and $\dim_D V = 2$. Then, we can take a basis $\underline{e}^V = (e_1^V, e_2^V)$ of V so that

$$(e_1^V, e_1^V) = (e_2^V, e_2^V) = 0, \text{ and } (e_1^V, e_2^V) = 1.$$

We take bases $\underline{e}$ of W and $\underline{e}'^{\square}$ of $W^{\square}$ as in §7.1.

Let $\mathcal{L}$ be a lattice

$$\left(\bigoplus_{i=1}^n e_1^V \varpi_D^{-1}\mathcal{O}_D \otimes e_i'\right) \oplus \left(\bigoplus_{i=1}^n e_2^V \mathcal{O}_D \otimes e_i'\right)$$

of $V \otimes W^{\triangle}$, and we denote by $1_{\mathcal{L}}$ the characteristic function of $\mathcal{L}$. Note that we have $|\mathcal{L}| = 1$ since $\mathcal{L}$ is self-dual.

Lemma 12.1. *We have*

$$\mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = q^{-2}\frac{(1-q^{-2})(1+q^{-2})(1+q^{-5})}{1-q^{-3}}.$$

Proof. Let $\mathcal{B}$ be the Iwahori subgroup given by

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(V) \;\middle|\; a,b,d \in \mathcal{O}_D, c \in \varpi_D \mathcal{O}_D \right\}.$$

Note that $|\mathcal{B}| = q^{-4}(1-q^{-2})$ and that $\mathcal{B}$ fixes the lattice $\mathcal{L}$. By [BT72, Théorèm 5.1.3], we have $G(V) = \mathcal{B}\cdot\mathcal{N}\cdot\mathcal{B}$ where $\mathcal{N}$ is the normalizer of the maximal F-split torus consisting of the diagonal matrices in $G(V)$. Moreover, we can take a system of representatives

$$\{a(t) \mid t \in \mathbb{Z}\} \cup \{w(t) \mid t \in \mathbb{Z}\}$$

for $\mathcal{B}\backslash G(V)/\mathcal{B}$, where

$$a(t) = \begin{pmatrix} \varpi_D^t & 0 \\ 0 & (-\varpi_D)^{-t} \end{pmatrix} \text{ and } w(t) = \begin{pmatrix} 0 & \varpi_D^t \\ \epsilon(-\varpi_D)^{-t} & 0 \end{pmatrix}.$$

Hence we have

$$\begin{aligned}
\mathcal{I}(1_{\mathcal{L}}, 1_{\mathcal{L}}) &= |\mathcal{B}| \cdot \sum_{t\in\mathbb{Z}} (|\mathcal{L}\cap a(t)\mathcal{L}| \cdot [\mathcal{B}a(t)\mathcal{B} : \mathcal{B}] + |\mathcal{L}\cap w(t)\mathcal{L}| \cdot [\mathcal{B}w(t)\mathcal{B} : \mathcal{B}]) \\
&= |\mathcal{B}| \cdot \sum_{t\in\mathbb{Z}} (q^{-3|t|} + q^{-6|t-1|+|1+3t|}) \\
&= q^{-4}(1-q^{-2}) \cdot (\frac{1+q^{-3}}{1-q^{-3}} + \frac{q^2+q^{-5}}{1-q^{-3}}) \\
&= q^{-2}\frac{(1-q^{-2})(1+q^{-2})(1+q^{-5})}{1-q^{-3}}.
\end{aligned}$$

Hence we have the lemma. □

Lemma 12.2. *We have*

$$\mathcal{E}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = m^\circ(\frac{1}{2})^{-1} \cdot \alpha_1(W)$$

where $m^\circ(s)$ is a function as in Lemma 7.8.

Proof. Note that $1_{\mathcal{L}}$ is a $K(\underline{e}'^{\square})$ fixed function with $1_{\mathcal{L}}(0) = 1$. Thus, we have $\mathcal{F}_{1_{\mathcal{L}}} = f^\circ_{-\frac{1}{2}}$ putting f°_s the unique $K(\underline{e}'^{\square})$ fixed section in $I(-\frac{1}{2}, 1)$ with $f^\circ_s(1) = 1$. By the Gindikin-Karperevich formula (see e.g. [Cas89]), we can take $\mathcal{F}^\dagger_{1_{\mathcal{L}}} = m^\circ(\frac{1}{2})^{-1} f^\circ_{\frac{1}{2}}$. Hence, we have

$$\begin{aligned}\mathcal{E}(1_{\mathcal{L}}, 1_{\mathcal{L}}) &= m^\circ(\frac{1}{2})^{-1} \int_{G(W)} f^\circ_\rho(\iota(g,1))\, dg \\ &= m^\circ(\frac{1}{2})^{-1} \cdot \alpha_1(W).\end{aligned}$$

□

Hence, by the above two lemmas, we have:

Proposition 12.3. *If $\epsilon = 1$, $V_0 = 0$ and $\dim_D V = 2$, then we have*

$$\alpha_2(V, W) = -|2|_F^{-\frac{15}{2}} \cdot |N(R(\underline{e}))|^{\frac{5}{2}} \cdot q^{-4} \cdot \frac{(1+q^{-1})(1-q^{-4})}{1-q^{-3}}.$$

13. Minimal cases (II)

In this section, we determine the constant $\alpha_2(V, W)$ and prove Proposition 10.3 (2).

Suppose that V is anisotropic. Recall that $\tau \in G(W^\square)$ is the Weyl element as in §7.1. For $\Phi \in \mathcal{S}(\mathfrak{u}_n)$, we define a section $f(s, \Phi, -) \in I(s, \chi_V)$ by

$$f(s, \Phi, g) := \begin{cases} 0 & (g \notin P(W^\triangle)\tau U(W^\triangle)), \\ \chi_{V, s+\rho}(\Delta(p)) \cdot \Phi(X) & (g = p\tau \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \in P(W^\triangle)\tau U(W^\triangle). \end{cases}$$

Here $G(W^\square)$ is embedded in $\mathrm{GL}_{2n}(D)$ by the basis $\underline{e}'^{\square}$. For $t \in \mathbb{Z}, \phi \in \mathcal{S}(V \otimes W^\triangledown)$, and $\Phi \in \mathcal{S}(\mathfrak{u}_n)$, we define $\phi_t \in \mathcal{S}(V \otimes W^\triangledown)$, and $\Phi_t \in \mathcal{S}(\mathfrak{u}_n)$ by

$$\phi_t(x) := q^{-4mnt}\phi(x\varpi_F^t), \text{ and } \Phi_t(X) := q^{-4n\rho t}\Phi(X\varpi^{2t}).$$

Then we have the following lemma:

Lemma 13.1. *(1) For $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, we have $\widehat{\phi_t} = q^{-4mnt}(\widehat{\phi})_{-t}$.*

(2) Let $\phi \in \mathcal{S}(V \otimes W^\triangledown)$, and let $\Phi \in \mathcal{S}(\mathfrak{u}_n)$ such that $M(\frac{1}{2}, \chi_V) f(\frac{1}{2}, \Phi, -) = F_\phi$. Then we have

$$M(\frac{1}{2}, \chi_V) f(\frac{1}{2}, \Phi_t, -) = q^{-4mnt} F_{\phi_{-t}}.$$

Proof. We have

$$\begin{aligned}\widehat{\phi_t}(x) &= \int_{V \otimes W^\triangledown} q^{-4mnt}\phi(y\varpi^t)\psi(\frac{\epsilon}{2}\langle\langle x, y\tau\rangle\rangle)\, dy \\ &= \int_{V \otimes W^\triangledown} \phi(y)\psi(\frac{\epsilon}{2}\langle\langle x\varpi^{-t}, y\tau\rangle\rangle)\, dy \\ &= q^{-4mnt}(\widehat{\phi})_{-t}(x).\end{aligned}$$

Hence we have (1).

$$\begin{aligned}
M(\frac{1}{2},\chi_V)f(\frac{1}{2},\Phi_t,\tau\begin{pmatrix}1&0\\X&1\end{pmatrix}) &= \int_{\mathfrak{u}} f(\frac{1}{2},\Phi_t,\tau\begin{pmatrix}1&0\\Y&1\end{pmatrix}\tau\begin{pmatrix}1&0\\X&1\end{pmatrix})\,dY\\
&= \int_{\mathfrak{u}} f(\frac{1}{2},\Phi_t,\begin{pmatrix}Y&0\\-\epsilon&\epsilon\cdot Y^{-1}\end{pmatrix}\tau\begin{pmatrix}1&0\\-\epsilon Y^{-1}+X&1\end{pmatrix})\,dY\\
&= q^{-4n\rho t}\int_{\mathfrak{u}} \chi_{\frac{1}{2}+\rho}(N(Y))^{-1} f(\frac{1}{2},\Phi,\tau\begin{pmatrix}1&0\\(-\epsilon Y^{-1}+X)\varpi^{2t}&1\end{pmatrix}))\,dY\\
&= q^{-4mnt}\int_{\mathfrak{u}} \chi_{\frac{1}{2}+\rho}(N(Y))^{-1} f(\frac{1}{2},\Phi,\tau\begin{pmatrix}1&0\\-\epsilon Y^{-1}+X\varpi^{2t}&1\end{pmatrix}))\,dY\\
&= q^{-4mnt}M(\frac{1}{2},\chi_V)f(\frac{1}{2},\Phi,\tau\begin{pmatrix}1&0\\X\varpi^{2t}&1\end{pmatrix})\,dY\\
&= q^{-4mnt}F_\phi(\tau\begin{pmatrix}1&0\\X\varpi^{2t}&1\end{pmatrix})\\
&= q^{-4mnt}\beta_V(\tau)\int_{V\otimes W^\triangledown}\phi(x)\psi(\frac{1}{4}\langle\langle x,x\begin{pmatrix}1&0\\X\varpi^{2t}&1\end{pmatrix}\rangle\rangle)\,dx\\
&= \beta_V(\tau)\int_{V\otimes W^\triangledown}\phi(x\varpi^{-t})\psi(\frac{1}{4}\langle\langle x,x\begin{pmatrix}1&0\\X&1\end{pmatrix}\rangle\rangle)\,dx\\
&= q^{-4mnt}F_{\phi_{-t}}(\tau\begin{pmatrix}1&0\\X&1\end{pmatrix}).
\end{aligned}$$

Hence we have (2). □

Proposition 13.2. *Let $\phi,\phi'\in\mathcal{S}(V\otimes W^\triangle)$. Then, for sufficiently large $t\in\mathbb{Z}$, we have*

$$\mathcal{I}(\phi_t,\phi') = q^{-4mnt}|G(V)|F_\phi(1)\overline{F_{\phi'}(\tau)}.$$

Proof. The Fourier transform on the space $\mathcal{S}(V\otimes W^\triangle)$ is given by the action of the Weyl element τ of $G(W^\square)$. Hence we have

$$\begin{aligned}
\mathcal{I}(\phi_t,\phi') &= \mathcal{I}(\widehat{\phi_t},\widehat{\phi'})\\
&= q^{-4mnt}\mathcal{I}((\widehat{\phi})_{-t},\widehat{\phi'})\\
&= q^{-4mnt}\int_{G(V)}((\widehat{\phi})_{-t},\overline{\omega^\square_\psi(h)\widehat{\phi'}})\,dh.
\end{aligned}$$

When t is sufficiently large, the support of $(\widehat{\phi})_{-t}$ is sufficiently small. Hence this integral is

$$\begin{aligned}
&q^{-4mnt}|G(V)|\widehat{(\widehat{\phi})_{-t}}(0)\overline{\widehat{\phi'}(0)}\\
&= q^{-4mnt}|G(V)|q^{4mnt}(\widehat{\widehat{\phi}})_t(0)\overline{\widehat{\phi'}(0)}\\
&= q^{-4mnt}|G(V)|\phi_t(0)\overline{\widehat{\phi'}(0)}\\
&= q^{-4mnt}|G(V)|F_\phi(1)\overline{F_{\phi'}(\tau)}.
\end{aligned}$$

Hence we have the proposition. □

Proposition 13.3. *Let $\phi,\phi'\in\mathcal{S}(V\otimes W^\triangle)$. Then, for sufficiently large $t\in\mathbb{Z}$, we have*

$$\mathcal{E}(\phi_t,\phi') = m^\circ(\rho)^{-1}\alpha_1(W)q^{-4mnt}F_\phi(1)\overline{F_{\phi'}(\tau)}.$$

Proof. When t is sufficiently large, the support of Φ_{-t} is sufficiently small. Then, by using Lemma 7.8, we have

$$\begin{aligned}\mathcal{E}(\phi_t, \phi') &= q^{-4mnt} \int_{G(W)} f(\frac{l}{2}, \Phi_{-t}, (g,1)) \overline{F_{\phi'}(g,1)}\, dg \\ &= m^\circ(\rho)^{-1}\alpha_1(W) q^{-4mnt} \int_{U(W^\triangle)} f(\frac{l}{2}, \Phi_{-t}, \tau u) \overline{F_{\phi'}(\tau u)}\, du \\ &= m^\circ(\rho)^{-1}\alpha_1(W) q^{-4mnt} \left(\int_{\mathfrak{u}_n} \Phi_{-t}(X)\, dX \right) \overline{F_{\phi'}(\tau)} \\ &= m^\circ(\rho)^{-1}\alpha_1(W) q^{-4mnt} F_\phi(1) \overline{F_{\phi'}(\tau)}.\end{aligned}$$

Hence we have the proposition. □

By Propositions 13.2 and 13.3, we have the following:

Proposition 13.4. *If V is anisotropic, then we have*

$$\alpha_2(V, W) = |G(V)| \cdot m^\circ(\rho) \cdot \alpha_1(W)^{-1}.$$

By Proposition 12.3 and this proposition, we obtain Proposition 10.3.

14. The behavior of the γ-factor under the local theta correspondence

The purpose of this section is to explain the behavior of the γ-factor under the local theta correspondence, which extends [GI14, Theorem 11.5]. Let V be a right ϵ-Hermitian space of dimension m, let W be a left $(-\epsilon)$-Hermitian space of dimension n. Note that, in this section, we allow the case where $l \neq 1$.

Theorem 14.1. *Let π be an irreducible representation of $G(W)$ and let ω be a character of $F^\times$. We denote $\sigma = \theta(\pi, V)$ and we assume $\sigma \neq 0$.*

(1) If $l > 0$, then we have

$$\frac{\gamma^V(s, \sigma \times \omega\chi_V, \psi)}{\gamma^W(s, \pi \times \omega\chi_W, \psi)} = \prod_{i=1}^{l} \gamma_F(s + \frac{l+1}{2} - i, \omega\chi_V\chi_W, \psi)^{-1}.$$

(2) If $l < 0$, then we have

$$\frac{\gamma^V(s, \sigma \times \omega\chi_V, \psi)}{\gamma^W(s, \pi \times \omega\chi_W, \psi)} = \prod_{i=1}^{-l} \gamma_F(s + \frac{-l+1}{2} - i, \omega\chi_V\chi_W, \psi).$$

The proof of Theorem 14.1 consists of four subsections (§§14.1-14.4). In the former three subsections, we reduce the theorem to the unramified cases by using properties of the doubling γ-factor. In the last subsection, we discuss the unramified cases to finish the proof of the theorem.

14.1. Multiplicative argument.

We put

$$f_D(s, V, W, \omega, \psi) = \begin{cases} \prod_{i=1}^{l} \gamma_F(s + \frac{l+1}{2} - i, \omega\chi_V\chi_W, \psi)^{-1} & (l > 0), \\ \prod_{i=1}^{-l} \gamma_F(s + \frac{-l+1}{2} - i, \omega\chi_V\chi_W, \psi) & (l < 0), \end{cases}$$

and we put

$$e_D(s, V, W, \pi, \omega, \psi) = \frac{\gamma^W(s, \sigma \times \omega\chi_W, \psi)}{\gamma^V(s, \pi \times \omega\chi_V, \psi)} \cdot f_D(s, V, W, \omega, \psi).$$

Then, Theorem 14.1 is equivalent to $e_D(s, V, W, \pi, \omega, \psi) = 1$. We allow D to be split. Of course, we have $e_D(s, V, W, \pi, \omega, \psi) = 1$ by [GI14, Theorem 11.5] when D is split.

Let $\{V_p\}_{p\geq 0}$ and $\{W_q\}_{q\geq 0}$ be Witt towers containing V and W respectively. Put $V = V_r, W = W_t, m_0 = \dim_D V_0, m_0 = \dim_D W_0$, and $l_p = l_{V_p,W}$. We denote by $\mathcal{J}_q(\pi)$ the set of the $G(W_0)$-part of the non-zero irreducible quotients of the Jacquet modules $J_P(\pi)$ for all parabolic subgroup P whose Levi subgroups contain $G(W_q)$ as a direct factor. We first state the multiplicativity:

Lemma 14.2. *We denote by $r(\pi)$ the first occurrence index of π (see §9.3). Suppose that $\mathcal{J}_q(\pi) \neq \varnothing$ and $p \geq r(\pi)$. Then, for an irreducible representation $\pi_q \in \mathcal{J}_q(\pi)$, we have*

$$e_D(V_p, W_q, \pi_q, \omega, \psi) = e_D(V, W, \pi, \omega, \psi).$$

Proof. First, we consider the case $q = t$ and $\pi_q = \pi$. We may assume that $r = r(\pi)$. Put $\sigma = \theta_\psi(\pi, V)$ and $\sigma' = \theta_\psi(\pi, V_p)$. Then, by Proposition 9.5 (1), we have

$$\begin{aligned}
&\gamma^{V_p}(s, \sigma' \times \omega, \psi) \cdot \gamma^V(s, \sigma \times \omega, \psi)^{-1} \\
&= \gamma^{GJ}_{\mathrm{GL}_{p-r}(D)}(s + (\frac{l_p}{2} + p - r), \omega, \psi) \cdot \gamma^{GJ}_{\mathrm{GL}_{p-r}(D)}(s - (\frac{l_p}{2} + p - r), \omega, \psi) \\
&= \prod_{i=1}^{p-r} \gamma^{GJ}_{D^\times}(s + \frac{l_p}{2} + (2i-1), \omega, \psi) \cdot \prod_{i=1}^{p-r} \gamma^{GJ}_{D^\times}(s - (\frac{l_p}{2} + (2i-1), \omega, \psi) \\
&= \prod_{i=1}^{2(p-r)} \gamma_F(s + \frac{l_p - 1}{2} + i, \omega, \psi) \cdot \prod_{i=1}^{2(p-r)} \gamma_F(s + \frac{-l_p + 1}{2} - i, \omega, \psi) \\
&= f_D(V_p, W, \pi, \omega, \psi) f_D(V, W, \pi, \omega, \psi)^{-1}.
\end{aligned}$$

Here, $\gamma^{GJ}_{\mathrm{GL}_u(D)}(s, \omega, \psi)$ is the γ-factor defined by

$$\epsilon^{GJ}_{\mathrm{GL}_u(D)}(s, \omega, \psi) \frac{L^{GJ}_{\mathrm{GL}_u(D)}(1 - s, \omega^{-1})}{L^{GJ}_{\mathrm{GL}_u(D)}(s, \omega)}$$

where $\epsilon^{GJ}_{\mathrm{GL}_u(D)}(s, -, \psi)$ and $L^{GJ}_{\mathrm{GL}_u(D)}(s, -)$ are ϵ-and L-factors defined in [GJ72], and ω denotes the composition $\omega \circ N$ of ω with the reduced norm N of $\mathrm{GL}_u(D)$. Thus we have

$$e_D(V_p, W, \pi, \omega, \psi) = e_D(V, W, \pi, \omega, \psi).$$

Second, we consider the general case. Put

$$t(\pi_q) = \min\{q' = 0, \dots, q \mid \mathcal{J}_{q'}(\pi_q) \neq \varnothing\}.$$

Then, any $\pi_{t(\pi)} \in \mathcal{J}_{t(\pi_q)}(\pi_q)$ is supercuspidal. Take a positive integer p' so that $p' \geq \max\{r + q - t, r(\pi) + q - t(\pi)\}$. Then, by the first part of this proof, we have

$$e_D(s, V_p, W_q, \pi_q, \omega, \psi) = e_D(s, V_{p'}, W_q, \pi_q, \omega, \psi).$$

Moreover, by using Proposition 9.5 (2) repeatedly, we can show that

$$e_D(s, V_{p'}, W_q, \pi_q, \omega, \psi) = e_D(s, V_{p'-(q-t(\pi))}, W_{t(\pi)}, \pi_{t(\pi)}, \omega, \psi).$$

By tracing the above discussions conversely, the right-hand side is equal to

$$e_D(s, V_{p'+(t-q)}, W, \pi, \omega, \psi) = e_D(s, V, W, \pi, \omega, \psi).$$

Thus we have the lemma. □

14.2. **Global argument.** In this subsection, we explain the global argument which we use in the proof of Theorem 14.1.

Lemma 14.3. *Let $\mathbb{F}$ be a number field, let $\mathbb{A}$ be the ring of its adeles, let $\mathbb{D}$ be a division quaternion algebra over $\mathbb{F}$, let $\underline{V}$ be a right ϵ-Hermitian space over $\mathbb{D}$, let $\underline{W}$ be a left $(-\epsilon)$-Hermitian space over $\mathbb{D}$, let Π be an irreducible cuspidal automorphic representation of $G(\underline{W})(\mathbb{A})$, let $\underline{\omega}$ be a Hecke character of $\mathbb{A}^\times/\mathbb{F}^\times$, and let $\underline{\psi}$ be a non-trivial additive character of $\mathbb{A}/\mathbb{F}$. Then, we have*

$$\prod_v e_{\mathbb{D}_v}(s, \underline{V}_v, \underline{W}_v, \Pi_v, \underline{\omega}_v, \underline{\psi}_v) = 1.$$

Proof. Consider the Witt tower $\{\underline{V}_p\}_{p=0}^\infty$ so that $\underline{V}_r = \underline{V}$. Denote by $r(\Pi)$ the first occurrence index of Π in $\{\underline{V}_p\}_{p=0}^\infty$, by Σ the theta correspondence $\theta(\Pi, \underline{W}_{r(\Pi)})$ of Π, and by S the set of the places where $\mathbb{D}_v$ is a division algebra. Then, we have $\theta_{\underline{\psi}}(\Pi, \underline{V}_{r(\Pi)})$ is cuspidal, and we have

$$\begin{aligned}\prod_v e_{\mathbb{D}_v}(s, \underline{V}_v, \underline{W}_v, \Pi_v, \underline{\omega}_v, \underline{\psi}_v) &= \prod_{v\in S} e_{\mathbb{D}_v}(s, \underline{V}_{r(\Pi)_v}, \underline{W}_v, \Pi_v, \underline{\omega}_v, \underline{\psi}_v)\\ &= \prod_{v\in S} \frac{\gamma^V(s, \Sigma\boxtimes\underline{\omega}\chi_{\underline{W}}, \underline{\psi})}{\gamma^W(\pi, \Pi\boxtimes\underline{\omega}\chi_{\underline{V}}, \underline{\psi})}\cdot f_{\mathbb{D}_v}(s, \underline{V}, \underline{W}, \underline{\omega}, \underline{\psi})\\ &\quad\times\frac{L^S(s, \Sigma\boxtimes\underline{\omega}\chi_{\underline{W}})L_f^S(s)}{L^S(s, \Pi\boxtimes\underline{\omega}\chi_{\underline{V}})}\cdot\frac{L^S(1-s, \Pi\boxtimes\underline{\omega}\chi_{\underline{V}})}{L^S(1-s, \Sigma\boxtimes\underline{\omega}\chi_{\underline{W}})L_f^S(1-s)}\\ &= 1\end{aligned}$$

where $L_f^S(s) = \prod_{v\notin S} L_{f,v}(s)$ with

$$L_{f,v}(s) = \begin{cases}\prod_{i=1}^{l} L_{\mathbb{F}_v}(s + \frac{l+1}{2} - i, \underline{\omega}_v\chi_{\underline{V}_v}\chi_{\underline{W}_v}) & (l > 0),\\ \prod_{i=1}^{-l} L_{\mathbb{F}_v}(s + \frac{-l+1}{2} - i, \underline{\omega}_v\chi_{\underline{V}_v}\chi_{\underline{W}_v})^{-1} & (l < 0).\end{cases}$$

Hence we have the lemma. □

14.3. **Globalization.**

Lemma 14.4. *Assume that D is a division algebra. Let F' be a non-Archimedean local field of characteristic zero, let ψ' be an additive non-trivial character of F', let D' be a division quaternion algebra over F', let V' be another right ϵ-Hermitian space of dimension m, and let W' be another left $(-\epsilon)$-Hermitian space of dimension n. Then, there exist*

- *a global field $\mathbb{F}$ and its places v_1, v_2 such that $\mathbb{F}_{v_1} = F, \mathbb{F}_{v_2} = F'$,*
- *a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ such that $\mathbb{D}_{v_1} = D$, $\mathbb{D}_{v_2} = D'$, and $\mathbb{D}_v$ is split for $v \neq v_1, v_2$,*
- *a left $(-\epsilon)$-hermitian spaces $\underline{W}$ over $\mathbb{D}$ such that $\underline{W}_{v_1} = W, W_{v_2} = W'$,*
- *a right ϵ-hermitian space $\underline{V}$ over $\mathbb{D}$ such that $\underline{V}_{v_1} = V, \underline{V}_{v_2} = V'$,*
- *a Hecke character $\underline{\omega}$ of $\mathbb{A}^\times$ such that $\underline{\omega}_{v_1} = \omega, \underline{\omega}_{v_2} = 1$,*
- *a non-trivial additive character $\underline{\psi}$ of $\mathbb{A}/\mathbb{F}$ such that $\underline{\psi}_{v_1} = \psi_{a_1^2}, \underline{\psi}_{v_2} = \psi_{a_2^2}$ for some $a_1 \in F^\times, a_2 \in F'^\times$.*

For representations, we use a Henniart type globalization:

Lemma 14.5. *Let $\mathbb{F}$ be a global field, let G be a reductive group over $\mathbb{F}$, let A be a maximal $\mathbb{F}$-split torus of the center of G, let $\underline{\chi}$ be a unitary character $A(\mathbb{A})/A(\mathbb{F}) \to \mathbb{C}^\times$, let v_0 be a fixed place of $\mathbb{F}$, let S be a finite set of non-Archimedean places of $\mathbb{F}$ such that $v_0 \notin S$. Suppose that an irreducible supercuspidal representation π_v of $G(\mathbb{F}_v)$ is given for each $v \in S$, and a compact open* *subgroup K_v is given for each non-Archimedean $v \notin S$. Then there is an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ such that*

- $\Pi|_{A(\mathbb{A})}$ *coincides with* $\underline{\chi}$,
- $\Pi_v \cong \pi_v$ *for* $v \in S$,
- Π_v *possess a non-zero* K_v*-fixed vector for* $v \notin S \cup \{v_0\}$.

Proof. Similar to [Hen84, Appendice I]. □

14.4. **Completion of the proof of Theorem 14.1.** Let π be an irreducible representation of $G(W)$, and let ω be a character of $F^\times$. By Lemma 14.2, and Corollary 9.6, we may assume that π and σ are irreducible supercuspidal representations. Take

- $F' = F$, $D' = D$,
- a $(-\epsilon)$-Hermitian space W' so that $\dim W' = n$, the dimension of the anisotropic kernel of W' is 0 or 1, and $\mathfrak{d}(W') \in \mathcal{O}_F^\times$,
- an ϵ-Hermitian spacce V' so that $\dim V' = m$, the dimension of the anisotropic kernel of V' is 0 or 1, and $\mathfrak{d}(V') \in \mathcal{O}_F^\times$,
- the special maximal compact subgroup $K_{W'}$ of $G(W')$ as in §5 below.

Moreover, take a global field $\mathbb{F}$, places v_1, v_2 of $\mathbb{F}$, a non-trivial additive character $\psi : \mathbb{A}/\mathbb{F} \to \mathbb{C}^\times$, a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$, an ϵ-Hermitian space $\underline{V}$ over $\mathbb{D}$, a $(-\epsilon)$-Hermitian space $\underline{W}$ over $\mathbb{D}$ as in Lemma 14.4. Let $\underline{\omega}$ be a character of $\mathbb{A}^\times/\mathbb{F}^\times$ such that $\underline{\omega}_{v_1} = \omega$ and $\underline{\omega}_{v_2} = 1$. Denote by $\{\underline{V}_i\}_{i=0}^\infty$ the Witt tower containing $\underline{V}$. Then, by Lemma 14.5, we can take an irreducible cuspidal automorphic representation Π of $G(\underline{W'})(\mathbb{A})$ so that $\Pi_{v_1} = \pi$, and Π_{v_2} possess a non-zero $K_{W'}$ fixed vector. Then we have

$$\begin{aligned} e_D(s, V, W, \pi, \omega, \psi) &= \prod_{v \neq v_1} e_{\mathbb{D}_v}(s, \underline{V}, \underline{W}, \Pi, \underline{\omega}, \underline{\psi})^{-1} \\ &= e_D(s, V', W', \Pi_{v_2}, 1, \psi)^{-1} \\ &= e_D(s, V'_p, W'_0, 1_{W'_0}, 1, \psi)^{-1} \end{aligned}$$

where p is a sufficiently large integer, and $1_{W'_0}$ is the trivial representation of $G(W'_0)$. By the above observation, it only suffices to consider the cases where $n = \dim W = 0, 1$ and $\pi = 1_W$.

Lemma 14.6. *We denote by 1_V (resp. 1_W) the trivial representation of $G(V)$ (resp. $G(W)$). Suppose that $n = 0$. Then we have $r(1_W) = 0$ and $\theta_\psi(1_W, V) = 1_V$.*

For the rest of this subsection, we consider the case $n = 1$. By using the accidental isomorphism and by a result of [GI16], we can describe the local theta correspondence in terms of L-parameters in the case $n = m = 1$.

Proposition 14.7. *Suppose that $n = m = 1$ and $\epsilon = 1$. Let π be a non-trivial irreducible representation of $G(W)$, and let ϕ be its L-parameter. Then, the representation $\Theta_\psi(\pi, V)$ is non-zero irreducible, and has L-parameter*

$$(\phi \otimes \chi_V \chi_W) \oplus \chi_W.$$

Proof. We use the accidental isomorphism:

$$G(V) \cong \mathrm{SU}_E(2), \text{ and } G(W) \cong \mathrm{U}'_E(1). \tag{14.1}$$

Here,

- E is the quadratic unramified extension field of F associated with the quadratic character χ_W of $F^\times$,

- $\mathrm{SU}_E(2)$ is the special unitary group preserving the hermitian form
$$(\ , \ )_E : E^2 \times E^2 \to E : (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}) \mapsto \overline{x_1} y_1 - \varpi_F \cdot \overline{x_2} y_2$$
where $\overline{x_i}$ denotes the conjugate of x_i with respect to E/F,
- $\mathrm{U}'_E(1)$ is the unitary group preserving the skew-hermitian form
$$\langle \ , \ \rangle_E : E \times E \to E : x, y \mapsto x\alpha\overline{y}$$
where $\alpha \in E$ is a non-zero trace zero element with $\mathrm{ord}_E(\alpha) = 0$.

By [Ike19, §7], the accidental isomorphisms (14.1) are compatible with the local theta correspondence. We know the description of the local theta correspondence
$$\mathrm{Irr}(\mathrm{U}'_E(1)) \to \mathrm{Irr}(\mathrm{U}_E(2))$$
via L-parameters ([GI16, Theorem 4.4]). Therefore, we have the claim. □

By tracing the converse of the global argument in the beginning of this subsection, we obtain:

Corollary 14.8. *Suppose that $n = 1$ and $\epsilon = 1$. Denote by $\{V_i\}_{i=0}^{\infty}$ the Witt tower containing V. Then we have $e_D(s, V_p, W, 1_W, 1, \psi) = 1$ for sufficiently large p.*

Similary, by using the accidental isomorphism, we have:

Lemma 14.9. *Suppose that $n = 1$ and $\epsilon = -1$. Denote by $\{V_i\}_{i=0}^{\infty}$ the Witt tower containing V. Then we have $e_D(s, V_p, W, 1_W, 1, \psi) = 1$ for sufficiently large p.*

Hence, we complete the proof of Theorem 14.1.

15. Local Rallis inner product formula

In this section, we discuss the local analogue of the Rallis inner product formula following [GI14], and describe the relation between $\alpha_2(V, W)$ and $\alpha_3(V, W)$.

We use the setting of §3, and we take a basis $\underline{e}$ of W as in §4. Suppose that $l = 1$ and π is an irreducible square-integrable representation of $G(W)$. Consider the map
$$\mathcal{P} : \omega_\psi \otimes \overline{\omega_\psi} \otimes \overline{\omega_\psi} \otimes \omega_\psi \otimes \overline{\pi} \otimes \pi \otimes \pi \otimes \overline{\pi} \to \mathbb{C}$$
defined by
$$\begin{aligned} &\mathcal{P}(\phi_1, \phi_2, \phi_3, \phi_4; v_1, v_2, v_3, v_4) \\ &= \int_{G(V)} (\sigma(h)\theta(\phi_1, v_1), \theta(\phi_2, v_2)) \cdot \overline{(\sigma(h)\theta(\phi_3, v_3), \theta(\phi_4, v_4))} \, dh. \end{aligned}$$
The integral defining $\mathcal{P}$ converges absolutely since σ is also square-integrable (Lemma 9.2). As in [GI14, §18], we compute $\mathcal{P}$ in two ways. First, we have
$$\begin{aligned} &\mathcal{P}(\phi_1 \dots, \phi_4, v_1, \dots, v_4) \\ &= \frac{1}{\deg \sigma} \cdot (\theta(\phi_1, v_1), \theta(\phi_3, v_3)) \cdot \overline{(\theta(\phi_2, v_2), \theta(\phi_4, v_4))} \\ &= \frac{1}{\deg \sigma} \cdot Z(-\frac{1}{2}, F_{\phi_1 \otimes \overline{\phi_3}}, \overline{\xi}_{v_1,, v_3}) \cdot \overline{Z(-\frac{1}{2}, F_{\phi_2 \otimes \overline{\phi_4}}, \overline{\xi}_{v_2, v_4})}. \end{aligned}$$
Second, as in [GI14, p.593-p.595], we have
$$\begin{aligned} &\mathcal{P}(\phi_1 \dots, \phi_4, v_1, \dots, v_4) \\ &= \frac{\alpha_2(V, W)}{\deg \pi} \cdot Z(\frac{1}{2}, F^{\dagger}_{\phi_1 \otimes \overline{\phi}_3}, \overline{\xi}_{v_1, v_3}) \cdot \overline{Z(-\frac{1}{2}, F_{\phi_2 \otimes \overline{\phi}_4}, \overline{\xi}_{v_2, v_4})} \cdot |2|_F^{2mn} \cdot |N(R(\underline{e}))|^{-m}. \end{aligned}$$

The local functional equation of the doubling zeta integral says that

$$\begin{aligned}
&Z(-\frac{1}{2}, F_{\phi_1\otimes\overline{\phi}_3}, \overline{\xi}_{v_1,v_3}) \\
&= \left(c(s,\chi_V,A_0,\psi)R(s,\chi_V,A,\psi)^{-1}\gamma(s+\frac{1}{2},\pi\times\chi_V,\psi)\right)\Big|_{s=\frac{1}{2}} \\
&\quad\times \pi(-1)\cdot Z(\frac{1}{2}, F^{\dagger}_{\phi_1\otimes\overline{\phi}_3}, \overline{\xi}_{v_1,v_3}).
\end{aligned}$$

By Theorem 14.1 and Proposition 7.2, we have

$$\begin{aligned}
&c(s,\chi_V,A_0,\psi)R(s,\chi_V,A,\psi)^{-1}\gamma(s+\frac{1}{2},\pi\times\chi_V,\psi) \\
&= e(G)\cdot|N(R(\underline{e}))|^{-s}\cdot|2|^{-2ns+n(n-\frac{1}{2})}\cdot\omega^{-1}(4) \\
&\times\frac{\gamma(s+\frac{1}{2},1,\psi)}{\gamma(2s,1,\psi)}\cdot\prod_{i=1}^{n-1}\gamma(2s-2i,1,\psi)^{-1}\cdot\gamma^V(s+\frac{1}{2},\sigma\times\chi_W,\psi) \\
&\times\begin{cases}\gamma(s-n+\frac{1}{2},\omega,\psi)^{-1} & -\epsilon=1 \\ \epsilon(\frac{1}{2},\chi_W,\psi)^{-1} & -\epsilon=-1.\end{cases}
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\gamma^V(1,\sigma\times\chi_W,\psi) &= \gamma(1,\sigma^\vee\times\chi_W,\psi) \\
&= \gamma(0,\sigma\times\chi_W,\overline{\psi})^{-1} \\
&= \gamma(0,\sigma\times\chi_W,\psi)^{-1}\times\begin{cases}\chi_W(-1) & (\epsilon=1), \\ \chi_V(-1) & (\epsilon=-1).\end{cases}
\end{aligned}$$

Combining above equations and Theorem 14.1, we obtain:

Proposition 15.1. *Suppose that $l=1$ and π is square integrable. Then, we have*

$$\frac{\deg\pi}{\deg\sigma} = \alpha_3(V,W)\omega_\pi(-1)\cdot\gamma^V(0,\sigma\times\chi_W,\psi)$$

where

$$\begin{aligned}
\alpha_3(V,W) =& \frac{1}{2}\cdot\alpha_2(V,W)\cdot e(G)\cdot|2|_F^{2n\rho-n(n-\frac{1}{2})}\cdot|N(R(\underline{e}))|^{-\rho}\cdot\prod_{i=1}^{n-1}\frac{\zeta_F(2i)}{\zeta_F(1-2i)} \\
&\times\begin{cases}\chi_V(-1)\gamma(1-n,\chi_V,\psi) & (-\epsilon=1), \\ \chi_W(-1)\epsilon(\frac{1}{2},\chi_W,\psi) & (-\epsilon=-1).\end{cases}
\end{aligned}$$

We write down the constant $\alpha_3(V,W)$ in the minimal cases.

Proposition 15.2. *(1) In the case $\epsilon=-1$ and V is anisotropic, we have*

$$\alpha_3(V,W) = (-1)^n\cdot\chi_V(-1)\cdot\epsilon(\frac{1}{2},\chi_V,\psi).$$

(2) In the case $\epsilon=1$ and either V or W is anisotropic, we have

$$\alpha_3(V,W) = \frac{1}{2}\cdot\chi_W(-1)\cdot\epsilon(\frac{1}{2},\chi_W,\psi).$$

Proof. For the case $m=0$, one can verify this proposition directly. Otherwise, we obtain the claim by Proposition 10.3. □

16. Plancherel measures

In this section, we recall some fundamental properties of the Plancherel measure, and we discuss how the Plancherel measure behaves under the theta correspondence.

16.1. **Definition.** Let G be a reductive group over F, let P be a parabolic subgroup of G, let M be a Levi subgroup of P, and let U be the unipotent radical of P. We denote by $X^*(M)$ the group of the algebraic characters of M, and by $E_{\mathbb{C}}^{\vee}$ the vector space $X^*(M) \otimes \mathbb{C}$. For a finite length representation π of $G(V)$ and for

$$\eta = \sum_{i=1}^{t} \chi_i \otimes s_i \in E_{\mathbb{C}}^{\vee}$$

we denote by $\pi \otimes \eta$ the representation given by

$$[\pi \otimes \eta](g) := \pi(g) \prod_{i=1}^{t} |\chi_i(g)|^{s_i}$$

for $g \in G(V)$. Take a maxmal compact subgroup K of G so that $G = PK$. Then for $f \in \mathrm{Ind}_P^G(\pi)$, we define $f_\eta \in \mathrm{Ind}_P^G(\pi \otimes \eta)$ by

$$f_\eta(muk) = \prod_{i=1}^{t} |\chi_i(m)|^{s_i} f(muk)$$

for $m \in M, u \in U, k \in K$. Denote by $\overline{P}$ the opposite parabolic subgroup of P, and by $\overline{U}$ the unipotent radical of $\overline{P}$. Waldspurger proved that for $f \in \mathrm{Ind}_P^G \pi$ the integral

$$[J_{\overline{P}|P}(\pi \otimes a) f_\eta](g) = \int_{\overline{U}} f_\eta(\overline{u}g)\, d\overline{u}$$

converges absolutely when a is contained in a certain open subset of $E_{\mathbb{C}}^{\vee}$, and it admits a meromorphic continuation to the whole complex plain $\mathbb{C}$ [Wal03]. Here, the measure $d\overline{u}$ is the normalized Haar measure as in §6.2. Therefore we have an intertwining operator

$$J_{\overline{P}|P}(\pi \otimes \eta) : \mathrm{Ind}_P^G(\pi \otimes \eta) \to \mathrm{Ind}_{\overline{P}}^G(\pi \otimes \eta)$$

for almost all $\eta \in E_{\mathbb{C}}^{\vee}$. Then, there exists a rational function $\mu(\eta, \pi)$ of η satisfying

$$J_{P|\overline{P}}(\pi \otimes \eta) \circ J_{\overline{P}|P}(\pi \otimes \eta) = \mu(\eta, \pi)^{-1}.$$

It is called the Plancherel measure.

Lemma 16.1. *Let S be a maximal F-split torus contained in the center of M. We denote by $\Delta_S(P) \subset X^*(S)$ the set of roots of P with respect to S. For $a \in \Delta_S(P)$, we denote by S_a the kernel of a in S, and by G_a the centralizer of S_a in G. Then, for an irreducible representation π of M and $\eta \in E_{\mathbb{C}}^{\vee}$, we have*

$$\mu(\eta, \pi) = \prod_{a \in \Delta(P)} \mu_a(\eta, \pi)$$

where $\mu_a(\eta, \pi)$ is the Plancherel measure defined by using $(M, P \cap G_a, G_a)$ instead of (M, P, G).

Proof. [Wal03, IV (5)]. □

Let $W' \subset W$ be $(-\epsilon)$-Hermitian spaces, and let X, X^* be totally isotropic subspaces of W so that $W = X + W' + X^*$ and $X + X^*$ is the orthogonal complement of W'. Now we consider the case where $G = G(W)$, and $P = P(X)$. The restriction to $X + W'$ gives the identification $M \cong \mathrm{GL}(X) \times G(W')$. Then, for a finite length representation π' of $G(W')$ and a finite length representation τ of $\mathrm{GL}(X)$, we abbreviate $\mu(N \otimes s, \pi' \boxtimes \tau)$ to $\mu(s, \pi' \boxtimes \tau)$. Here, N denotes the reduced norm of $\mathrm{End}(X)$.

16.2. **Global Property.** In this subsection, we recall the global property of the Plancherel measure for inner forms of general linear group and quaternionic unitary groups. Let $\mathbb{F}$ be a global field, and let $\mathbb{D}'$ be a division central algebra over $\mathbb{F}$. Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{F}$. We first discuss it for $\mathrm{GL}_{t_1}(\mathbb{D}') \times \mathrm{GL}_{t_2}(\mathbb{D}')$.

Lemma 16.2. *Let ${\mathbb{D}'}^t = {\mathbb{D}'}^{t_1} \oplus {\mathbb{D}'}^{t_2}$ be a vector space over $\mathbb{D}'$. Then, $M(\mathbb{A}) = \mathrm{GL}_{t_1}(\mathbb{D}'_{\mathbb{A}}) \times \mathrm{GL}_{t_2}(\mathbb{D}'_{\mathbb{A}})$ is a Levi subgroup of a maximal parabolic subgroup of $\mathrm{GL}_t(\mathbb{D}')$. Then, for an irreducible cuspidal representation $\Pi \boxtimes \Xi$ of $M(\mathbb{A})$ and for $\eta = (s_1, s_2) \in \mathbb{C}^2 \cong E_{\mathbb{C}}$, we have*

$$\prod_{v \in S} \mu_v(\eta, \Pi_v \boxtimes \Xi_v) = \frac{L^S(1 - s_1 + s_2, \Pi \boxtimes \Xi^\vee)}{L^S(s_1 - s_2, \Pi^\vee \boxtimes \Xi)} \cdot \frac{L^S(1 + s_1 - s_2, \Pi^\vee \boxtimes \Xi)}{L^S(-s_1 + s_2, \Pi \boxtimes \Xi^\vee)}.$$

Here, S is a finite set of places of $\mathbb{F}$ such that

- *S contain all Archimedean places,*
- *$\mathbb{D}'_v$ is split for $v \notin S$, and*
- *Π_v, Σ_v are unramified for $v \notin S$,*

and we denote

$$L^S(s, \Pi \boxtimes \Xi) = \prod_{v \notin S} L^{\mathrm{RS}}(s, \Pi_v \times \Xi_v)$$

where the right-hand side is the infinite product of the Rankin-Selberg L-factors (see [JPSS83]*).*

Second, we discuss it for quaternionic unitary groups.

Lemma 16.3. *Assume that $\mathbb{D}'$ is a division quaternion algebra over $\mathbb{F}$. Let $\underline{W}$ be a left $(-\epsilon)$-Hermitian space over $\mathbb{D}$, let $\underline{X}, \underline{X}^*$ be two left $\mathbb{D}'$-vector spaces so that $\dim \underline{X} = \dim \underline{X}^* = r'$, and let $\underline{W'} = \underline{X} + \underline{W} + \underline{X}^*$ a $(-\epsilon)$-Hermitian space equipped with the $(-\epsilon)$-Hermitian form*

$$\langle\ ,\ \rangle' : (\underline{X} + \underline{W} + \underline{X}^*) \times (\underline{X} + \underline{W} + \underline{X}^*) \to \mathbb{D}'$$

defined by

$$\langle (x_1, w_1, y_1), (x_2, w_2, y_2) \rangle' = x_1 \cdot J_{r'} \cdot {}^t y_2^* + \langle w_1, w_2 \rangle - \epsilon y_1 \cdot J_{r'} \cdot {}^t x_2^*.$$

Here, we recall that $J_{r'}$ is the anti-diagonal matrix defined in §4. Then, $M = \mathrm{GL}_{r'}(\mathbb{D}') \times G(\underline{W})$ is a Levi subgroup of a maximal parabolic subgroup of $G(\underline{W'})$. Then, for an irreducible cuspidal automorphic representation $\Pi \boxtimes \Xi$ of $M(\mathbb{A})$, we have

$$\prod_{v \in S} \mu_v(s, \Pi_v \boxtimes \Xi_v) = \frac{L^S(1 - s, \Pi \boxtimes \Xi^\vee)}{L^S(s, \Pi^\vee \boxtimes \Xi)} \cdot \frac{L^S(1 + s, \Pi^\vee \boxtimes \Xi)}{L^S(-s, \Pi \boxtimes \Xi^\vee)}$$
$$\times \frac{L^S(1 - 2s, \Xi^\vee, \wedge^2)}{L^S(2s, \Xi, \wedge^2)} \cdot \frac{L^S(1 + 2s, \Xi, \wedge^2)}{L^S(-2s, \Xi^\vee, \wedge^2)}.$$

Here, S is a finite set of places of $\mathbb{F}$ such that

- *S contain all Archimedean places,*
- *$\mathbb{D}'_v$ is split for $v \notin S$, and*
- *Π_v, Σ_v are unramified for $v \notin S$,*

and we denote

$$L^S(s, \Xi^\vee, \wedge^2) = \prod_{v \notin S} L(s, \Xi_v, \wedge^2)$$

where the right-hand side is an infinite product of local exterior-square L-factor.

16.3. **Plancherel measures for inner forms of general linear groups.** In this subsection, we denote by D' a central division algebra over F. Let t_1 and t_2 be positive integers, and let $t = t_1 + t_2$. We consider the case where $M = \mathrm{GL}_{t_1}(D') \times \mathrm{GL}_{t_2}(D')$ and $G = \mathrm{GL}_t(D')$. Then, we have an identification $\mathbb{C}^2 \cong E_{\mathbb{C}}^{\vee}$ by

$$(\eta_1, \eta_2) \mapsto N_1 \otimes \eta_1 + N_2 \otimes \eta_2$$

where N_i denotes the reduced norm of $\mathrm{GL}_{t_i}(D')$ for $i = 1, 2$ respectively.

Lemma 16.4. *Let τ_i be irreducible representations of $\mathrm{GL}_{t_i}(D')$ for $i = 1, 2$. For $\eta \in \mathbb{C}^2$ and $z \in \mathbb{C}$, we have*

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \mu(\eta - (z, z), \tau_1 \boxtimes \tau_2)$$

Proof. Let P be a F-rational parabolic subgroup of $\mathrm{GL}_t(D')$ having the Levi subgroup M. Then, we have

$$J_{\overline{P}|P}((\tau_1 \boxtimes \tau_2) \otimes \eta) \otimes |N|^{-z} = J_{\overline{P}|P}((\tau_1 \boxtimes \tau_2) \otimes (\eta - (z, z))).$$

Hence we have the lemma. □

Lemma 16.5. *Let $u_1, \ldots, u_k$ be positive integers for $i = 1, \ldots, k$ so that $u_1 + \cdots + u_k = t_1$, and $\rho_1, \ldots, \rho_k$ be irreducible representations of $\mathrm{GL}_{u_1}(D'), \ldots, \mathrm{GL}_{u_k}(D')$ respectively. Moreover, let P_1 be an F-rational parabolic subgroup of $\mathrm{GL}_{t_1}(D')$ having the Levi subgroup $\mathrm{GL}_{u_1}(D') \times \cdots \times \mathrm{GL}_{u_k}(D')$. Then, for an irreducible constituent τ_1 of $\mathrm{Ind}_{P_1}^{\mathrm{GL}_{t_1}(D')} \rho_1 \boxtimes \cdots \boxtimes \rho_k$, and for an irreducible representation τ_2 of $\mathrm{GL}_{t_2}(D')$, we have*

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \prod_{i=1}^{k} \mu(\eta_i, \sigma_i \boxtimes \tau_2)$$

for $\eta \in E_{\mathbb{C}}^{\vee}$. Here η_i denotes the image of η by the map

$$X^*(M) \otimes \mathbb{C} \to X^*(\mathrm{GL}_{u_i}(D') \times \mathrm{GL}_{t_2}(D')) \otimes \mathbb{C}$$

induced by the restriction for $i = 1, \ldots, k$.

Proof. Let S' be the center of $(\mathrm{GL}_{u_1}(D') \times \cdots \mathrm{GL}_{u_k}(D')) \times \mathrm{GL}_{t_2}(D')$, and let S be the center of $\mathrm{GL}_{t_1}(D') \times \mathrm{GL}_{t_2}(D')$. Then, we have

$$\Delta_S(P) = \Delta_{S'}(P) \setminus \Delta_{S'}(P_1 \times \mathrm{GL}_{t_2}(D')),$$

and by Lemma 16.1, we have

$$\begin{aligned} \mu(\eta, \tau_1 \boxtimes \tau_2) &= \prod_{a \in \Delta_S(P)} \mu_a(\eta, \tau_1 \boxtimes \tau_2) \\ &= \prod_{a \in \Delta_S(P)} \mu_a(\eta, (\sigma_1 \boxtimes \cdots \boxtimes \sigma_k) \boxtimes \tau_2) \\ &= \prod_{i=1}^{k} \mu(\eta_i, \sigma_i \boxtimes \tau_2). \end{aligned}$$

Hence we have the lemma. □

In particular, we obtain a formula of the Plancherel measure.

Proposition 16.6. *Let τ_i be an irreducible representation of $\mathrm{GL}_{t_i}(D)$ for $i = 1, 2$. Then we have*

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \gamma(s_1 - s_2, \tau_1 \boxtimes \tau_2^{\vee}, \psi)\gamma(s_2 - s_1, \tau_1^{\vee} \boxtimes \tau_2, \overline{\psi})$$

for $\eta = (s_1, s_2) \in \mathbb{C}^2$.

Proof. We can embed τ_1 into

$$\mathrm{Ind}_{B_1}^{\mathrm{GL}_{t_1}(D)} \sigma_{11}|N_{11}|^{a_1} \boxtimes \cdots \boxtimes \sigma_{1\lambda_1}|N_{1\lambda_1}|^{a_{\lambda_1}}$$

where B_1 is a parabolic subgroup of $\mathrm{GL}_{t_1}(D)$, $N_{11}, N_{12}, \ldots$ denote the reduced norms, $\sigma_{11}|N_{11}|^{a_1} \boxtimes \cdots \boxtimes \sigma_{1\lambda_1}|N_{1\lambda_1}|^{a_{\lambda_1}}$ is a representation of the Levi subgroup $\mathrm{GL}_{t_{11}}(D) \times \cdots \times \mathrm{GL}_{t_{1\lambda_1}}(D)$ with complex numbers $a_1, \ldots, a_{\lambda_1}$ and irreducible square integrable representations $\sigma_{11}, \ldots, \sigma_{1\lambda_1}$. Similarly, we can embed τ_2 into

$$\mathrm{Ind}_{B_2}^{\mathrm{GL}_{t_2}(D)} \sigma_{21}|N_{21}|^{b_1} \boxtimes \cdots \boxtimes \sigma_{2\lambda_2}|N_{2\lambda_2}|^{b_{\lambda_2}}.$$

Then we have

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \prod_{i,j} \mu(\eta + (a_i, b_j), \sigma_{1i} \boxtimes \sigma_{2j})$$

By [Zel80, Theorem 9.3], an irreducible square integrable representation is generic. Thus, by [Sha90], we have

$$\begin{aligned}\mu(\eta + (a_i, b_j), \sigma_{1i} \boxtimes \sigma_{2j}) &= \gamma(s_1 + a_i - s_2 - b_j, \sigma_{1i} \boxtimes \sigma_{2j}^\vee, \psi) \\ &\quad \times \gamma(s_2 + b_j - s_1 - a_i, \sigma_{1i}^\vee \boxtimes \sigma_{2j}, \overline{\psi}) \\ &= \gamma(s_1 - s_2, \sigma_{1i}|N_{1i}|^{a_i} \boxtimes (\sigma_{2j}|N_{2j}|^{b_j})^\vee, \psi) \\ &\quad \times \gamma(s_2 - s_1, (\sigma_{1i}|N_{1i}|^{a_i})^\vee \boxtimes \sigma_{2j}|N_{2j}|^{b_j}, \overline{\psi}).\end{aligned}$$

Hence, by the multiplicativity of the γ-factor, we have

$$\mu(\eta, \tau_1 \boxtimes \tau_2) = \gamma(s_1 - s_2, \tau_1 \boxtimes \tau_2^\vee, \psi)\gamma(s_2 - s_1, \tau_1^\vee \boxtimes \tau_2, \overline{\psi}).$$

□

16.4. **The behavior of the Plancherel measure under the theta correspondence.** Now we consider the Plancherel measures for quaternionic unitary groups. Let V be an m-dimensional right ϵ-Hermitian space, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space. Note that, in this section, we allow the case where $l \neq 1$.

Proposition 16.7. *Let π be an irreducible representation of $G(W)$, let $\sigma := \theta_\psi(\pi; V)$ and let τ be an irreducible representation of $\mathrm{GL}(X)$. Then we have*

$$\frac{\mu(s, \pi \boxtimes \tau\chi_V)}{\mu(s, \sigma \boxtimes \tau\chi_W)} = \gamma(s - \frac{l-1}{2}, \tau, \psi) \cdot \gamma(-s - \frac{l-1}{2}, \tau^\vee, \overline{\psi}).$$

The remaining part of this subsection is devoted to the proof of Proposition 16.7. Put

$$u_D(s; W, V, X, \pi, \tau) = \frac{\mu(s, \pi \boxtimes \tau\chi_V)}{\mu(s, \sigma \boxtimes \tau\chi_W)} \gamma(s - \frac{l-1}{2}, \tau, \psi)^{-1} \gamma(-s - \frac{l-1}{2}, \tau^\vee, \overline{\psi})^{-1}.$$

We will use global argument to prove Proposition 16.7, so that **we allow D to be split** in the rest of this section. We want to show $u_D(W, V, X, \pi, \tau) = 1$ for all D, W, V, X, π, τ.

Lemma 16.8. *Let $\{W_i\}_{i \geq 0}$ be a Witt tower consisting of $(-\epsilon)$-Hermitian spaces and let $\{V_j\}_{j \geq 0}$ be a Witt tower consisting of ϵ-Hermitian spaces. We suppose that $V = V_r$ and $W = W_t$.*

(1) If D is split, then we have

$$u_D(s; W, V, X, \pi, \tau) = 1.$$

(2) Suppose that π is a subrepresentation of $\mathrm{Ind}_{P_{t',t}}^{G(W)} \pi' \boxtimes \rho_{s_0}\chi_V$ where t' is an integer so that $\max\{t(\pi), r\} \leq t' \leq t$, $s_0 \in \mathbb{C}$, π' is an irreducible representation of $G(W_{t'})$, and ρ is an irreducible supercuspidal representation of $\mathrm{GL}_{t-t'}(D)$. Then, we have

$$u_D(s; W, V, X, \pi, \tau) = u_D(s; W_{t'}, V_{r'}, X, \pi', \tau)$$

where $r' = r - (t - t')$.

(3) *Let* X', X'' *be two subspaces of* X *so that* $X = X' + X''$. *Suppose that* τ *is an irreducible subquotient of induced representation* $\mathrm{Ind}_{P(X')}^{\mathrm{GL}(X)} \tau' \boxtimes \tau''$ *where* τ' *(resp.* τ''*) is an irreducible representation of* $\mathrm{GL}(X')$ *(resp.* $\mathrm{GL}(X'')$*). Then, we have*

$$u_D(s; W, V, X, \pi, \tau) = u_D(s; W, V, X', \pi, \tau')u_D(s; W, V, X'', \pi, \tau'').$$

(4) *If* $r(\pi)$ *denotes the first occurrence index, then we have*

$$u_D(s; W, V_{r(\pi)}, X, \pi, \tau) = u_D(s; W, V, X, \pi, \tau).$$

Proof. The claim (1) is proved in [GI14, Theorrem 12.1]. Then, we prove (2). By [GI14, Proposition B.3], we have

$$\begin{aligned}\mu(s, \tau\chi_V \otimes \pi) &= \mu((s, s_0), \tau\chi_V \boxtimes \rho\chi_V)\mu((s, -s_0), \tau\chi_V \boxtimes \rho^\vee\chi_V)\mu(s, \tau \boxtimes \pi') \\ &= \mu((s, s_0), \tau \boxtimes \rho)\mu((s, -s_0), \tau \boxtimes \rho^\vee)\mu(s, \tau \boxtimes \pi').\end{aligned}$$

Hence, by Corollary 9.6 (with replacing V and W, σ and π), we have

$$\frac{\mu(s, \tau\chi_V \otimes \pi)}{\mu(s, \tau\chi_W \otimes \sigma)} = \frac{\mu(s, \tau\chi_V \otimes \pi')}{\mu(s, \tau\chi_W \otimes \sigma')}.$$

Thus, we have (2). We prove (3) in the similar way by using [GI14, Lemma B.2]. Finally, we prove (4). Put $t^\pi = r - r(\pi)$. Then, by using the local functional equation of the doubling γ-factor ([Kak20b, Theorem 5.7(4)]), we have

$$\begin{aligned}\mu(s, \sigma \boxtimes \tau\chi_W) &= \mu(s, |N|^{\frac{l}{2}+t^\pi} \boxtimes \tau\chi_W) \cdot \mu(s, |N|^{-\frac{l}{2}-t^\pi} \boxtimes \tau\chi_W) \cdot \mu(s, \sigma' \boxtimes \tau\chi_W) \\ &= \gamma(s, |N|^{\frac{l}{2}+t^\pi} \boxtimes \tau^\vee\chi_W, \psi) \cdot \gamma(-s, |N|^{-\frac{l}{2}-t^\pi} \boxtimes \tau\chi_W, \overline{\psi}) \\ &\quad \times \gamma(s, |N|^{-\frac{l}{2}-t^\pi} \boxtimes \tau^\vee\chi_W, \psi) \cdot \gamma(-s, |N|^{\frac{l}{2}+t^\pi} \boxtimes \tau\chi_W, \overline{\psi}) \\ &\quad \times \mu(s, \sigma' \boxtimes \tau\chi_W) \\ &= \frac{\prod_{i=1}^{2t^\pi} \gamma(s + \frac{l}{2} + i - \frac{1}{2}, \tau^\vee\chi_W, \psi)}{\prod_{i=1}^{2t^\pi} \gamma(s + \frac{l}{2} + i + \frac{1}{2}, \tau^\vee\chi_W, \psi)} \cdot \frac{\prod_{i=1}^{2t^\pi} \gamma(s - \frac{l}{2} + \frac{1}{2} - i, \tau^\vee\chi_W, \psi)}{\prod_{i=1}^{2t^\pi} \gamma(s - \frac{l}{2} + \frac{3}{2} - i, \tau^\vee\chi_W, \psi)} \\ &\quad \times \mu(s, \sigma' \boxtimes \tau\chi_W) \\ &= \frac{\gamma(s + \frac{l+1}{2}, \tau^\vee\chi_W, \psi)}{\gamma(s + \frac{l_0+1}{2}, \tau^\vee\chi_W, \psi)} \cdot \frac{\gamma(s - \frac{l_0-1}{2}, \tau^\vee\chi_W, \psi)}{\gamma(s - \frac{l-1}{2}, \tau^\vee\chi_W, \psi)} \cdot \mu(s, \sigma' \boxtimes \tau\chi_W) \\ &= \frac{\gamma(-s - \frac{l_0-1}{2}, \tau\chi_W, \overline{\psi})}{\gamma(-s - \frac{l-1}{2}, \tau\chi_W, \overline{\psi})} \cdot \frac{\gamma(s - \frac{l_0-1}{2}, \tau^\vee\chi_W, \psi)}{\gamma(s - \frac{l-1}{2}, \tau^\vee\chi_W, \psi)} \cdot \mu(s, \sigma' \boxtimes \tau\chi_W) \\ &= \frac{\gamma(-s - \frac{l_0-1}{2}, \tau\chi_W, \psi)}{\gamma(-s - \frac{l-1}{2}, \tau\chi_W, \psi)} \cdot \frac{\gamma(s - \frac{l_0-1}{2}, \tau^\vee\chi_W, \overline{\psi})}{\gamma(s - \frac{l-1}{2}, \tau^\vee\chi_W, \overline{\psi})} \cdot \mu(s, \sigma' \boxtimes \tau\chi_W).\end{aligned}$$

Hence we have

$$\begin{aligned} u_D(s;W,V,X,\pi,\tau) &= \frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma\boxtimes\tau\chi_W)}\gamma(s-\frac{l-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l-1}{2},\tau^\vee,\overline{\psi})^{-1} \\ &= \frac{\mu(s,\sigma'\boxtimes\tau\chi_W)}{\mu(s,\sigma\boxtimes\tau\chi_W)}\cdot\frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma'\boxtimes\tau\chi_W)} \\ &\quad\times\gamma(s-\frac{l-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l-1}{2},\tau^\vee,\overline{\psi})^{-1} \\ &= \frac{\mu(s,\pi\boxtimes\tau\chi_V)}{\mu(s,\sigma'\boxtimes\tau\chi_W)}\gamma(s-\frac{l_0-1}{2},\tau,\psi)^{-1}\gamma(-s-\frac{l_0-1}{2},\tau^\vee,\overline{\psi})^{-1} \\ &= u_D(s;W,V_{r(\pi)},X,\pi,\tau). \end{aligned}$$

Thus we have (4). □

Now we prove Proposition 16.7. By Corollary 9.6 and Lemma 16.8 (2)(3), we may assume that π, σ, and τ are supercuspidal. Take

- a global field $\mathbb{F}$ and two distinct places v_1, v_2 of $\mathbb{F}$ so that $\mathbb{F}_{v_1}=\mathbb{F}_{v_2}=F$,
- a non-trivial additive character $\underline{\psi}$ of the ring of adeles $\mathbb{A}$ of $\mathbb{F}$,
- a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ so that $\mathbb{D}_{v_1}=\mathbb{D}_{v_2}=D$ and $\mathbb{D}_v$ is split for all $v\neq v_1,v_2$,
- an ϵ-Hermitian space $\mathbb{V}$ over $\mathbb{D}$ so that $\mathbb{V}_{v_1}=\mathbb{V}_{v_2}=V$,
- a Witt tower $\{\mathbb{V}_i\}_{i=0}^\infty$ containing $\mathbb{V}$,
- a $-\epsilon$-Hermitian space $\mathbb{W}$ over $\mathbb{D}$ so that $\mathbb{W}_{v_1}=\mathbb{W}_{v_2}=W$,
- an irreducible cuspidal automorphic representation Π of $G(\mathbb{W})(\mathbb{A})$ so that $\Pi_{v_1}=\pi$,
- a vector space $\mathbb{X}$ over $\mathbb{D}$ so that $\dim_{\mathbb{D}}\mathbb{X}=\dim_D X$,
- an irreducible cuspidal automorphic representation Ξ of $\mathrm{GL}(\mathbb{X})(\mathbb{A})$ so that $\Xi_{v_1}=\tau$,
- a finite subset S of places so that $v_1,v_2\in S$, all Archimedean places are contained in S and Π_v, Ξ_v are unramified for all places $v\notin S$.

Let $r(\Pi)$ be the first occurrence index of the theta correspondence of Π to the Witt tower $\{\mathbb{V}_i\}_{i=0}^\infty$. Then, $\Theta_{\underline{\psi}}(\Pi,\mathbb{V}_{r(\Pi)})$ is a non-zero irreducible cuspidal automorphic representation. We denote by π' (resp. τ') the representation Π_{v_2} (resp. Ξ_{v_2}). Hence we have

$$\begin{aligned} (16.1)\qquad & u_D(s;V,W,X,\pi,\tau)\cdot u_D(s;V,W,X,\pi',\tau') \\ &= u_{\mathbb{D}_{v_1}}(s;\mathbb{V}_{v_1},\mathbb{W}_{v_1},\mathbb{X}_{v_1},\Pi_{v_1},\Xi_{v_1})\cdot u_{\mathbb{D}_{v_2}}(s;\mathbb{V}_{v_2},\mathbb{W}_{v_2},\mathbb{X}_{v_2},\Pi_{v_2},\Xi_{v_2}) \\ &= u_{\mathbb{D}_{v_1}}(s;(\mathbb{V}_{r(\Pi)})_{v_1},\mathbb{W}_{v_1},\mathbb{X}_{v_1},\Pi_{v_1},\Xi_{v_1})\cdot u_{\mathbb{D}_{v_2}}(s;(\mathbb{V}_{r(\Pi)})_{v_2},\mathbb{W}_{v_2},\mathbb{X}_{v_2},\Pi_{v_2},\Xi_{v_2}) \\ &\quad\times\prod_{v\neq v_1,v_2}u_{\mathbb{D}_v}(s;(\mathbb{V}_{r(\Pi)})_v,\mathbb{W}_v,\mathbb{X}_v,\Pi_v,\Xi_v) \\ &= 1 \end{aligned}$$

Applying (16.1) when Π and Ξ are chosen so that $\pi'=\pi$ and $\tau'=\tau$, we have $u_D(s;V,W,X,\pi,\tau)^2=1$. Hence $u_D(s;V,W,X,\pi,\tau)=\pm1$. It remains to determine the signature. By Lemma 16.8 (4), we may assume that σ is also supercuspidal. Moreover, we may assume that $\mathrm{JL}(\tau)$ is also an irreducible supercuspidal representation of $\mathrm{GL}_{2\dim X}(F)$. Then, the Godement-Jacquet L-factor of τ is 1. First, we have

$$\mu(0,\pi\boxtimes\tau\chi_V)>0,\ \text{and}\ \mu(0,\sigma\boxtimes\tau\chi_W)>0.$$

On the other hand, putting

$$\epsilon(s+\frac{1}{2},\tau,\psi)=a_\psi(\tau)\cdot q^{-\lambda s}$$

with $a_\psi(\tau) \in \mathbb{C}^\times, \lambda \in \mathbb{Z}$, we have

$$\epsilon(-s+\frac{1}{2}, \tau^\vee, \overline{\psi}) = a_\psi(\tau)^{-1} \cdot q^{\lambda s},$$

and we have

$$\begin{aligned}&\gamma(-\frac{l-1}{2}, \tau, \psi)\gamma(-\frac{l-1}{2}, \tau^\vee, \psi)\\ &= a_\psi(\tau) q^{\lambda l/2} \cdot \frac{L(\frac{l+1}{2}, \tau^\vee)}{L(-\frac{l-1}{2}, \tau)} \cdot a_\psi(\tau)^{-1} q^{\lambda l/2} \cdot \frac{L(\frac{l+1}{2}, \tau)}{L(-\frac{l-1}{2}, \tau^\vee)}\\ &= q^{\lambda l} > 0.\end{aligned}$$

Thus, the signature of $u_D(s; V, W, X, \pi, \tau)$ turns out to be 1. This completes the proof of Proposition 16.7.

17. Accidental isomorphims

In this section, we explain accidental isomorphisms for quaternionic (similitude) unitary groups, and we prove an explicit formula of the Plancherel measure for some irreducible supercuspidal representations. Let $\mathbb{F}$ be a global field, let $\mathbb{D}$ be a division quaternion algebra over $\mathbb{F}$, and $\underline{V}$ be an anisotropic ϵ-Hermitian spaces over $\mathbb{D}$. Put

$$G'(\underline{V}) = \begin{cases} G(\underline{V}) & m = 1, \\ \widetilde{G}(\underline{V}) & m = 2, 3. \end{cases}$$

Here $\widetilde{G}(\underline{V})$ denotes the similitude group of $\underline{V}$. Then it is known that $G'(\underline{V})$ is isomorphic to a certain more familiar group.

- Suppose first that $\epsilon = -1$ and $m = 1$. Let $\underline{V'}$ be a two-dimensional quadratic space such that $\chi_{\underline{V'}} = \chi_{\underline{V}}$. Then we have $G'(\underline{V}) = \mathrm{SO}(\underline{V'})$.
- Suppose that $\epsilon = -1$ and $m = 2$. Let $\mathbb{E}$ be a quadratic extension field of $\mathbb{F}$ such that $\chi_{\mathbb{E}} = \chi_{\underline{V}}$. Then, $G'(\underline{V})$ is an inner form of the quasi-split similitude special orthogonal group $\mathrm{GSO}(4, \chi_{\mathbb{E}})$ which is isomorphic to $\mathrm{GL}_2(\mathbb{E})/\mathbb{E}^1$. Thus, we have $G'(\underline{V}) \cong \mathbb{B}^\times/\mathbb{E}^1$ for some division quaternion algebra $\mathbb{B}$ over $\mathbb{E}$.
- Finally, suppose that $\epsilon = -1$ and $m = 3$. Then $G'(\underline{V})$ is an inner form of the split similitude special orthogonal group GSO_6 which is isomorphic to $\mathrm{GL}_4 \times \mathrm{GL}_1^\times /\{(z, z^{-2}) \mid z \in \mathrm{GL}_1\}$. Thus, we have $G'(\underline{V}) \cong \mathbb{D}_4^\times \times \mathbb{F}^\times/\{(z, z^{-2}) \mid z \in \mathbb{F}^\times\}$ for some central division algebra $\mathbb{D}_4$ with $[\mathbb{D}_4 : \mathbb{F}] = 16$.

Therefore, we can apply the Jacquet-Langlands correspondence to the study of irreducible representations of $G'(\underline{V})$: let Σ be an irreducible cuspidal automorphic representation of $G'(\underline{V})(\mathbb{A})$.

- Suppose that $\epsilon = 1$ and $m = 1$. Then we can regard Σ as a representation of $\mathbb{D}^1(\mathbb{A})$, and we define the Jacquet-Langlands correspondence representation $\mathrm{JL}(\Sigma)$ of $\mathrm{SL}_2(\mathbb{A}) = \mathrm{Sp}_2(\mathbb{A})$.
- Suppose that $\epsilon = -1$ and $m = 1$. Then, by identification $G'(\underline{V}) = \mathrm{SO}(\underline{V'})$ as above, we can regard Σ as a representation of $\mathrm{SO}(\underline{V'})$, and we denote it by $\mathrm{JL}(\Sigma)$.
- Suppose that $\epsilon = -1$ and $m = 2, 3$. Then, we can define a representation $\mathrm{JL}(\Sigma)$ of $\mathrm{GSO}(4, \chi_{\mathbb{E}})(\mathbb{A})$ or $\mathrm{GSO}_6(\mathbb{A})$ respectively by the isomorphisms explained above.

On the other hand, let F be a local field of characteristic 0, let D be the division quaternion algebra over F, and let V be an ϵ-Hermitian space over D. We define $G'(V)$ as in the global case. Then, for irreducible representation σ of $G'(V)$, we define $\mathrm{JL}(\sigma)$ in the similar way by using the local Jacquet-Langlands correspondence. In this section, we use these accidental isomorphisms to analyze the Plancherel measure.

Proposition 17.1. *Let F be a local field, let D be a division quaternion algebra over F, let V be an m-dimensional ϵ-Hermitian space. Denote by V_0 be the anisotropic kernel of V, and write $V = X + V_0 + X^*$ where X, X^* are totally isotropic subspace so that $X + X^*$ is the orthogonal complement of V_0. For an irreducible representation σ of $G(V_0)$ and an irreducible supercuspidal representation τ of $\mathrm{GL}(X)$, there is a rational function $\Upsilon(s)$ such that all zeros of $\Upsilon(s)$ lie in $\{\Re s \leq 0\}$ and*

$$\mu(s, \sigma \boxtimes \tau) = \frac{\Upsilon(s)}{\Upsilon(1+s)} \cdot \frac{\gamma(2s, \mathrm{JL}(\tau), \wedge^2, \psi)}{\gamma(1+2s, \mathrm{JL}(\tau), \wedge^2, \psi)}. \tag{17.1}$$

In particular, if $\mathrm{JL}(\tau)$ is supercuspidal and the image of the L-parameter $\phi_\tau : \mathrm{SL}_2(\mathbb{C}) \times W_F \to \mathrm{GL}_{2r}(\mathbb{C})$ is contained in $\mathrm{Sp}_{2r}(\mathbb{C})$, then $\mu(s, \sigma \boxtimes \tau)$ has at least one pole in $\mathbb{R}_{>0}$. Here, $\gamma(s, \mathrm{JL}(\tau), \wedge^2, \psi)$ is the Langlands-Shahidi γ-factor ([Sha90]*).*

Proof. Take

- a supercuspidal representation σ' of $G'(V_0)$ such that $\sigma'|_{G(V_0)} \supset \sigma$,
- a global field $\mathbb{F}$ and places v_1, v_2 of $\mathbb{F}$ such that $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$,
- a division quaternion algebra $\mathbb{D}$ over $\mathbb{F}$ such that $\mathbb{D}_{v_1} = \mathbb{D}_{v_2} = D$, and for all place $v \neq v_1, v_2$, $\mathbb{D}_v$ is split,
- an anisotropic ϵ-Hermitian space $\mathbb{V}_0$ such that $\mathbb{V}_{0v_1} = \mathbb{V}_{0v_2} = V_0$, and for all place $v \neq v_1, v_2$, $G(\mathbb{V}_0)$ is quasi-split,
- a non-Archimedean place $v_3 \neq v_1, v_2$,
- an irreducible cuspidal automorphic representation Σ of $G'(\mathbb{V}_0)(\mathbb{A})$ such that $\Sigma_{v_1} = \Sigma_{v_2} = \sigma'$ and Σ_{v_3} is supercuspidal,
- a vector spaces $\mathbb{X}$, $\mathbb{X}^*$ over $\mathbb{D}$ such that $\dim_{\mathbb{D}} \mathbb{X} = \dim_{\mathbb{D}} \mathbb{X}^* = \dim_D X$,
- a cuspidal representation Ξ of $\mathrm{GL}(\mathbb{X})(\mathbb{A})$ such that $\Xi_{v_1} = \Xi_{v_2} = \tau$ and Ξ_{v_3} is supercuspidal.

Then $\mathrm{JL}(\Sigma)$ and $\mathrm{JL}(\Xi)$ are cuspidal, hence globally generic. Hence,

$$\begin{aligned}
\mu(s, \sigma \boxtimes \tau)^2 &= \mu(s, \sigma' \boxtimes \tau)^2 \\
&= \prod_{v \neq v_1, v_2} \mu_v(s, \Sigma \boxtimes \Xi)^{-1} \\
&= \prod_{v \neq v_1, v_2} \mu_v(s, \mathrm{JL}(\Sigma) \boxtimes \mathrm{JL}(\Xi))^{-1} \\
&= \mu(s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau))^2.
\end{aligned}$$

Moreover, by the positivity,

$$\begin{aligned}
\mu(s, \sigma \boxtimes \tau) &= \mu(s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau)) \\
&= \frac{\gamma(s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau), \psi)}{\gamma(1+s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau), \psi)} \\
&\quad \times \frac{\gamma(2s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau \cdot \chi_W), \wedge^2, \psi)}{\gamma(1+2s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau), \wedge^2, \psi)}.
\end{aligned}$$

Since V_0 is anisotropic, $\mathrm{JL}(\sigma)$ is a discrete series representation. Moreover, $\mathrm{JL}(\tau)$ is also a discrete series representation. Hence, by [Sha90, Proposition 7.2], $\gamma(s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau), \mathrm{std}, \psi)$ has no zero in $\{\Re s \leq 0\}$. Hence, putting

$$\Upsilon(s) := \gamma(s, \mathrm{JL}(\sigma') \boxtimes \mathrm{JL}(\tau), \mathrm{std}, \psi),$$

we have the equation (17.1).

Now, we suppose that $\mathrm{JL}(\tau)$ is supercupidal and the image of the L-parameter $\phi_\tau : \mathrm{SL}_2(\mathbb{C}) \times W_F \to \mathrm{GL}_{2r}(\mathbb{C})$ is contained in $\mathrm{Sp}_{2r}(\mathbb{C})$. Then, by a result of Jiang-Nien-Qin [JNQ10], we can conclude that $\gamma(s, \tau, \wedge^2, \psi)$ has a pole at $s = 1$. Let $\mathrm{Fr} \in W_F$ be a Frobenius element. Then, by [GR10, Lemma3], $\phi_\tau(\mathrm{Fr})$ has finite order, hence, $[\wedge^2 \circ \phi_\tau](\mathrm{Fr})$ is a unitary operator. Thus all poles of $L(s, \tau, \wedge^2)$ lie in $\{\Re s = 0\}$, and we can conclude that $\gamma(s, \tau, \wedge^2, \psi)$ has a pole at $s = 1$, and all poles of $\gamma(s, \tau, \wedge^2, \psi)$ lie in $\{\Re s = 1\}$. Hence, the ratio

$$\frac{\gamma(2s, \mathrm{JL}(\tau), \wedge^2, \psi)}{\gamma(1 + 2s, \mathrm{JL}(\tau), \wedge^2, \psi)}$$

has a pole at $s = \frac{1}{2}$. Put

$$\mathcal{P} = \{s_0 \geq \frac{3}{2} \mid \Upsilon(s) \text{ has a pole at } s = s_0\}.$$

If $\mathcal{P} = \varnothing$, then $\mu(s, \sigma \boxtimes \tau)$ has a pole at $s = \frac{1}{2}$ since all zeros of $\Upsilon(s)$ lie in $\{\Re s \leq 0\}$. If $\mathcal{P} \neq \varnothing$, then the ratio $\Upsilon(s)/\Upsilon(1 + s)$ has a pole at $s = \sup \mathcal{P}$. Hence we have the proposition. □

Finally, we note here that there exists at least one irreducible supercuspidal representation τ of $\mathrm{GL}_r(D)$ such that the Jacquet-Langlands correspondence $\mathrm{JL}(\tau)$ has an L-parameter whose image is contained in $\mathrm{Sp}_{2r}(\mathbb{C}) \times W_F$ (for a construction, see [Mie20, §4]).

18. Induction argument

In this section, we prove the compatibility of $\alpha_3(V, W)$ with the induction on the dimensions of V, W with $l = 1$. Now, we explain more precisely. Let V be an m-dimensional right ϵ-Hermitian space, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space. We assume that $l = 2n - 2m - \epsilon = 1$. Note that we allow V and W to be 0. Consider

- an ϵ-Hermitian space V' containing V and its totally isotropic subspaces X, X^* so that $\dim_D X = \dim_D X^* = t$, $X + V + X^* = V'$ and $X + X^*$ is the orthogonal complement of V,
- a $(-\epsilon)$-Hermitian space W' containing W and its totally isotropic subspaces Y, Y^* so that $\dim_D Y = \dim_D Y^* = t$, $Y + W + Y^* = W'$ and $Y + Y^*$ is the orthogonal complement of W.

We put $n' = n + 2t$ and $m' = m + 2t$. Then, we will prove

$$\alpha_3(V', W') = \alpha_3(V, W) \tag{18.1}$$

in this section. First, we prove (18.1) with some hypotheses, which will be proved in the latter part of this section. Let Q (resp. P) be the maximal parabolic subgroup of $G(V')$ (resp. $G(W')$) preserving X (resp. Y). Then, we can identify the Levi subgroup L_Q (resp. M_P) of Q (resp. P) with $\mathrm{GL}(X) \times G(V)$ (resp. $\mathrm{GL}(Y) \times G(W)$).

Proposition 18.1. *Suppose that there are $s_0 > 0$, an irreducible supercuspidal representation π of $G(W)$, an irreducible supercuspidal representation σ of $G(V)$, and a non-trivial irreducible supercuspidal representation τ of $\mathrm{GL}(X) \cong \mathrm{GL}(Y)$ so that*

- $\sigma \cong \theta_\psi(\pi, V)$,
- $\mathrm{Ind}_P^{G(W')} \pi \boxtimes \tau_{s_0}\chi_V$ *is reducible, and*
- $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_W$ *is reducible.*

Then, $\mathrm{Ind}_P^{G(W')} \pi \boxtimes \tau_{s_0}\chi_V$ *and* $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_W$ *have unique irreducible square integrable representations π' and σ' respectively, and σ' coincides with $\theta_\psi(\pi', V')$. Moreover, we have $\alpha_3(V', W') = \alpha_3(V, W)$.*

We proof this proposition in the former part of this section. Suppose that a quadraple (s_0, π, σ, τ) as in the proposition is given. By Lemma 9.2 and Proposition 9.5, the representation $\theta_\psi(\pi', V')$ is the unique square-integrable irreducible subquotient of $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_W$, which is nothing other than σ'. To prove the last assertion, we use the following proposition, which is due to a result of Heiermann [Hei04].

Proposition 18.2. *Let $s_0 > 0$, let π be an irreducible supercuspidal representation of $G(W)$ and let τ be a supercuspidal representation of $\mathrm{GL}_t(D)$. Suppose that $\mu(s, \pi \boxtimes \tau\chi_V)$ has a pole at $s = s_0$. Then we have the following:*

(1) The induced representation $\mathrm{Ind}_Q^{G(W')} \pi \boxtimes \tau_{s_0}\chi_V$ is reducible and it has a unique irreducible square integrable constituent π'. Moreover we have,

$$\begin{aligned}\deg \pi' =& 2t \log q \cdot \deg \pi \deg \tau \cdot \mathrm{Res}_{s=s_0} \mu(s, \pi \boxtimes \tau\chi_V) \\ &\times \gamma(G(W')/P) \cdot \frac{|K_{M_P}|}{|K_{G(W')}|} \cdot |U_P \cap K_{W'}| \cdot |\overline{U_P} \cap K_{W'}|.\end{aligned}$$

Here, $\gamma(G(W')/P)$ is the constant defined by

$$\gamma(G(W')/P) = \int_{\overline{U}} \delta_P(\overline{u})\, d\overline{u}$$

where $\overline{U}$ is the unipotent radical of the opposite parabolic subgroup $\overline{P}$ of P, and f° is the unique $K_{W'}$-invariant section of the representation $\mathrm{Ind}_P^{G(W')} \delta_P^{\frac{1}{2}}$ induced by the square root of the modular character δ_P so that $f^\circ(1) = 1$.

(2) The induced representation $\mathrm{Ind}_Q^{G(V')} \sigma \boxtimes \tau_{s_0}\chi_V$ is also reducible, and it has a unique irreducible square integrable constituent σ'. Moreover we have

$$\begin{aligned}\deg \sigma' =& 2t \log q \cdot \deg \sigma \deg \tau \cdot \mathrm{Res}_{s=s_0} \mu(s, \sigma \boxtimes \tau\chi_W) \\ &\times \gamma(G(V')/Q) \cdot \frac{|K_{L_Q}|}{|K_{G(V')}|} \cdot |U_Q \cap K_{V'}| \cdot |\overline{U_Q} \cap K_{V'}|.\end{aligned}$$

Here, $\gamma(G(V')/Q)$ is the constant defined similarly as in (1).

Proof. Similar to [GI14, Proposition 20.4]. □

Now, take π as in Proposition 18.4, and put $\sigma = \theta(\pi, V)$. Then, by Proposition 18.2, we have

$$\begin{aligned}\frac{\deg \pi'}{\deg \sigma'} &= \frac{\deg \pi}{\deg \sigma} \cdot \frac{\mathrm{Res}_{s=s_0} \mu(s, \pi \boxtimes \chi_V)}{\mathrm{Res}_{s=s_0} \mu(s, \sigma \boxtimes \chi_W)} \cdot \frac{\gamma(G(W)/P)}{\gamma(G(V)/Q)} \cdot \frac{|K_{G(V')}||K_{M_P}|}{|K_{G(W')}||K_{L_Q}|} \\ &= \frac{\deg \pi}{\deg \sigma} \cdot \gamma(s_0 - \frac{l-1}{2}, |\ |^{s_0}, \psi)\gamma(-s_0 - \frac{l-1}{2}, |\ |^{s_0}, \overline{\psi}) \\ &\times \frac{|U_P \cap K_{W'}| \cdot |\overline{U_P} \cap K_{W'}|}{|U_Q \cap K_{V'}| \cdot |\overline{U_Q} \cap K_{V'}|} \cdot \frac{|\mathcal{B}^+_{V'}||\mathcal{B}^+_{M_P}|}{|\mathcal{B}^+_{W'}||\mathcal{B}^+_{L_Q}|} \\ &\times \frac{\prod_{\alpha \in \Sigma_{\mathrm{red}}(\overline{P})} [X_\alpha \cap K_{W'} : X_\alpha \cap \mathcal{B}^+_{W'}]^{-1}}{\prod_{\beta \in \Sigma_{\mathrm{red}}(\overline{Q})} [X_\beta \cap K_{V'} : X_\beta \cap \mathcal{B}^+_{V'}]^{-1}}.\end{aligned}$$

Here, we denote by $\mathcal{B}^+$ by the pro-unipotent radical of $\mathcal{B}$, by $\Sigma_{\mathrm{red}}(\overline{P})$ (resp. $\Sigma_{\mathrm{red}}(\overline{Q})$) the set of positive reduced root with respect to the opposite parabolic subgroup $\overline{P}$ (resp. $\overline{Q}$) of P (resp. Q), and by X_α (resp. X_β) the root subrgroup associated with $\alpha \in \Sigma_{\mathrm{red}}(\overline{P})$ (resp. $\beta \in \Sigma_{\mathrm{red}}(\overline{Q})$).

Lemma 18.3. *We have*

$$\frac{|\mathcal{B}_{V'}^{+}||\mathcal{B}_{M_P}^{+}|}{|\mathcal{B}_{W'}^{+}||\mathcal{B}_{L_Q}^{+}|} = q^{2t}.$$

Proof. Since $|\mathcal{B}_{M_P}^{+}| = |\mathcal{B}_{W}^{+}||\mathcal{B}_{\mathrm{GL}_r(D)}^{+}|$ and $|\mathcal{B}_{L_Q}^{+}| = |\mathcal{B}_{V}^{+}||\mathcal{B}_{\mathrm{GL}_r(D)}^{+}|$, we have

$$\begin{aligned}\frac{|\mathcal{B}_{V'}^{+}||\mathcal{B}_{M_P}^{+}|}{|\mathcal{B}_{W'}^{+}||\mathcal{B}_{L_Q}^{+}|} &= \frac{|\mathcal{B}_{V'}^{+}||\mathcal{B}_{W}^{+}|}{|\mathcal{B}_{W'}^{+}||\mathcal{B}_{V}^{+}|} \\ &= \begin{cases} q^{(n'^2-n^2)-(m'^2-m'-m^2+m)-\frac{1}{2}(a'_{V'}-a'_{V})} & (-\epsilon = 1), \\ q^{(n'^2-n'-n^2+n)-(m'^2-m^2)+\frac{1}{2}(a'_{W'}-a'_{W})} & (-\epsilon = -1) \end{cases}\end{aligned}$$

where

$$a'_W = \begin{cases} 0 & (\chi_W : \text{ unramified}) \\ -1 & (\chi_W : \text{ ramified}). \end{cases}$$

One can show that both coincide with q^{2t}. Hence we have the lemma. □

Moreover, we have

$$\begin{aligned}\frac{\prod_{\alpha\in\Sigma_{\mathrm{red}}(\overline{P})}[X_\alpha \cap K_{W'} : X_\alpha \cap \mathcal{B}_{W'}^{+}]^{-1}}{\prod_{\beta\in\Sigma_{\mathrm{red}}(\overline{Q})}[X_\beta \cap K_{V'} : X_\beta \cap \mathcal{B}_{V'}^{+}]^{-1}} &= q^{-2(n_0-m_0)t} \\ &= q^{-(1+\epsilon)t},\end{aligned}$$

and

$$\begin{aligned}\frac{|U_P \cap K_{W'}|\cdot|\overline{U_P}\cap K_{W'}|}{|U_Q \cap K_{V'}|\cdot|\overline{U_Q}\cap K_{V'}|} &= q^{-2(nt+\frac{1}{2}t(t-\epsilon))}\cdot q^{2(mt+\frac{1}{2}t(t+\epsilon))} \\ &= q^{-2(n-m)+2\epsilon t} \\ &= q^{-(1-\epsilon)t}.\end{aligned}$$

Hence we have

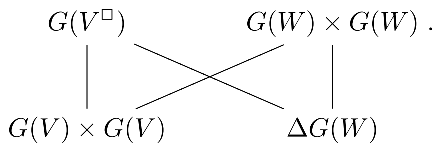
$$\begin{aligned}\frac{\deg\pi'}{\deg\sigma'} &= \frac{\deg\pi}{\deg\sigma}\cdot\gamma(s_0-\frac{l-1}{2},\tau,\psi)\gamma(-s_0-\frac{l-1}{2},\tau^\vee,\overline{\psi}) \\ &= \frac{\deg\pi}{\deg\sigma}\cdot\gamma(s_0,\tau,\psi)\gamma(-s_0,\tau^\vee,\overline{\psi})\end{aligned}$$

since $l = 1$. Thus we have Proposition 18.1.

Now, we prove the existence of the quadruple (s_0, π, σ, τ) as in Proposition 18.1 when either V or W is anisotropic.

Proposition 18.4. *Suppose that V is anisotropic. Then, there exists an irreducible supercuspidal representation π of $G(W)$ such that $\Theta_\psi(\pi, V) \neq 0$.*

Proof. We use the following see-saw diagram to prove:



More precisely, for an irreducible representation σ of $G(V)$, we have

$$\Theta_\psi(\sigma, W) \neq 0 \Leftrightarrow \mathrm{Hom}_{\Delta G(W)}(\Theta_\psi(\sigma, W) \otimes \Theta_\psi(\sigma, W)^\vee, 1_W) \neq 0$$
$$\Leftrightarrow \sigma \boxtimes \sigma^\vee \text{ appears as a quotient of } \Theta_\psi(1_W, V^\square)|_{G(V)\times G(V)}.$$

In the cases where W is anisotropic, the proposition is clear by the above observation. Then we suppose that W is isotropic. This only occurs in the case where $\epsilon = -1$. Thus we have $\chi_W = 1$. Hence, we have the isomorphism

$$R_s : I^V(s, 1) \to \mathcal{S}(G(V))$$

by $[R_s f_s](g) = f_s(\iota(g, 1))$ for $f_s \in I^V(s, 1)$ and $g \in G(V)$.

Lemma 18.5. *For $u \in U(V^\triangle)$ there is a unique element $g_u \in G(V)$ such that $\iota(g_u, 1) \in P(V^\triangle)\tau u$ for some $p \in P(V^\triangle)$. Moreover, $u \mapsto g_u$ gives a homeomorphism*

$$U(V^\triangle) \to G(V) \setminus \{1\}.$$

By this lemma, if we take a non-zero function $\varphi \in \mathcal{S}(G(V))$ so that $\overline{\mathrm{supp}(\varphi)} \not\ni 1$ and $\varphi(g) \geq 0$ for all $g \in G(V)$, then the integral defining $M(s, 1)(R_s^{-1}\varphi)$ converges and $M(s, 1)(R_s^{-1}\varphi) \neq 0$ for all $s \in \mathbb{C}$. On the other hand, if we denote by W_i the i-dimensional $(-\epsilon)$-Hermitian space with $\chi_{W_i} = 1$ and by l_i the integer $2i - 2m - \epsilon$, then we have

$$\Theta_\psi(1_{W_i}, V^\square) = \ker M(-\frac{l_i}{2}, 1)$$

for $i = 0, \ldots, n-1$ by [Yam11, Theorems 1.3, 1.4]. Thus, we have proved that

$$\sum_{i=0}^{n-1} R_{-l_i/2}(\Theta_\psi(1_{W_i}, V^\square)) \subsetneqq \mathcal{S}(G(V)).$$

Hence, there is an irreducible representation σ of $G(V)$ such that $n^+(\sigma) \geq n$ and $n^-(\sigma) \geq n+1$. Since we have assumed $l = 1$, the conservation relation (Proposition 9.3) says that $n^+(\sigma) + n^-(\sigma) = 2n + 1$. Thus, we have $n^+(\sigma) = n$, and we have the lemma by putting $\pi = \Theta_\psi(\sigma, W)$. □

Proposition 18.6. *Suppose that W is anisotropic and V is isotropic. Then, there is an irreducible representation π of $G(W)$ such that $\theta_\psi(\pi, V)$ is non-zero supercuspidal.*

Proof. The situation in this proposition occurs only in the case where $\epsilon = 1$, $\dim W = 3$, $\dim V = 2$ and $\chi_W = \chi_V = 1$. Then, as in §17, we have the accidental isomorphism

$$\widetilde{G}(W) \cong D_4^\times \times F^\times / \{(a, a^{-2}) \mid a \in F^\times\}$$

where D_4 is a central division algebra of F so that $[D_4 : F] = 16$. Now, we denote by π_0 an irreducible representation of $D_4^\times$ obtained by as follows: let π_1 be an irreducible supercuspidal representation of $\mathrm{GL}_4(F)$ so that the image of its L-parameter is contained in $\mathrm{Sp}_4(\mathbb{C}) \times W_F$ (see [Mie20, §4]). Then we denote by π_0 the irreducible representation of $D_4^\times$ associated with π_1 by the Jacquet-Langlands correspondence. Since the central character of π_0 is trivial, we have the irreducible representation $\pi_0 \boxtimes 1$ of $D_4^\times \times F^\times / \{(a, a^{-2}) \mid a \in F^\times\}$. We may regard it as a representation of $\widetilde{G}(W)$ by the accidental isomorphism. We denote by π an irreducible component of the restriction of $\pi_0 \boxtimes 1$ to $G(W)$. Then, the square exterior γ-factor $\gamma(s, \phi_{\pi_0}, \wedge^2, \psi)$ has a pole at $s = 1$ (see [JNQ10]). Hence we have $\Theta_\psi(\pi, V) \neq 0$ (see [GT14, Theorem 6.1], and [GT14, Proposition 3.3]). Moreover, since $\pi \neq 1$, we have $m(\pi) > 0$. This forces that $m(\pi) = 2$, and $\theta_\psi(\pi, V)$ is supercuspidal. Hence we have the proposition. □

Corollary 18.7. *There exist (s_0, π, σ, τ) as in Proposition 18.1 when either V or W is anisotropic.*

Proof. Suppose first that V is anisotropic. Take π as Proposition 18.4, and put $\sigma = \theta_\psi(\pi, V)$. Moreover, take (τ, s_0) as the latter part of Proposition 17.1. Since $\mathrm{JL}(\tau)$ is an irreducible supercuspidal represetation of $\mathrm{GL}_{2r}(F)$, the Godement-Jaquet L-function $L(s,\tau)$ is 1. Hence the poles of $\mu(s, \pi \boxtimes \tau\chi_V)$ is equal to that of $\mu(s, \sigma \boxtimes \tau\chi_W)$ by Proposition 16.7. Therefore, $\mu(s, \sigma \boxtimes \tau\chi_W)$ has a pole at $s = s_0$. Hence, the quadruple (s_0, π, σ, τ) satisfies the assumption of Proposition 18.1.

Then, suppose that W is anisotropic and V is isotropic. Take π as in Proposition 18.6, and put $\sigma = \theta_\psi(\pi, V)$. Moreover, take τ as an irreducible supercuspidal representation of $\mathrm{GL}_r(D)$ so that $\mathrm{JL}(\tau)$ is a symplectic non-zero supercuspidal representation of $\mathrm{GL}_{2r}(F)$ (see [Mie20, §4]). Then, by Proposition 17.1, $\mu(s, \pi \boxtimes \tau\chi_V)$ has a pole at a positive real number s_0. Hence, the quadraple (s_0, π, σ, τ) satisfies the assumption 18.1. Hence we have the corollary. □

Corollary 18.7 completes the proof of (18.1).

19. Determination of α_1 and α_2

In this section, we complete the formulas of $\alpha_1(W)$ and $\alpha_2(V, W)$ even when both V and W are isotropic. Let V be an m-dimensional right ϵ-Hermitian space, and let W be an n-dimensional left $(-\epsilon)$-Hermitian space. We assume that $2n - 2m - \epsilon = 1$. Take a basis $\underline{e} = (e_1, \ldots, e_n)$ for W. Note that in this section, we do not suppose that $R(\underline{e})$ is of the form (4.1). First, we have:

Theorem 19.1.

$$\alpha_2(V, W) = |2|_F^{-2n\rho + n(n-\frac{1}{2})} \cdot |N(R(\underline{e}))|^\rho \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(1-2i)}{\zeta_F(2i)}$$
$$\times \begin{cases} 2(-1)^n \gamma(1-n, \chi_V, \psi)^{-1} \epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ 1 & (-\epsilon = -1). \end{cases}$$

Proof. We note first that there is at least one irreducible square irreducible integrable representation π of $G(W)$ such that $\Theta_\psi(\pi, V) \neq 0$ (this has been proved in §18 by replacing V with V'). Then, comparing the formula of $\alpha_3(V, W)$ of Proposition 15.1 with its definition in Theorem 11.1, we obtain

$$\frac{1}{2} \cdot \alpha_2(V, W) \cdot e(G) \cdot |2|_F^{2n\rho - n(n-\frac{1}{2})} \cdot |N(R(\underline{e}))|^{-\rho} \cdot \prod_{i=1}^{n-1} \frac{\zeta_F(2i)}{\zeta_F(1-2i)}$$
$$\times \begin{cases} \chi_V(-1)\gamma(1-n, \chi_V, \psi) & (-\epsilon = 1), \\ \chi_W(-1)\epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1) \end{cases}$$
$$= \begin{cases} (-1)^n \chi_V(-1)\epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ \frac{1}{2} \cdot \chi_W(-1)\epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases}$$

Hence, we have the claim. □

Suppose that $-\epsilon = -1$. We denote by W^u a $(-\epsilon)$-Hermitian space so that $\dim W^u = n$ and $W^{(u)}$ possesses a basis $\underline{e}^{(u)}$ with $R(\underline{e}^{(u)}) \in \mathrm{GL}_n(\mathcal{O}_D)$. Then, by Theorem 19.1, we have:

Corollary 19.2.

$$\alpha_2(V, W) = |N(R(\underline{e}))|^\rho \cdot \alpha_2(V, W^u).$$

Proof. Since $|N(R(\underline{e}^{(u)})| = 1$, the claim follows from Theorem 19.1. □

On the other hand, we may identify $W^{u\square}$ with $W^\square$ by identifying e'_i with $e_i^{(u)\prime}$ for $i = 1, \ldots, 2n$. Then we can compare $\mathcal{I}^{W^u}$ with $\mathcal{I}^W$:

Lemma 19.3. *For $\phi, \phi' \in \mathcal{S}(V \otimes W^{\triangle}) = \mathcal{S}(V \otimes W^{u\triangle})$, we have*

$$\mathcal{I}^W(\phi, \phi') = \mathcal{I}^{W^u}(\phi, \phi').$$

Proof. By writing down the definitions, we have the equation. □

Therefore, we have

$$\begin{aligned}\frac{\alpha_1(W)}{\alpha_1(W^u)} &= \frac{\mathcal{E}^{W^u}(\phi, \phi')}{\mathcal{E}^W(\phi, \phi')} \\ &= \frac{\alpha_2(V, W^u)}{\alpha_2(V, W)} \\ &= |N(R(\underline{e}))|^{-\rho}.\end{aligned}$$

Thus, we have a general formula of $\alpha_1(W)$:

Proposition 19.4. *In the case $-\epsilon = -1$, we have*

$$\alpha_1(W) = |2|_F^{2n\rho} \cdot |N(R(\underline{e}))|^{-\rho} \cdot q^{-(2\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor)} \cdot \prod_{i=1}^{n} (1 + q^{-(2i-1)}).$$

Proof. We already have a formula of $\alpha_1(W)$ either when $n_0 = 0$ or $n_0 = 1$ and χ_W is unramified (Proposition 7.6). Hence, we have the proposition by Lemma 19.3. □

20. Formal degree conjecture for the non-split inner forms of Sp_4, GSp_4

The Langlands conjecture for the non-split inner forms of GSp_4 and Sp_4 has been established by Gan-Tantono [GT14] and Choiy [Cho17]. Thus, the refined formal degree conjecture for these groups can be stated unconditionally. In this section, we prove the refined formal degree conjecture for the non-split inner forms of Sp_4 and GSp_4 as an application of Theorem 11.1. We denote by $G_{1,1}$, $H_{1,1}$, and $H_{3,0}$ the isometry groups of

- the two dimensional Hermitian space W with $\chi_W = 1$,
- the two dimensional skew-Hermitian space W with $\chi_W = 1$,
- the three dimensional skew-Hermitian space W with $\chi_W = 1$

respectively. We also denote by $\widetilde{G}_{1,1}$, $\widetilde{H}_{1,1}$, and $\widetilde{H}_{3,0}$ their similitude groups respectively. Note that in this section we regard $H_{1,1}, H_{3,0}$ as inner forms of quasi-split special orthogonal groups $\mathrm{SO}_{2,2}$ and $\mathrm{SO}_{3,3}$ (see Remark 3.1). In this section, we assume that G is one of $G_{1,1}, H_{1,1}, H_{3,0}$, and we assume that $\widetilde{G}$ is the corresponding similitude group. We denote by $\mathfrak{p} : \widehat{\widetilde{G}} \to \widehat{G}$ the projection of [Lab85, Theorem 8.1]. Let $\widetilde{\phi}$ be an L-parameter for $\widetilde{G}$. We denote by $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ the L-parameter given by the composition $\mathfrak{p} \circ \widetilde{\phi}$. According to [Cho17, §7.3], the L-parameter ϕ of $\widetilde{G}_{1,1}$ is classified into one of the following "Case I-(a), Case I-(b), Case II, Case III";

- Case I-(a): the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{1,1}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 2, and the action of $\mathrm{Hom}(W_F, \mathbb{C}^1)$ is not transitive;
- Case I-(b): the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{1,1}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 2, and the action of $\mathrm{Hom}(W_F, \mathbb{C}^1)$ is transitive;
- Case II: the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{1,1}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 1;
- Case III: the parameter $\widetilde{\phi}$ comes from that of $\widetilde{H}_{3,0}$, and the cardinality of the L-packet $\Pi_{\widetilde{\phi}}$ is equal to 1.

Denote by $X(\widetilde{\phi})$ the stabilizer

$$\{a \in H^1(W_F, \widehat{\mathrm{GL}_1}) \mid a\widetilde{\phi} = \widetilde{\phi} \text{ as } L\text{-parameters } \}.$$

Then we have an exact sequence

$$\mathcal{S}_{\widetilde{\phi}}(\widetilde{G}) \to \mathcal{S}_{\phi}(G) \to X(\widetilde{\phi}) \to 1$$

where $\mathcal{S}_\phi(G)$ is the component group $\pi_0(S_\phi(G))$ of $S_\phi(G) := C_\phi(G)/Z(\widehat{G})^{\Gamma_{F^s/F}}$ (similarly for $\mathcal{S}_{\widetilde{\phi}}(\widetilde{G})$). In the case $S_\phi(G)$ is a finite group (An L-parameter associated with an irreducible square integrable representation satisfies this.), the first map is injective. Put $C'_{\widetilde{\phi}}(\widetilde{G}) = C_{\widehat{\phi}}(\widetilde{G}) \cap \widehat{\widetilde{G}/A}$, and put $Z'(\widetilde{G}) = Z(\widetilde{G}) \cap \widehat{\widetilde{G}/A}$ where A is the maximal F-split torus of the center of $G(W)$. Then, we have $S_\phi(\widehat{\widetilde{G}}) = C'_\phi(\widetilde{G})/Z'(\widetilde{G})^{\Gamma_{F^s/F}}$.

20.1. **Restriction of representations from $\widetilde{G}$ to G.** It is known that such restriction problems have much information of Langlands parameters for G. We only use the following lemma:

Lemma 20.1. *Let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$. Then, we have a decomposition*

$$\widetilde{\pi}|_G = (\bigoplus_{i=1}^{t} \pi_i)^{\oplus k}$$

where $\pi_1, \dots, \pi_t$ are irreducible representations of G and

$$k = \begin{cases} \frac{1}{2}\dim\eta & G = G_{1,1} \text{ and } \widetilde{\pi} \text{ has the } L\text{-parameter of Case I-(b)}, \\ \dim\eta & \text{otherwise.} \end{cases}$$

Proof. It is obtained by [Cho17, Theorems 5.1, 6.1, 7.5]. □

In this paper, we need this lemma to prove the following two lemmas.

Lemma 20.2. *Let π be a square integrable irreducible representation of G, let (ϕ, η) be its Langlands parameter, let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$ so that its restriction $\widetilde{\pi}|_G$ to G contains π, and let $(\widetilde{\phi}, \widetilde{\eta})$ be the Langlands parameter of $\widetilde{\pi}$. Then, we have*

$$\deg\widetilde{\pi} = \frac{\dim\widetilde{\eta}}{\dim\eta} \cdot \frac{\#C_\phi(G)}{\#C'_{\widetilde{\phi}}(\widetilde{G})} \cdot \deg\pi, \ \textit{and}\ \mathrm{Ad}\circ\widetilde{\phi} = \mathrm{Ad}\circ\phi.$$

Proof. Put

$$X(\widetilde{\pi}) = \{\chi \in \mathrm{Hom}(F^\times, \mathbb{C}^\times) \mid (\chi\circ\lambda)\widetilde{\pi} \cong \widetilde{\pi}\}.$$

Then the reciprocity map of the local class field theory induces an embedding $X(\widetilde{\pi}) \to X(\widetilde{\phi})$. Moreover, we have

$$[X(\widetilde{\phi}) : X(\widetilde{\pi})] = \begin{cases} 2 & G = G_{1,1} \text{ and } \widetilde{\pi} \text{ has the } L\text{-parameter of Case I-(b)}, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, by [GI14, Lemma 13.2] and by Lemma 20.1, we have

$$\begin{aligned}\deg\pi &= \frac{\#Z'(\widehat{\widetilde{G}})}{\#Z(\widehat{G})}\cdot\frac{k}{\#X(\widetilde{\pi})}\cdot\deg\widetilde{\pi}\\ &= \frac{\#Z'(\widehat{\widetilde{G}})}{\#Z(\widehat{G})}\cdot\frac{\dim\eta\cdot\#\mathcal{S}_{\widetilde{\phi}}(\widetilde{G})}{\#\mathcal{S}_{\phi}(G)}\cdot\deg\widetilde{\pi}\\ &= \frac{\dim\eta\cdot\#C'_{\widetilde{\phi}}(\widetilde{G})}{\#C_{\phi}(G)}\cdot\deg\widetilde{\pi}.\end{aligned}$$

Moreover, since the projection $\mathfrak{p}:\widehat{\widetilde{G}}\to\widehat{G}$ factors through the adjoint map Ad, we have

$$\begin{aligned}\mathrm{Ad}\circ\widetilde{\phi} &= \mathrm{Ad}\circ\mathfrak{p}\circ\phi\\ &= \mathrm{Ad}\circ\phi.\end{aligned}$$

Hence, we have the lemma. □

Lemma 20.3. *Let π be a square integrable irreducible representation of $G_{1,1}$, and let σ be an irreducible representation of either $H_{1,1}$ or $H_{3,0}$ associated with π by the local theta correspondence. We assume that $\sigma\neq 0$. We denote by (ϕ_π,η_ϕ) (resp. $(\phi_\sigma,\eta_\sigma)$) the Langlands parameter associated with π (resp. σ). Then, we have*

$$\frac{\dim\eta_\sigma}{\dim\eta_\pi}=\begin{cases}\frac{1}{2} & \pi\text{ has the L-parameter of Case I-(b)},\\ 1 & \text{otherwise}\end{cases}\tag{20.1}$$

and we have

$$\frac{\#C'_{\phi_\sigma}}{\#C'_{\phi_\pi}}=\begin{cases}\frac{1}{2} & \pi\text{ has the L-parameter of Case I-(b), III},\\ 1 & \text{otherwise}\end{cases}$$

Proof. Note that discrete series parameters are of Case I and Case III. By [GT14, Proposition 3.3] and Lemma 20.1, we have (20.1). The remaining equality is obtained by the case-by-case discussion in [Cho17, p. 1867 - p.1874]. □

20.2. **Refined formal degree conjecture.** In this section, we discuss the refined formal degree conjecture [GR10, Conjecture 7.1]. We first prove it for inner forms of GL_N. Note that $\#C'_\phi(\mathrm{GL}_N)=N$ if ϕ is a discrete parameter for GL_N.

Lemma 20.4. *Let G be an inner form of GL_N, and let π be a square-integrable irreducible representation of G. Then, we have*

$$\deg(\pi)=c_\pi(-1)^{N-1}\cdot\frac{1}{N}\cdot\gamma(0,\pi,\mathrm{Ad},\psi).$$

Here, Ad *is the adjoint representation of ${}^LG$ on $\mathfrak{sl}_N(\mathbb{C})$.*

Proof. By [HII08, §3.1], we have

$$\deg(\pi)=\frac{1}{N}\cdot|\gamma(0,\pi,\mathrm{Ad},\psi)|.$$

Denote by $\mathrm{JL}(\pi)$ the Jacquet-Langlands correspondence of π to $\mathrm{GL}_N(F)$. Note that $c_{\pi^*}(-1) = c_\pi(-1)$. Then, by [GI14, Proposition 14.1], we have

$$\begin{aligned}\frac{\gamma(0,\pi,\mathrm{Ad},\psi)}{|\gamma(0,\pi,\mathrm{Ad},\psi)|} &= \frac{\gamma(0,\mathrm{JL}(\pi),\mathrm{Ad},\psi)}{|\gamma(0,\mathrm{JL}(\pi),\mathrm{Ad},\psi)|}\\ &= c_{\mathrm{JL}(\pi)}(-1)^{N-1}\\ &= c_\pi(-1)^{N-1}.\end{aligned}$$

Thus, by positivity of $\deg\pi$, we have the lemma. □

Let G' be one of $G_{1,1}$, $H_{1,1}$, $H_{3,0}$, $\widetilde{G}_{1,1}$, $\widetilde{H}_{1,1}$, and $\widetilde{H}_{3,0}$. Then the refined formal degree conjecture for G' is true:

Theorem 20.5. *Let π be a square integrable irreducible representation of G', and let (ϕ,η) be its Langlands parameter. Then we have*

$$\deg\pi = c_\pi(-1)\cdot\frac{\dim\eta}{\#C'_\phi(G')}\cdot\gamma(0,\mathrm{Ad}\circ\phi,\psi).$$

Proof. When G' is either $\widetilde{H}_{1,1}$ or $\widetilde{H}_{3,0}$, we have the claim because of the accidental isomorphisms

$$\begin{aligned}\widetilde{H}_{1,1} &= D^\times\times\mathrm{GL}_2(F)/\{(t,t^{-1}\cdot I_2)\mid t\in F^\times\},\\ \widetilde{H}_{3,0} &= D_4^\times\times F^\times/\{(t,t^{-2})\mid t\in F^\times\}\end{aligned}$$

as in §17. Here, D_4 is a central division algebra of F with $[D_4:F]=16$. Hence, we have the claim for $H_{1,1}$ and $H_{3,0}$ by Lemma 20.2. When $G'=G_{1,1}$, we have the claim by Theorem 11.1 and Lemma 20.3 since

$$\frac{\gamma(s,\pi,\mathrm{Ad},\psi)}{\gamma(s,\sigma,\mathrm{Ad},\psi)} = \gamma(s,\sigma\times\chi_W,\psi).$$

Hence, we also have the claim for $\widetilde{G}_{1,1}$. Thus we have the theorem. □

21. Appendix: an explicit formula of zeta integrals

In [Kak20b, Proposition 8.3], the author computed the doubling zeta integral of right $K(\underline{e}'^{\square})$-invariant sections. However, the formula does not tell us about the constant term and a certain multiplier polynomial factor $S(T)$. In this section, we complete the formula by applying the formula of $\alpha_1(W)$. Note that there are two errors in [Kak20b, Proposition 8.3]. We also point out and correct them. In this section, we assume **the residue characteristic of F is not** 2. Finally, we note that the results in this Appendix are not used in this paper but had been used in the previous version. In the case $q \not| 2$, we can prove by Proposition 7.6 them.

Fix a basis $\underline{e}$ of W as in §4. We denote by $\underline{e}_0$ the basis $e_{r+1},\dots,e_{r+n_0}$ for W_0. Moreover, we may suppose that

$$R_0 = R(\underline{e}_0) = \begin{cases} 1 & (-\epsilon=1, n_0=1),\\ \alpha & (-\epsilon=-1,n_0=1),\\ \varpi_D^{-1} & (-\epsilon=-1,n_0=1),\\ \mathrm{diag}(\varpi_D^{-1},\alpha\varpi_D^{-1}) & (-\epsilon=-1,n_0=2,\chi_W \text{ is unramified}),\\ \mathrm{diag}(\alpha,\varpi_D^{-1}) & (-\epsilon=-1,n_0=2,\chi_W \text{ is ramified}),\\ \mathrm{diag}(\alpha,\varpi_D^{-1},\alpha\varpi_D^{-1}) & (-\epsilon=-1,n_0=3).\end{cases}$$

Here, α is defined in §2. We recall that we have put $n_0=\dim W_0$ and $r=\frac{n-n_0}{2}$. By this basis, we regard $G(W)$ as a subgroup of $\mathrm{GL}_n(D)$. Then, put

$$C_1 := \{g\in G(W)\cap\mathrm{GL}_n(\mathcal{O}_D)\mid (g-1)R(\underline{e}'^{\square})\in\mathrm{M}_n(\mathcal{O}_D)\}.$$

Note that C_1 is an open compact subgroup of $G(W)$. Let X_i be a subspace of X spanned by $e_1, \ldots, e_i$. We denote by $\mathfrak{f}$ the flag

$$\mathfrak{f} : 0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X,$$

and by B the minimal parabolic subgroup preserving $\mathfrak{f}$.

Proposition 21.1. *We have $G(W) = B \cdot C_1$.*

Proof. We use the setting and the notation of §5 in the proof of this proposition. By [BT72, Théorème (5.1.3)], we have the decomposition

$$G(W) = B \cdot N_{G(W)} \cdot \mathcal{B}.$$

Since $B \supset Z_{G(W)}(S)$, we can take a representative system $w_1, \ldots, w_t$ for $B \backslash (B \cdot N_{G(W)}(S))$ so that $w_i \in C_1$ for $i = 1, \ldots, t$. Moreover, $X_{a,0} \subset C_1$ for $a \in \Phi^+$ and $X_{a,\frac{1}{2}} \subset C_1$ for $a \in \Phi^-$. Hence, by Lemma 5.3, we have

$$\begin{aligned} B \cdot N_{G(W)}(S) \cdot \mathcal{B} &= \bigcup_{i=1}^{t} B \cdot w_i \cdot Z_{G(W)}(S)_1 \cdot \prod_{a \in \Phi^+} X_{a,0} \cdot \prod_{a \in \Phi^-} X_{a,\frac{1}{2}} \\ &= \bigcup_{i=1}^{t} B \cdot Z_{G(W)}(S)_1 \cdot w_i \cdot \prod_{a \in \Phi^+} X_{a,0} \cdot \prod_{a \in \Phi^-} X_{a,\frac{1}{2}} \\ &\subset B \cdot C_1. \end{aligned}$$

Thus we have the proposition. □

Let σ_0 be the trivial representation of $G(W_0)$, let s_i be a complex number for $i = 1, \ldots, r$, let σ_i be the character $|\ |^{s_i}$ of $\mathrm{GL}_1(D)$ for $i = 1, \ldots, r$. Then, $\sigma = \otimes_{i=0}^{r} \sigma_i$ is a character of the Levi subgroup of B. Let π be an irreducible subquotient representation of $\mathrm{Ind}_B^{G(W)}(\sigma)$ having a non-zero C_1-fixed vector. Then, we have the following formula of a zeta integral with a certain section and a matrix coefficient:

Proposition 21.2. *Let $f_s^\circ \in I(s,1)^{K(\underline{e}'^{\square})}$ be a non-zero $K(\underline{e}'^{\square})$-invariant section with $f_s^\circ(1) = 1$, let ξ° be the C_1-fixed matrix coefficient of π. Then, we have*

$$Z(f_s^\circ, \xi^\circ) = |C_1| \cdot \frac{S(q^{-s})}{d^W(s)} \prod_{i=0}^{r} L^{W_i}(s + \frac{1}{2}, \sigma_i)$$

for some self-reciprocal monic polynomial $S(T)$ of degree

$$f_W = \begin{cases} 1 & (-\epsilon = -1, n_0 = 2, \chi_W \text{ is unramified}), \\ 0 & (\text{ otherwise}). \end{cases}$$

Here we set

$$d^W(s) = \begin{cases} \zeta_F(s + n + \frac{1}{2}) \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta_F(2s + 2n + 1 - 4i) & \text{if } -\epsilon = 1, \\ \prod_{i=1}^{\lceil n/2 \rceil} \zeta_F(2s + 2n + 3 - 4i) & \text{if } -\epsilon = -1. \end{cases}$$

Note that if $n_0 = 0$, then $L^{W_0}(s, 1_{W_0} \times 1)$ denotes

$$\begin{cases} \zeta_F(s) & \text{if } -\epsilon = 1, \\ 1 & \text{if } -\epsilon = -1. \end{cases}$$

Note that we will determine $S(T)$ and $|C_1|$ later (Propositions 21.4 and 21.5).

Remark 21.3. *Proposition 21.2 differs from* [Kak20b, Proposition 8.3] *at the definition of* f_W *in the case* $n_0 = 3$ *and the definition of* $L^{W_0}(s, 1_{W_0} \times 1)$ *in the case* $n_0 = 0$, $-\epsilon = 1$. *The former is caused by an error of the computation of the* γ*-factor, which is modified by* (21.3). *And the latter is caused by a typo.*

Proof. We can deform the doubling zeta integral to the summation

$$Z^W(f_s^\circ, \xi^\circ) = \int_{C_1} \xi^\circ(g)\, dg + \int_{G(W)-C_1} f_s^\circ((g,1))\xi^\circ(g)\, dg.$$

If s_0 is a sufficiently large real number so that $Z^W(f_s^\circ, \xi^\circ)$ converges absolutely, then, by [Kak20b, Lemma 8.4], we have

$$\begin{aligned}\left| \int_{G(W)-C_1} f_s^\circ((g,1))\xi^\circ(g) \right| &\leq \int_{G(W)-C_1} |\Delta((g,1))|^{s-s_0} |f_{s_0}^\circ((g,1))\xi^\circ(g)|\, dg \\ &\leq q^{-(\Re s - s_0)} \int_{G(W)} |f_{s_0}^\circ((g,1))\xi^\circ(g)|\, dg\end{aligned}$$

for $\Re s > s_0$. Thus we have

$$\lim_{\Re s \to \infty} Z^W(f_s^\circ, \xi^\circ) = |C_1|. \tag{21.1}$$

Put

$$\Xi(q^{-s}) := \frac{Z^W(f_s^\circ, \xi^\circ)}{\prod_{i=0}^r L^{W_i}(s + \frac{1}{2}, \sigma_i \times 1)}.$$

Then, by the "g.c.d property" ([Yam14, Theorem 5.2] and [Yam14, Lemma 6.1]) concludes that $\Xi(q^{-s})$ is a polynomial in q^{-s} and q^s. Moreover, by (21.1), it is a polynomial of q^{-s} with the constant term $|C_1|$. Put $D(q^{-s}) := d^W(s)$. Once we prove the equation

$$\Xi(q^{-s})D(q^s) = (q^{-s})^{f_W} \cdot \Xi(q^s)D(q^{-s}), \tag{21.2}$$

one can deduce that

$$\Xi(q^{-s}) = |C_1| \cdot S(q^{-s})D(q^{-s})$$

for some monic self-reciprocal monic polynomial of degree f_W since $q^{-ts}D(q^s)$ is a polynomial of q^{-s} which is coprime to $D(q^{-s})$ for sufficiently large t, which proves the proposition.

In the following, we prove the equation (21.2). By the definition of the γ-factor, we have

$$R(s, 1, A, \psi) \cdot Z^W(M^*(s, 1, A, \psi) f_s^\circ, \xi^\circ) = \pi(-1) \cdot \gamma^W(s + \frac{1}{2}, \pi \boxtimes 1, \psi) Z^W(f_s^\circ, \xi^\circ).$$

Note that $\pi(-1) = 1$ and by comparing this with the equation

$$R(s, 1, A, \psi)M^*(s, 1, A, \psi) f_s^\circ = q^{-n's}|N(R(\underline{e}))|^{-s}\epsilon(\frac{1}{2}, \chi_W, \psi) \cdot \frac{D(q^{-s})}{D(q^s)} f_{-s}^\circ$$

where

$$n' = \begin{cases} 2\lceil \frac{n}{2} \rceil & (-\epsilon = 1), \\ 2\lfloor \frac{n}{2} \rfloor & (-\epsilon = -1), \end{cases}$$

we obtain

$$\begin{aligned}\Xi(q^{-s})D(q^s) &= D(q^{-s})\Xi(q^s) \\ &\quad \times |N(R(\underline{e}))|^{-s} q^{-n's} \cdot \frac{\epsilon(\frac{1}{2}, \chi_W, \psi)}{\gamma(s + \frac{1}{2}, \pi \boxtimes 1, \psi)} \cdot \frac{\prod_{i=0}^r L^{W_i}(-s + \frac{1}{2}, \sigma_i^\vee \times 1)}{\prod_{i=0}^r L^{W_i}(s + \frac{1}{2}, \sigma_i \times 1)}.\end{aligned}$$

Moreover, by Lemma 7.5, we have

$$\gamma^W(s+\frac{1}{2},\pi\times 1,\psi)=q^{-\lambda s}\cdot\epsilon^W(\frac{1}{2},\chi_W,\psi)\prod_{i=0}^{r}\frac{L^{W_i}(-s+\frac{1}{2},\sigma_i^\vee\times 1)}{L^{W_i}(s+\frac{1}{2},\sigma\times 1)} \tag{21.3}$$

where

$$\lambda=\begin{cases}2\lceil\frac{n}{2}\rceil & -\epsilon=1,\\ 2\lfloor\frac{n}{2}\rfloor & -\epsilon=-1,\ n\not\equiv 3 \bmod 4,\ \chi_W:\ \text{unramified},\\ 2\lfloor\frac{n}{2}\rfloor+1 & \text{otherwise.}\end{cases}$$

Therefore,

$$\begin{aligned}\Xi(q^{-s})D(q^s)&=D(q^{-s})\Xi(q^s)\cdot q^{-(n'-\lambda)s}\cdot|N(R(\underline{e}))|^{-s}\\&=D(q^{-s})\Xi(q^s)\cdot(q^{-s})^{f_W}.\end{aligned}$$

Hence we have the equation (21.2), and we have the proposition. □

For the polynomial $S(T)$, we have the following:

Proposition 21.4. *We have*

$$S(T)=\begin{cases}T^2+(q^{\frac{1}{2}}+q^{-\frac{1}{2}})T+1 & (-\epsilon=-1,\ n_0=2,\ \chi_W\ \textit{is unramified}),\\ 1 & (\ \textit{otherwise}).\end{cases}$$

Proof. We have $f_W=0$ in the cases other than $-\epsilon=-1$, $n_0=2$, and χ_W is unramified. Thus the proposition is clear for the second case. Consider the case $n=n_0=2$ and χ_W is unramified. Since $G(W)$ is compact, $Z(f_s^\circ,\xi^\circ)$ is a polynomial in q^{-s}. In other words,

$$S(q^{-s})\frac{\zeta_F(s+\frac{3}{2})L(s+\frac{1}{2},\chi_W)}{\zeta_F(2s+3)}$$

is a polynomial. Thus, we can conclude that $(1+q^{-\frac{1}{2}}T)$ divides $S(T)$. Such a self-reciprocal polynomial is only $(1+q^{-\frac{1}{2}}T)(1+q^{\frac{1}{2}}T)$. Hence we have

$$S(T)=T^2+(q^{\frac{1}{2}}+q^{-\frac{1}{2}})T+1.$$

Now, suppose that $-\epsilon=-1$, $n>n_0=2$, and χ_W is unramified. We recall a certain intertwining operator associated with the parabolic induction. Let $Q(X^\square)$ be the parabolic subgroup of $G(W^\square)$ preserving $X^\square$, let $U(X^\square)$ be the unipotent radical of $Q(X^\square)$, let M be the Levi-subgroup of $Q(X^\square)$, and let $I^X(s,1)$ be the space of smooth functions f on $\mathrm{GL}(X^\square)$ satisfying

$$f(pg)=|N(p|_{X^\triangle})|^{-(s+r)}|N(p|_{X^\triangledown})|^{s+r}f(g)$$

for $p\in P'(X^\triangle)$ and $g\in\mathrm{GL}(X^\square)$. Here, we denote by $P'(X^\triangle)$ the parabolic subgroup of $\mathrm{GL}(X^\square)$ preserving $X^\triangle$, by $p|_{X^\triangle}$ (resp. $p|_{X^\triangledown}$) the restriction of p to $X^\triangle$ (resp. $X^\triangledown$), and by N the reduced norm of $\mathrm{End}(X^\triangle)$ (resp. $\mathrm{End}(X^\triangledown)$). For a coefficient ξ of an irreducible representation of $\mathrm{GL}(X)$ and a section $f\in I(s,1)$, we define the doubling zeta integral by

$$Z^X(f,\xi)=\int_{\mathrm{GL}(X)}f(\iota_X(g,1))\xi(g)\,dg$$

where $\iota_X:\mathrm{GL}(X)\times\mathrm{GL}(X)\to\mathrm{GL}(X^\square)$ is the embedding induced by the natural action of $\mathrm{GL}(X)\times\mathrm{GL}(X)$ on $X^\square$. Then, there is an intertwining map

$$\Psi(s):I^W(s,1)\to\mathrm{Ind}_{Q(X^\square)}^{G(W^\square)}(I^X(s,1)\otimes I^{W_0}(s,1)\otimes|\Delta_{(X,W_0):W}|):f_s\mapsto(g\mapsto[\Phi(s)f_s]_g)$$

(see [Yam14, Proposition 4.1]). Although we omit the definition, we note the relation

$$[\Phi(s)f_s^\circ]_e=J(s)f_s'^\circ\otimes f_s''^\circ$$

where $f_s'^{\circ}$ (resp. $f_s''^{\circ}$) is the unique $\mathrm{GL}_r(\mathcal{O}_D)$-invariant section of $I^X(s,1)$ (resp. the unique $K(\underline{e}_0'^{\square})$-invariant section of $I^{W_0}(s,1)$) so that $f_s'^{\circ}(1)=1$ (resp. $f_s''^{\circ}(1)=1$), and

$$J(s)=\int_{U(X^{\square})\cap Q(W^{\triangle})\backslash U(X^{\square})} f_s^{\circ}(u)\,du.$$

Moreover, by Proposition 21.1, we have

$$\begin{aligned} Z^W(f_s^{\circ},\xi^{\circ}) &= |C_1|\int_Q f_s^{\circ}((g,1))\,dg \\ &= |C_1|\int_M [\Psi(s)f_s^{\circ}]((m,1))\,dm \\ &= |C_1|J(s)Z^{W_0}(f_s'^{\circ},\xi'^{\circ})Z^X(f_s''^{\circ},\xi''^{\circ}) \\ &= |C_1|J(s)S(q^{-s})\frac{L^{W_0}(s+\frac{1}{2},1_W\times 1)}{d^{W_0}(s)}\cdot\frac{L^X(s+\frac{1}{2},\sigma)}{d^X(s)} \\ &= |C_1|S^{W_0}(q^{-s})\frac{J(s)}{d^{W_0}(s)d^X(s)}L^W(s+\frac{1}{2},1_W\times 1). \end{aligned}$$

Thus, we obtain

$$S^W(q^{-s})=S^{W_0}(q^{-s})\times J(s)\frac{d^W(s)}{d^{W_0}(s)d^X(s)}.$$

However, since $J(s)$ does not have a pole in $\Re s>-1$ ([Yam14, Lemma 5.1]) and $d^W(s), d^{W_0}(s), d^X(s)$ has neither a pole nor a zero at $s=\pi\sqrt{-1}\pm\frac{1}{2}$, we can conclude that $S^W(X)$ is diveded by $(1+q^{\pm\frac{1}{2}}T)$. Thus, we have $S^W(T)=S^{W_0}(T)$. Hence, we finish the proof of the proposition. □

Finally, by the formula of $\alpha_1(W)$ (Proposition 19.4), we can determine the volume $|C_1|$ of C_1:

Proposition 21.5. *(1) In the case $-\epsilon=1$, we have*

$$|C_1|=|K_W|=q^{-2\lfloor n/2\rfloor\lceil n/2\rceil-\lceil n/2\rceil}\prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})(1-q^{-2i}).$$

(2) In the case $-\epsilon=-1$, we have

$$|C_1|=|N(R(\underline{e}))|^{-\rho}q^{-(2\lfloor n/2\rfloor\lceil n/2\rceil-\lfloor n/2\rfloor)}$$
$$\times\begin{cases}\prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor}(1-q^{-2i}) & (n_0=0),\\ \prod_{i=1}^{\lceil n/2\rceil}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor}(1-q^{-2i}) & (n_0=1,\chi_W:\textit{unramified}),\\ \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor}(1-q^{-2i}) & (n_0=1,\chi_W:\textit{ramified}),\\ \prod_{i=1}^{\lfloor n/2\rfloor-1}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor-1}(1-q^{-2i}) & (n_0=2,\chi_W:\textit{unramified}),\\ \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor-1}(1-q^{-2i}) & (n_0=2,\chi_W:\textit{ramified}),\\ \prod_{i=1}^{\lfloor n/2\rfloor}(1+q^{-(2i-1)})\prod_{i=1}^{\lfloor n/2\rfloor-1}(1-q^{-2i}) & (n_0=3).\end{cases}$$

Proposition 21.4 and Proposition 21.5 give a completion of the formula in Proposition 21.2.

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