Gromov–Hausdorff limits of compact Heisenberg manifolds with sub-Riemannian metrics

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Abstract

In the dissertation, we study a moduli space of left invariant metrics on compact Heisenberg manifolds. It is an analogy of the classical moduli space of flat metrics on tori. A key idea is a new volume form on compact Heisenberg manifolds, called a minimal Popp's volume form, which is continuous under the canonical topology of the moduli space.

In this setting, we show a version of Mahler's compactness theorem for compact Heisenberg manifolds (Theorem 6.3). To be precise, a subspace of the moduli space is precompact in the canonical topology if total measure with respect to the minimal Popp's volume is uniformly bounded above and the systole is uniformly bounded below. Moreover we show that the canonical topology of the moduli space coincides with the Gromov–Hausdorff topology (Theorem 7.1). This concludes that non-collapsed Gromov–Hausdorff limits of compact Heisenberg manifolds are isometric to again compact Heisenberg manifolds.

1 Introduction

A sub-Riemannian manifold is a triple (M, D, g), where M is a connected smooth manifold, $D \subset TM$ is a distribution, and g is a metric on D. Recently sub-Riemannian geometry is actively studied from a viewpoint of geometric analysis and optimal control theory. In the dissertation, we study Gromov–Hausdorff limits of a sequence of sub-Riemannian manifolds.

The model of our work is the following Mahler's compactness theorem.

Theorem 1.1 (Theorem 2 and 3 of [5])**.** *Let $\mathcal{M}(T^n)$ be the moduli space of flat metrics on $\mathbb{T}^n$. A subset $\mathcal{A} \subset \mathcal{M}(\mathbb{T}^n)$ is precompact in the canonical topology if and only if there are $v, s > 0$ such that for any metric in $\mathcal{A}$,*

(1) the total measure is bounded above by v,

(2) the systole is bounded below by s.

This theorem implies that every non-collapsed limit of a sequence of n-dimensional flat tori is isometric to a flat torus of the same dimension.

In sub-Riemannian geometry, the Heisenberg Lie group H_n is a model of flat space. Indeed, the tangent cone of contact sub-Riemannian manifolds are isometric to the Heisenberg Lie group [6], and has zero curvature in the Bott connection [1]. Hence their quotient by uniform discrete subgroups Γ, compact Heisenberg manifolds, are analogies of flat tori.

When the metric is Riemannian, a version of the Mahler's compactness theorem is showed by Boldt. Let $\mathcal{M}_R(\Gamma\backslash H_n)$ be the moduli space of left invariant Riemannian metrics on compact Heisenberg manifold $\Gamma\backslash H_n$. The definition is given by Gordon–Wilson in [3].

Theorem 1.2 (Precisely in Theorem 6.1)**.** *A subset $\mathcal{A} \subset \mathcal{M}_R(\Gamma\backslash H_n)$ is precompact in the quotient topology if and only if the assumptions (A-1)-(A-4) on metric tensor hold.*

We will explain the assumptions (A-1)-(A-4) later since they are described by using the parametrization of the moduli space $\mathcal{M}_R(\Gamma\backslash H_n)$.

To generalize the sub-Riemannian setting, we study the moduli space of left invariant sub-Riemannian metrics on $\Gamma\backslash H_n$, denoted by $\mathcal{M}(\Gamma\backslash H_n)$. The Riemannia moduli space $\mathcal{M}_R(\Gamma\backslash H_n)$ is densely embedded into $\mathcal{M}(\Gamma\backslash H_n)$. It is easy to see that a subset in $\mathcal{M}(\Gamma\backslash H_n)$ is precompact if and only if slightly different conditions (A-1)-(A-3) and (A-4)' hold.

Our question is how to describe these conditions with geometric assumptions on an upper bound of the volume and a lower bound of the systole. When the metric is Riemannian, then the canonical Riemannian volume form is defined by the wedge of dual coframe of an orthonormal frame. When the metric is sub-Riemannian (not Riemannian), then the Popp's volume is a well known volume form invariant by isometry. It is appropriate to describe conditions with the canonical Riemannian volume and the Popp's volume. However, an upper bound of the above two volume do not give a necessary condition for a subset in the moduli space $\mathcal{M}(\Gamma\backslash H_n)$ being precompact. Indeed if a sequence of Riemannian metrics conveges to a sub-Riemannian one, then the associated Riemannian volume forms diverge to the infinity. This implies that a subset with a uniform upper bound of the Riemannian volume form does not contain such sequences.

To address this issue, we introduce a new volume form which is well posed on Lie groups. Let $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ be a connected Lie group endowed with a left invariant sub-Riemannian metric. For a subspace $\mathfrak{v}' \subset \mathfrak{v}$, Let $vol(\mathfrak{v}')$ be the Popp's volume form of the restricted sub-Riemannian structure $(\mathfrak{v}', \langle\cdot,\cdot\rangle_{\mathfrak{v}'\otimes\mathfrak{v}'})$.

Definition 1.1. *For $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$, define the minimal Popp's volume $minvol(\mathfrak{v}, \langle\cdot,\cdot\rangle)$ by the minimum of $vol(\mathfrak{v}')$, where $\mathfrak{v}'$ is a subspace of $\mathfrak{v}$.*

The minimum exists since the Grassmannian manifold is compact.

By using the minimal Popp's volume, we can describe the condition for a subset in $\mathcal{M}(\Gamma\backslash H_n)$ being precompact in the canonical quotient topology.

Theorem 1.3. *Let X be a subset in $\mathcal{M}(\Gamma\backslash H_n)$. If there are constants $C_1, C_2 > 0$ such that for any $(\mathfrak{v}, \langle\cdot,\cdot\rangle) \in \mathcal{M}(\Gamma\backslash H_n)$, $\int_{\Gamma\backslash H_n} minvol(\mathfrak{v}, \langle\cdot,\cdot\rangle) \leq C_1$ and $systole(\Gamma\backslash H_n, \mathfrak{v}, \langle\cdot,\cdot\rangle) \geq C_2$. Then X is precompact in the quotient topology.*

For its proof, we show the assumptions in Theorem 1.3 implies the condition (A-1)-(A-4)' by Boldt.

2 Moduli space and Mahler's compactness theorem for flat tori

In this section, the author explains the classical theory of flat tori. it is well known that the moduli space of flat metrics on the torus $\mathbb{T}^n$ is parametrized by

$$\mathcal{M}(\mathbb{T}^n) \simeq GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})/O(n).$$

Here we identify a matrix $A \in GL_n(\mathbb{R})$ to the inner product $\langle \cdot, \cdot \rangle_A$ such that its orthonormal basis is $\{Ae_1, \dots, Ae_n\}$.

Remark 2.1. *The original parametrization is written by $O(n)\backslash GL_n(\mathbb{R})/GL_n(\mathbb{Z})$. It is homeomorphic to our parametrization via the mapping $A \mapsto A^{-1}$.*

The classical Mahler's compactness theorem asserts the following.

Theorem 2.1 (Theorem 2 and 3 of [5])**.** *A subset $\mathcal{X} \subset \mathbb{T}^\backslash$ is precompact in the quotient topology if and only if there are constants $C_1, C_2 > 0$ such that for any $[A] \in X$*

(1) $\det(A) \geq C_1$*,*

(2) $\|z\|_A \geq C_2$ *for all* $z \in \mathbb{Z}^n \setminus \{0\}$*.*

It is easy to see that the first condition is equivalent to the total measure of the torus is less than or equal to C_1^{-1}, and the systole is greater than or equal to C_2.

3 Moduli space of compact nilmanifolds with left-invariant sub-Riemannian metrics

Let N be a simply connected nilpotent Lie group, $\mathfrak{n}$ the associated Lie algebra, and Γ a lattice in N. Fix a basis of $\mathfrak{n}$ by $\{X_1, \dots, X_n\}$. Define a subset $\mathcal{X}_k \subset M(\dim \mathfrak{n})$ by

$$\mathcal{X}_k = \left\{ A \in M(n) \middle| \begin{array}{c} KerA = Span\{X_{i_1}, \dots, X_{i_k}\}, \\ \{i_1, \dots, i_k\} \subset \{1, \dots, n\}, \\ ImA \text{ is bracket generating} \end{array} \right\}.$$

Here a subspace $\mathfrak{v} \subset \mathfrak{n}$ is bracket generating if there is $r \in \mathbb{N}$ such that

$$\mathfrak{v} + [\mathfrak{v}, \mathfrak{v}] + \cdots \underbrace{[\mathfrak{v}, [\mathfrak{v}, \cdots, \mathfrak{v}] \cdots]}_{r} = \mathfrak{n}.$$

From a matrix $A \in \mathcal{X}_k$, we obtain the sub-Riemannian structure $\{ImA, \langle \cdot, \cdot \rangle_A\}$ such that the orthonormal basis is $\{AX_1, \dots, AX_n\}$. The moduli space of left-invaraint sub-Riemannian metrics on $\Gamma\backslash N$ is given as follows.

Theorem 3.1.

$$\mathcal{M}(\Gamma\backslash N) \simeq (Stab(\Gamma)\cdot Inn(N))_* \backslash \bigcup_k \mathcal{X}_k/O(n),$$

where $Stab(\Gamma) < Aut(N)$ is the stabilizer of Γ and $Inn(N)$ is the group of inner automorphisms.

For its proof we use the affiness of isometries on nilpotent Lie group shown by Kivioja–Le Donne [4].

When N is the $(2n+1)$-dimensional Heisenberg Lie group H_n, we obtain more explicit parametrization of the moduli space as follows. Let $\mathfrak{h}_n$ be the associated Lie algebra, and fix a basis $\{X_1,\dots,X_n,Y_1,\dots,Y_n,Z\}$ of $\mathfrak{h}_n$ so that $[X_i,Y_i]=Z$ and the other brackets are zero. Let $D_n=\{\boldsymbol{r}=(r_1,\dots,r_n)\}$ be the set of n-tuple of natural numbers such that r_i divides r_{i+1} for all $i=1,\dots,n-1$. For $\boldsymbol{r}\in D_n$, let $\Gamma_{\boldsymbol{r}}$ be a lattice in H_n given by

$$\Gamma_{\boldsymbol{r}} = \langle \exp(r_1X_1),\dots,\exp(r_nX_n),\exp(Y_1),\dots,\exp(Y_n),\exp(Z)\rangle\,,$$

where $\exp:\mathfrak{h}_n\to H_n$ is the exponential map. Similar to the construction of $\mathcal{M}_R(\Gamma_{\boldsymbol{r}}\backslash H_n)$ in Remark 2.6 of [3], the moduli space $\mathcal{M}(\Gamma_{\boldsymbol{r}}\backslash H_n)$ is given as follows.

Theorem 3.2. *The moduli space of left-invariant sub-Riemannian metrics on $\Gamma_{\boldsymbol{r}}\backslash H_n$ is*

$$\mathcal{M}(\Gamma_{\boldsymbol{r}}\backslash H_n)\simeq \Pi_{\boldsymbol{r}}\backslash GL_{2n}(\mathbb{R})\times\mathbb{R}/O(2n)\times O(1),$$

where $GL_{2n}(\mathbb{R})\times\mathbb{R} = \left\{\begin{pmatrix}\tilde{A} & 0\\ 0 & \rho_A\end{pmatrix} \;\middle|\; \tilde{A}\in GL_{2n}(\mathbb{R}),\ \rho_A\in\mathbb{R}\right\}$,

$\Pi_{\boldsymbol{r}} = \iota\left(G_{\boldsymbol{r}}\cap \widetilde{Sp}_{2n}(\mathbb{R})\right)$,

$G_{\boldsymbol{r}} = diag(r_1,\dots,r_n,1,\dots,1)GL_{2n}(\mathbb{Z})diag(r_1,\dots,r_n,1,\dots,1)^{-1}$,

$$\widetilde{Sp}_{2n}(\mathbb{R}) = \left\{P\in GL_{2n}(\mathbb{R}) \;\middle|\; {}^tP\begin{pmatrix}O & I_n\\ -I_n & O\end{pmatrix}P = \epsilon(P)\begin{pmatrix}O & I_n\\ -I_n & O\end{pmatrix},\ \epsilon(P)=\pm1\right\},$$

$$\iota:\widetilde{Sp}_{2n}(\mathbb{R})\ni P\mapsto \begin{pmatrix}P & 0\\ 0 & \epsilon(P)\end{pmatrix}\in GL_{2n+1}(\mathbb{R})\simeq Aut_{\mathbb{R}}(\mathfrak{h}_n).$$

A metric is Riemannian if and only if $\rho_A\neq 0$. In particular, $\mathcal{M}_R(\Gamma_{\boldsymbol{r}}\backslash H_n)\simeq \Pi_{\boldsymbol{r}}\backslash GL_{2n}(\mathbb{R})\times\mathbb{R}\setminus\{0\}/O(2n)\times O(1)$.

Notice that there is the quotient topology on $\mathcal{M}(\Gamma_{\boldsymbol{r}}\backslash H_n)$ induced from $GL_{2n}(\mathbb{R})\times\mathbb{R}$.

4 Length minimizing paths on the Heisenberg group

In this section, the author describes length minimizing geodesics on the Heisenberg Lie group. In sub-Riemannian geometry, there are two kind of length minimizing paths, normal geodesics and abnormal geodesics. On the Heisenberg Lie group, it is well known that every length minimizing path is normal geodesic. As in Riemannian geometry, normal geodesic on a sub-Riemannian manifold (M, E, g) is given by the projection of the sub-Riemannian Hamiltonian flow on the cotangent bundle, where the Hamiltonian is

$$H(p) = \frac{1}{2}\sum_{k=1}^{m}\langle p \mid f_k(x)\rangle \quad \text{for } \; p \in T_x^*M,$$

where $\{f_1, \ldots, f_m\}$ is an orthonormal frame around x.

In the left-invariant sub-Riemannian metrics on H_n, the (global) orthonormal frame is $\{AX_1, \ldots, AY_n, AZ\}$ (AZ might be zero). This allows us to give an explicit form of geodesics in H_n with a matrix A.

For an explicit formula, we need the following j-operator. Let $\mathfrak{v}_0 = Span\{X_1, \ldots, Y_n\}$ be the subspace in $\mathfrak{h}_n$.

Definition 4.1. *For $A \in GL_{2n}(\mathbb{R}) \times \mathbb{R}$, let $j_A : \mathfrak{v}_0 \to \mathfrak{v}_0$ be the linear map defined by*

$$\langle j_A(U), V\rangle_A = Z^*([U, V]) \quad \text{for all } \; U, V \in \mathfrak{v}_0,$$

where Z^* is the dual covector of Z.

The matrix representation of j_A is given by ${}^t\tilde{A}\begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$ in the basis $\{AX_1, \ldots, AY_n\}$. Since the j-operator is skew symmetrizable, its eigenvalues are purely imaginary. We denote them by $\pm\sqrt{1}\lambda_1(A), \ldots, \pm\sqrt{-1}\lambda_n(A)$ with the order $0 < \lambda_1(A) \leq \cdots \leq \lambda_n(A)$.

We will denote by d_A the associated distance function on $\Gamma_{\boldsymbol{r}}\backslash H_n$ and $\tilde{d}_A$ the one on H_n.

Lemma 4.1. *For $A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \rho_A \end{pmatrix}$ and $p \in \mathbb{R}$, the distance from e to $\exp(pZ)$ is given by*

$$\tilde{d}_A(e, \exp(pZ)) = \frac{2}{\lambda_n(A)}\sqrt{|p|\pi\lambda_n(A) - 4\pi^2\rho_A^2}.$$

5 The minimal Popp's volume form on the Heisenberg Lie group

In this section, the author gives the explicit formula of the minimal Popp's volume. It is well known that the canonical Riemannian volume form is the wedge of the dual coframe of the orthonormal frame. In our setting, the canonical Riemannian volume associated to A is written by

$$v_R(A) = (AX_1)^* \wedge \cdots \wedge (AZ)^* = \det(\tilde{A})^{-1}\rho_A^{-1}X_1^* \wedge \cdots \wedge Z^*.$$

In sub-Riemannian geometry, the Popp's volume form is a generalization of the canonical Riemannian volume form. It is explicitly written by

$$v_{sR}(A) = \det(\tilde{A})^{-1}\delta(A)^{-1}X_1^* \wedge \cdots \wedge Z^*,$$

where $\delta(A)$ is the Hilbert–Schmidt norm of the operator j_A. We omit the detailed definition here.

Remark 5.1.
- *$v_R(A)$ and $v_{sR}(A)$ are well posed up to sign.*
- *Since the eigenvalue of j_A are $\pm\sqrt{-1}\lambda_i(A)$'s, $\delta(A) = \sqrt{2\sum_{k=1}^{n}\lambda_k(A)^2}$.*

Recall that the minimal Popp's volume is the infimum of the Popp's volume (including the canonical Riemannian volume) associated to the restricted sub-Riemannian structure $(\mathfrak{v}', \langle\cdot,\cdot\rangle_{\mathfrak{v}'\otimes\mathfrak{v}'})$, $\mathfrak{v}' \subset ImA$.

The explicit formula of the minimal Popp's volume is given as follows.

Proposition 5.1. *For $A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \rho_A \end{pmatrix}$, the minimal Popp's volume minvol(A) is written as follows;*

- *If $|\rho_A| \geq \delta(A)$, then minvol$(A) = \det(\tilde{A})^{-1}\rho_A^{-1}X_1^* \wedge \cdots \wedge Z^*$.*
- *If $|\rho_A| \leq \delta(A)$, then minvol$(A) = \det(\tilde{A})^{-1}\delta(A)^{-1}X_1^* \wedge \cdots \wedge Z^*$.*

The total measure of a compact Heisenberg manifold $\Gamma_{\boldsymbol{r}}\backslash H_n$ with respect the minimal Popp's volume is controled by its diameter. This proposition does not hold for the canonical Riemannian volume without the Ricci curvature upper bound.

Proposition 5.2. *For any $d > 0$, there is $V(d) > 0$ such that if the diameter of a compact Heisenberg manifold is bounded by d, then the minimal Popp's volume form is smaller than $V(d)X_1^* \wedge \cdots \wedge Z^*$.*

6 Mahler's compactness theorem for compact Heisenberg manifold

In this section, the author shows the main theorem. We use the idea of Boldt's result for left-invariant Riemannian metrics on compact Heisenberg manifold.

Theorem 6.1 (Corollary 3.14 in [2])**.** *A subset $\mathcal{A} \subset \mathcal{M}_R(\Gamma_{\boldsymbol{r}}\backslash H_n)$ is precompact in the quotient topology if and only if there are positive constants $C_1, \ldots, C_5 > 0$ such that for any $[A] \in \mathcal{A}$ the following four conditions hold;*

(A-1) $\inf\{\|z\|_A \mid z \in \mathfrak{v}_0 \cap \log(\Gamma_{\boldsymbol{r}})\} \geq C_1$,

(A-2) $|\det(\tilde{A})| \geq C_2$,

(A-3) $\lambda_n(A) \leq C_3$,

(A-4) $C_4 \le |\rho_A| \le C_5$.

It is easily generalized to the sub-Riemannian setting as follows.

Theorem 6.2. *A subset $\mathcal{A} \subset \mathcal{M}(\Gamma_{\boldsymbol{r}} \backslash H_n)$ is precompact in the quotient topology if and only if there are positive constants $C_1, \ldots, C_4 > 0$ such that for any $[A] \in \mathcal{A}$ the following four conditions hold;*

(A-1) $\inf\{\|z\|_A \mid z \in \mathfrak{v}_0 \cap \log(\Gamma_{\boldsymbol{r}})\} \ge C_1$,

(A-2) $|\det(\tilde{A})| \ge C_2$,

(A-3) $\lambda_n(A) \le C_3$,

(A-4) $|\rho_A| \le C_5$.

The idea is to obtain the conditions (A-1)-(A-4)' from a lower bound of the systole and an upper bound of the minimal Popp's volume form.

Proposition 6.1. *Denote by s the systole of $(\Gamma_{\boldsymbol{r}} \backslash H_n, ImA, \langle \cdot, \cdot \rangle_A)$. Then we have;*

(1) $\inf\{\|z\|_A \mid z \in \mathfrak{v}_0 \cap \log(\Gamma_{\boldsymbol{r}})\} \ge s$,

(2) $\delta(A) \le \frac{\sqrt{2}n}{s}$,

(3) $|\rho_A| \le C_s = \max\left\{\sqrt{\frac{n}{\sqrt{2}\pi s}}, \frac{1}{s}\right\}$.

Proposition 6.2. *Denote by v the total measure with respect to the minimal Popp's volume form of $(\Gamma_{\boldsymbol{r}} \backslash H_n, ImA, \langle \cdot, \cdot \rangle_A)$. Then we have*

$$\det(\tilde{A}) \ge \frac{\tilde{C}_s}{v} \prod_{i=1}^{n} r_i,$$

where $\tilde{C}_s = \min\left\{C_s^{-1}, \frac{s}{\sqrt{2}n}\right\}$.

These propositions show the following theorem.

Theorem 6.3. *A subset $\mathcal{A} \subset \mathcal{M}(\Gamma_{\boldsymbol{r}} \backslash H_n)$ is relatively compact in the quotient topology if there are positive constants $s, v > 0$ such that for any $[A] \in \mathcal{A}$,*

(1) $systole(\Gamma_{\boldsymbol{r}} \backslash H_n, d_A) \ge s$,

(2) $\int_{\Gamma_{\boldsymbol{r}} \backslash H_n} minvol(A) \le v$.

The converse assertion follows in an easy way.

7 The quotient topology and the Gromov–Hausdorff topology of the moduli space

In this last section, the author shows that thequotient topology coincides with the GRomov–Hausdorff topology. This is a consequence of the geodesic equation and Proposition 5.2.

Theorem 7.1. *Let* $Id : (\mathcal{M}(\Gamma_{\boldsymbol{r}}\backslash H_n), \mathfrak{O}_q) \to (\mathcal{M}(\Gamma_{\boldsymbol{r}}\backslash H_n), \mathfrak{O}_{GH})$ *be the identity map, where* $\mathfrak{O}_q$ *is the quotient topology and* $\mathfrak{O}_{GH}$ *is the Gromov–Hausdorff topology. Then the identity map is a homeomorphism.*

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