A summary of "A study on anabelian geometry of complete discrete valuation fields"

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Anabelian geometry has been developed over a much wider class of fields than Grothendieck, who is the originator of anabelian geometry, conjectured. So, it is natural to ask the following question:

What kinds of fields are suitable for the base fields of anabelian geometry?

Let $\mathcal{C}$ be a subcategory of the category of fields (e.g., the category of number fields, the category of p-adic local fields (i.e., finite extensions of $\mathbb{Q}_p$), etc.). We consider the above question from the following two points of view:

(I) Does the Grothendieck conjecture hold for hyperbolic curves over fields belonging to $\mathcal{C}$?

(II) Does the Grothendieck conjecture (i.e., an analogue of the theorem of Neukirch-Uchida) hold for fields themselves belonging to $\mathcal{C}$?

For example, number fields affirm both (I) and (II). On the other hand, finite fields affirm (I) and do not affirm (II). p-adic local fields affirm (I), and, in some sense, (II). Though it is known that the analogue of the theorem of Neukirch-Uchida does not hold for the absolute Galois groups of p-adic local fields as it is, Mochizuki proved a certain analogue of the theorem for the absolute Galois groups with ramification filtrations. For (I), more generally, the Grothendieck conjecture for hyperbolic curves over generalized sub-p-adic fields (i.e., fields isomorphic to subfields of fields finitely generated over the quotient field of the Witt ring with coefficients in an algebraic closure of $\mathbb{F}_p$) was proved by Mochizuki. Moreover, for (II), Abrashkin proved an analogue of the theorem of Neukirch-Uchida (for the absolute Galois groups with ramification filtrations) for higher local fields with (first) residue characteristics at least 3 (and for complete discrete valuation fields of any positive characteristics with finite residue fields).

For fields affirming (I) (resp. (II)), it is natural to ask whether or not the fields affirm (II) (resp. (I)). So, firstly, we consider whether or not higher local fields, which affirm (II) as above, affirm (I). Secondly, we note that p-adic local fields affirm (I), and, in some sense, (II) (for the absolute Galois groups with ramification filtrations), and generalized sub-p-adic fields affirm (I). Therefore, it is natural to ask, more generally, whether or not mixed-characteristic complete discrete valuation fields with perfect residue fields affirm (II) (for the absolute Galois groups with ramification filtrations). We consider these problems. Moreover, for both kinds of fields, we consider mono-anabelian reconstruction algorithms of various invariants of the fields from their absolute Galois groups (with ramification filtrations, in the latter case).

In summary, we treat the following problems:

(A) Which invariants of higher local fields are reconstructed from the absolute Galois groups in the sense of mono-anabelian reconstruction?

(B) Does the Grothendieck conjecture hold for hyperbolic curves over higher local fields?
(C) Which invariants of mixed-characteristic complete discrete valuation fields with perfect residue fields are reconstructed from the absolute Galois groups with ramification filtrations in the sense of mono-anabelian reconstruction?
(D) Does the analogue of the theorem of Neukirch-Uchida for mixed-characteristic complete discrete valuation fields with perfect residue fields and the absolute Galois groups with ramification filtrations hold?

For (A), we prove the following theorem:

Theorem A
Let K be a higher local field, $d \in \mathbb{Z}_{\geq 0}$ the dimension of K (as a higher local field), $K = K_d, K_{d-1}, \cdots, K_1, K_0$ a chain of residue fields of K (which is well-defined up to isomorphism) and G_K the absolute Galois group of K. Then we may determine group-theoretically from G_K which of the following statements holds:

(i) *K is a finite field.*
(ii) *K is of characteristic 0, and $K_d, \cdots, K_0$ are perfect fields.*
(iii) *K is of positive characteristic and is not a finite field.*
(iv) *K is of characteristic 0, and K_i is an imperfect field for some $0 < i < d$.*

In the case where one of (ii)∼(iv) holds, there exist mono-anabelian reconstruction algorithms of the following invariants from G_K:

- *the characteristic p of K_0;*
- *the cardinality of K_0;*
- *the dimension of K (as a higher local field);*
- $\mathrm{Ker}(G_K \twoheadrightarrow G_{K_0})$ *(where G_{K_0} is the absolute Galois group of K_0).*

Moreover, in the case where (ii) holds, there exist mono-anabelian reconstruction algorithms of the following invariants from G_K:

- $[K_1 : \mathbb{Q}_p]$*;*
- $\mathrm{Ker}(G_K \twoheadrightarrow G_{K_1})$ *(where G_{K_1} is the absolute Galois group of K_1).*

In particular, the isomorphism class of K is completely determined by G_K in the case where one of the following holds:

- *K satisfies (iii);*
- *K satisfies (ii), and the profinite group $G_K/\mathrm{Ker}(G_K \twoheadrightarrow G_{K_1})$ (which is automatically isomorphic to the absolute Galois group of a p-adic local field) is of GSMLF-type (cf.* [H, Definition 6.8]*).*

For (B), we prove the following theorem:

Theorem B
Higher local fields of characteristic 0 which are mixed-characteristic as complete discrete valuation fields are Kummer-faithful.

Hoshi proved that isomorphisms (which satisfy certain conditions) between the étale fundamental groups of affine hyperbolic curves over Kummer-faithful fields arise from isomorphisms of schemes. Although the above theorem does not necessarily affirm the question (B), this theorem, together with Hoshi's result, ensures that, to some extent, higher local fields of characteristic 0 which are mixed-characteristic as complete discrete valuation fields are suitable for the base fields of anabelian geometry.

For (C), we prove the following theorem:

Theorem C

Let K be a mixed-characteristic complete discrete valuation field with perfect residue field, $\mathfrak{G}_K$ the filtered absolute Galois group of K with the ramification filtration, and G_K the underlying profinite group of $\mathfrak{G}_K$ (i.e., the absolute Galois group of K). Then there exist mono-anabelian reconstruction algorithms of the following invariants from $\mathfrak{G}_K$:

- *the characteristic p of the residue field of K;*
- *the absolute ramification index e_K of K;*
- *the largest nonnegative integer a_K such that K contains a primitive p^{a_K}-th root of unity;*
- *the p-adic cyclotomic character $\chi_p : G_K \to \mathbb{Z}_p^\times$.*

By this theorem, in some special cases, the filtered absolute Galois group $\mathfrak{G}_K$ of a mixed-characteristic complete discrete valuation field K with perfect residue field and the isomorphism class of the residue field of K determine the isomorphism class of K (which gives an answer to (D)):

Theorem D

Let k be a perfect (resp. an algebraically closed) field of positive characteristic, K a mixed-characteristic complete discrete valuation field with residue field k, $\mathfrak{G}_K = \{G_K^v\}_{v\in\mathbb{R}_{\geq -1}}$ the filtered absolute Galois group of K with the ramification filtration, and G_K the underlying profinite group of $\mathfrak{G}_K$ (i.e., the absolute Galois group of K). Set $p := \operatorname{char} k$. Suppose that one of the following condition holds:

(i) $p \neq 2$.
(ii) $a_K \geq 2$ *(cf. Theorem C).*
(iii) $e_K = 1$ *(resp. e_K is prime to p) (cf. Theorem C).*

(Note that p, a_K and e_K are reconstructed from $\mathfrak{G}_K$ in the sense of mono-anabelian reconstruction by Theorem C.) Suppose, moreover, that there exists an open subgroup H of G_K satisfying the following conditions:

(a) $[G_K : H] = [I(\mathfrak{G}_K) : H \cap I(\mathfrak{G}_K)]$ *(where $I(\mathfrak{G}_K)$ is the closure of $\bigcup_{\varepsilon\in\mathbb{R}_{>0}} G_K^{-1+\varepsilon}$).*
(b) $e_L = p^{a_L-1}(p-1)$ *(resp. $e_L = p^{a_L-1}(p-1)n$, where n is a positive integer prime to p), where L is a finite extension of K corresponding to H, and a_L, e_L are defined similarly to a_K, e_K (note that a_L and e_L are reconstructed from H and $\mathfrak{G}_K$ in the sense of mono-anabelian reconstruction by Theorem C).*

Then the isomorphism class of K is completely determined by $\mathfrak{G}_K$ and the isomorphism class of k.

References

[H] Yuichiro Hoshi, Topics in the anabelian geometry of mixed-characteristic local fields, *Hiroshima Math. J.* **49** (2019), no. 3, 323–398.