A summary of "L^p-Kato class measures and their relations with Sobolev embedding theorems"

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This is a summary of the paper [8], entitled "L^p-Kato class measures and their relations with Sobolev embedding theorems".

In the paper, we discuss relationships between continuous embeddings of Dirichlet spaces into Lebesgue spaces and the integrability of the associated resolvent kernels. The Sobolev inequality has been studied for various settings; Euclidean space, Riemannian manifolds, Lie groups, and so on (see [10, 3] for example). It is known that the Sobolev inequality is equivalent to the ultra-contractivity of the associated transition semigroup [14], the Nash type inequality [4], and the capacity isoperimetric inequality [7, 6]. The notion of L^p-Kato class is a generalization of the Kato class of measures. Kato class is introduced to analyse the Schrödinger semigroups and analyse integral kernels of semigroups given by Feynman-Kac functionals (see [1, 2] for example). Dynkin class, the 1-order version of the set of Green-bounded measures, is larger than the Kato class and has also been studied. An embedding result of the Dirichlet spaces into L^2 spaces with respect to Dynkin class measures is obtained by Stollmann and Voigt [12] via the operator theory, and later Shiozawa and Takeda [11] proved it in terms of Dirichlet forms. In this paper, we introduce the L^p-version of Dynkin and Kato classes and present relationships between those and Sobolev embeddings of the Dirichlet spaces into L^{2p} spaces.

Let E be a locally compact, separable metric space and let m be a Radon measure on E with $\mathrm{supp}[m] = E$. Let ∂ be a point added to E so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of E. The point ∂ serves as the cemetery point for E. Suppose $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E; m)$ and $X = (\Omega, X_t, \zeta, \mathbb{P}_x)$ is an associated m-symmetric Hunt process. For $\alpha > 0$ and $u \in \mathcal{F}$, we simply write $\mathcal{E}_\alpha(u, u) = \|u\|^2_{\mathcal{E}_\alpha} := \mathcal{E}(u, u) + \alpha \int_E u^2 dm$. We always take the quasi-continuous version of the element u of $\mathcal{F}$. We assume that the transition kernel $(P_t)_{t>0}$ of X satisfies the absolute continuity condition: $P_t(x, dy)$ is absolutely continuous with respect to $m(dy)$ for each $t > 0$ and $x \in E$. Then $(P_t)_{t>0}$ admits a heat kernel $p_t(x, y)$ which is jointly measurable on $(0, \infty) \times E \times E$ such that $p_t(x, y) = p_t(y, x)$ and $p_{t+s}(x, y) = \int_E p_s(x, z)p_t(z, y)m(dz)$ for all $s, t > 0$, $x, y \in E$. For each $\alpha > 0$, we write the α-order resolvent kernel of X by $r_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y)dt$.

Definition 1.1. Let $p \in [1, \infty)$ and $\delta \in (0, 1]$. For a positive Radon measure μ on E, μ is said to be of *p-Dynkin class with respect to* X (write $\mu \in \mathcal{D}^p(X)$) if

$$\sup_{x\in E} \int_E r_\alpha(x, y)^p \mu(dy) < \infty \quad \text{for some } \alpha > 0,$$

μ is said to be of *p-Kato class with respect to* X (write $\mu \in \mathcal{K}^p(X)$) if

$$\lim_{\alpha\uparrow\infty} \sup_{x\in E} \int_E r_\alpha(x, y)^p \mu(dy) = 0,$$

and μ is said to be of *p-Kato class with order* δ (write $\mu \in \mathcal{K}^{p,\delta}(X)$) if

$$\sup_{x\in E} \left(\int_E r_\alpha(x, y)^p \mu(dy) \right)^{\frac{1}{p}} = O(\alpha^{-\delta}) \quad \text{as } \alpha \to \infty.$$

Clearly $\mathcal{K}^{p,\delta}(X) \subset \mathcal{K}^p(X) \subset \mathcal{D}^p(X)$. For example, suppose $E = \mathbb{R}^d$, m is the Lebesgue measure on $\mathbb{R}^d$ and X is a Brownian motion on $\mathbb{R}^d$. Let $p \in [1, \infty)$ with $d - p(d - 2) > 0$ and μ be a positive Radon measure on $\mathbb{R}^d$. By the same way as the proof of [1, Theorem 4.5], $\mu \in \mathcal{K}^p(X)$ if and only if

$$\lim_{\alpha\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<\alpha}\frac{\mu(dy)}{|x-y|^{p(d-2)}}=0,\quad d\geq 3,$$

$$\lim_{\alpha\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<\alpha}\big(-\log|x-y|\big)^p\mu(dy)=0,\quad d=2,$$

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leq 1}\mu(dy)<\infty,\quad d=1.$$

In particular, when $d=1$, $\mathcal{K}^1(X)=\mathcal{K}^p(X)$ for any $p>1$. We can also see that $m\in\mathcal{K}^{p,\frac{d-p(d-2)}{2p}}(X)$ for such p.

We now state the main results. The first theorem gives Sobolev embeddings of the Dirichlet spaces into the L^{2p} spaces with respect to p-Dynkin or p-Kato class measures.

Theorem 1.2. *Let $p\in[1,\infty)$ and $\mu\in\mathcal{D}^p(X)$.*

(i) It holds that

$$\|u\|^2_{L^{2p}(E;\mu)}\leq\left(\sup_{x\in E}\int_E r_\alpha(x,y)^p\mu(dy)\right)^{\frac{1}{p}}\mathcal{E}_\alpha(u,u)\tag{1.1}$$

for any $u\in\mathcal{F}$ and $\alpha>0$. In particular, the Hilbert space $(\mathcal{F},\mathcal{E}_1)$ is continuously embedded into $L^{2p}(E;\mu)$.

(ii) If $\mu\in\mathcal{K}^p(X)$, then it holds that

$$\|u\|^2_{L^{2p}(E;\mu)}\leq\varepsilon\mathcal{E}_1(u,u)+K(\varepsilon)\|u\|^2_{L^2(E;m)}\tag{1.2}$$

for any $u\in\mathcal{F}$ and $\varepsilon>0$, where K is a positive function on $(0,\infty)$ satisfying $\varepsilon^{-1}K(\varepsilon)\uparrow\infty$ as $\varepsilon\downarrow 0$.

(iii) If $\mu\in\mathcal{K}^{p,\theta}(X)$ for some $\theta\in(0,1]$, then (1.2) holds for a function $K(\varepsilon)=A\varepsilon^{-\frac{1-\theta}{\theta}}$, where A is a positive constant. In particular, it holds that

$$\|u\|_{L^{2p}(E;\mu)}\leq B\sqrt{\mathcal{E}_1(u,u)}^{(1-\theta)}\|u\|^\theta_{L^2(E;m)}\tag{1.3}$$

for any $u\in\mathcal{F}$, where B is an another positive constant.

(1.1) can be viewed as a p-version of Stollmann-Voigt's inequality. Note that (1.2) is similar to the notion of compactly boundedness in [13], and that (1.3) is so-called the interpolation type inequality.

On the other hand, the second theorem gives kinds of converse assertions, that is, belonging to p-Dynkin or p-Kato classes of a measure follows from Sobolev type embeddings of the Dirichlet spaces into the $L^{2p'}$ spaces with respect to the measure for $p'>p$.

Theorem 1.3. *Let $p'\in(1,\infty)$ and let μ be a measure in $\mathcal{D}^1(X)$.*

(i) Suppose the following Sobolev type inequality holds: there exists a constant $S>0$ such that

$$\|u\|^2_{L^{2p'}(E;\mu)}\leq S\mathcal{E}_1(u,u)$$

for all $u\in\mathcal{F}$. Then $\mu\in\mathcal{D}^p(X)$ for any $p\in[1,p')$.

(ii) Suppose the following Sobolev type inequality holds: there exists a function $K:(0,\infty)\to(0,\infty)$ with $\varepsilon^{-1}K(\varepsilon)\uparrow\infty$ as $\varepsilon\downarrow 0$, such that

$$\|u\|^2_{L^{2p'}(E;\mu)}\leq\varepsilon\mathcal{E}_1(u,u)+K(\varepsilon)\|u\|^2_{L^2(E;m)}$$

for any $u\in\mathcal{F}$ and $\varepsilon>0$. Then $\mu\in\mathcal{K}^p(X)$ for any $p\in(1,p')$.

(iii) Suppose the following Sobolev type inequality holds: there exist constants $A>0$ and $\theta\in(0,1]$ such that

$$\|u\|_{L^{2p'}(E;\mu)}\leq A\sqrt{\mathcal{E}_1(u,u)}^{(1-\theta)}\|u\|^\theta_{L^2(E;m)}$$

for any $u\in\mathcal{F}$. Then $\mu\in\mathcal{K}^{p,\theta(1-\delta)}(X)$ for any $p\in(1,p')$ with $\delta=1-\frac{p'}{p'-1}\frac{p-1}{p}$.

We give an example of Theorems 1.2 and 1.3. Suppose $E = \mathbb{R}^d$, m is the Lebesgue measure on $\mathbb{R}^d$ and X is a Brownian motion on $\mathbb{R}^d$. the classical Sobolev embedding theorem on $\mathbb{R}^d$ gives that, $H^1(\mathbb{R}^d)$ is continuously embedded into $L^{2p}(\mathbb{R}^d)$ for $p \in [1,\infty)$ with $d - p(d-2) \geq 0$. By combining this with Theorem 1.3 (i), the Lebesgue measure on $\mathbb{R}^d$ is in $\mathcal{D}^p(X)$ for $p \in [1,\infty)$ with $d - p(d-2) > 0$. In particular, when $d \geq 3$, by setting $p^* = d/(d-2)$ the critical Sobolev embedding theorem gives that $H^1(\mathbb{R}^d)$ is continuously embedded into $L^{2p}(\mathbb{R}^d)$ for $p \in [1, p^*)$, but the Lebesgue measure on $\mathbb{R}^d$ does not belong to $\mathcal{D}^{p^*}(X)$. We now assume $d \geq 1$, $d - p'(d-2) \geq 0$ and $1 < p < p'$. The classical Gagliardo-Nirenberg interpolation inequality says that, by setting $\theta = 1 - \frac{d(p'-1)}{2p'} = \frac{d-p'(d-2)}{2p'}$, there exists a positive constant C such that

$$\|u\|_{L^{2p'}(\mathbb{R}^d)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^d)}^{1-\theta}\|u\|_{L^2(\mathbb{R}^d)}^{\theta}$$

holds for all $u \in H^1(\mathbb{R}^d)$. By applying Theorem 1.3 (iii) and then by letting $p' \downarrow p$, we can see that the Lebesgue measure on $\mathbb{R}^d$ belongs to $\mathcal{K}^{p,\frac{d-p(d-2)}{2p}-\varepsilon}(X)$ for all $\varepsilon \in (0, \frac{d-p(d-2)}{2p})$, which is a slightly weaker condition than the example above Theorem 1.2. By combining with the Gagliardo-Nirenberg inequality, we can also see that the fact that the Lebesgue measure on $\mathbb{R}^d$ belongs to $\mathcal{K}^{p,\frac{d-p(d-2)}{2p}}(X)$.

Finally, we give an application of the p-Kato class to the continuity of the intersection measure in time. We assume that $p \geq 2$ is an integer and the reference measure m is in p-Dynkin class. Let $X^{(1)}, \ldots, X^{(p)}$ be independent Hunt processes with distribution X. We write $\zeta^{(1)}, \ldots, \zeta^{(p)}$ as their life times and write $x_0^{(1)}, \ldots, x_0^{(p)}$ as their starting points, respectively. Before stating the result, we review the construction of the intersection measure. For detail, see [5, 9] for example. Fix bounded Borel sets $J^{(1)}, \ldots, J^{(p)} \subset [0,\infty)$ and write $J = \prod_{i=1}^p J^{(i)}$. For each $\varepsilon > 0$, we define *the approximated (mutual) intersection measure* $\ell_{J,\varepsilon}^{\mathrm{IS}}$ of $X^{(1)}, \ldots, X^{(p)}$ with respect to the (multi-parameter) time interval J by

$$\langle \ell_{J,\varepsilon}^{\mathrm{IS}}, f\rangle = \int_E f(x)\left[\prod_{i=1}^p \int_{J^{(i)}} p_\varepsilon(x, X_s^{(i)})ds\right]m(dy)$$

for $f \in \mathcal{B}_b(E)$, where, for convenience we regard $p_\varepsilon(x, X_s^{(i)}) = 0$ when $s \geq \zeta^{(i)}$. Then, there exists a random measure ℓ_J^{IS} on E such that, $\ell_{J,\varepsilon}^{\mathrm{IS}}$ converges vaguely to ℓ_J^{IS} in $\mathcal{M}(E)$ and that

$$\lim_{\varepsilon\to 0}\mathbb{E}\big[|\langle f, \ell_{J,\varepsilon}^{\mathrm{IS}}\rangle - \langle f, \ell_J^{\mathrm{IS}}\rangle|^k\big] = 0$$

for any integer $k \geq 1$ and $f \in C_0(E)$, where $\mathcal{M}(E)$ is the set of Radon measures on E equipped with the vague topology. We call the limit ℓ_J^{IS} as *the (mutual) intersection measure of* $X^{(1)}, \ldots, X^{(p)}$ *with respect to* J. For $\boldsymbol{t} = (t_1, \ldots, t_p) \in [0,\infty)^p$ we simply denote the approximated intersection measure and the intersection measure with respect to $[\boldsymbol{0}, \boldsymbol{t}] := \prod_{i=1}^p [0, t_i]$ as $\ell_{\boldsymbol{t},\varepsilon}^{\mathrm{IS}}$ and $\ell_{\boldsymbol{t}}^{\mathrm{IS}}$, respectively. The intersection measure ℓ_J^{IS} enjoys the so-called Le Gall's moment formula: for any $f \in \mathcal{B}_b(E)$ with compact support and for any integer $k \geq 1$, it holds that

$$\begin{aligned}&\mathbb{E}\big[\langle f, \ell_J^{\mathrm{IS}}\rangle^k\big]\\ &= \int_{E^k} f(x_1)\cdots f(x_k)\prod_{i=1}^p\left\{\sum_{\sigma\in\mathfrak{S}_k}\int_{(J^{(i)})^k_<}\prod_{j=1}^k p_{s_j - s_{j-1}}(x_{\sigma(j-1)}, x_{\sigma(j)})ds_1\cdots ds_k\right\}m(dx_1)\cdots m(dx_k),\end{aligned}$$

where $(J^{(i)})^k_< := \{(s_1, \ldots, s_k) \in (J^{(i)})^k; s_1 < \cdots < s_k\}$ and $\mathfrak{S}_k$ is the set of permutations of $\{1, \ldots, k\}$. For convenience we set $\sigma(0) = 0$ for $\sigma \in \mathfrak{S}_k$ and set $x_0 = x_0^{(i)}$.

The following result enables us to treat $\{\ell_{\boldsymbol{t}}^{\mathrm{IS}} : \boldsymbol{t} \in [0,\infty)^p\}$ as a measure-valued continuous stochastic process:

Theorem 1.4. *Assume $m \in \mathcal{K}^{p,\delta}(X)$ for some $\delta \in (0,1]$. Then it holds that*

(i) the $\mathcal{M}(E)$-valued process $\{\ell_{\boldsymbol{t}}^{\mathrm{IS}} : \boldsymbol{t} \in [0,\infty)^p\}$ has a continuous modification,

(ii) for any $f \in \mathcal{B}_b(E)$, the real-valued process $\{\langle f, \ell_{\boldsymbol{t}}^{\mathrm{IS}}\rangle : \boldsymbol{t} \in [0,\infty)^p\}$ has a modification whose paths are locally γ-Hölder continuous of every order $\gamma \in (0,\delta)$.

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