NISHINO'S RIGIDITY, LOCALLY PSEUDOCONVEX MAPS, AND HOLOMORPHIC MOTIONS

TAKEO OHSAWA

1. Introduction

Function theory of several complex variables (=SCV) was born in the 19th century, when the profound theory of algebraic functions in one variable was established after the discovery of elliptic functions. The principal motivation for SCV was to clarify the nature of periodic functions in n complex variables.

In SCV, the nature of analytic functions differs from that of one variable at the point that poles and zeros of analytic functions are never isolated. By this reason, a large portion of the basic theory of SCV had to be built on the notions absent in the classical one variable theory. They are pseudoconvexity and coherence, whose essential feature was clarified by Oka [O-1,2,3,4,6] and Cartan [C-1,2,3] through a geometric characterization of holomorphic convexity and basic existence theorems on Stein manifolds.

Stein manifolds and compact complex manifolds are two extreme classes of complex manifolds. Hopf [H] found a class of compact complex manifolds without nonconstant meromorphic functions and Kodaira [K-1] characterized projective algebraic manifolds by a differential geometric condition. With these backgrounds, the basic theory of SCV has been generalized on complex manifolds and analytic spaces by various methods (cf. [G-1,2,3], [K-N], [A-V], [Hö]) and provides a sound foundation for the studies of

2010 *Mathematics Subject Classification.* Primary 32E40; Secondary 32T05 .

automorphic functions (cf. [S]). As a result, SCV is now related to many different fields such as partial differential equations, algebraic geometry, diferential geometry, number theory, mathematical physics, representation theory, and dynamical systems.

On the other hand, there was also a remarkable development in the one variable theory, particularly in the value distribution of meromorphic functions after the disovery of theorems of Picard that improved the Casorati-Weierestrass theorem. Stimulated by this progress, Nevanlinna's estimates for the order functions and Ahlfors's geometric approaches have been successfully extended to several variables (cf. [K-O], [C-G], [K-2], [D-2]). Value distribution theory studies the relations between various defect charateristics of transendental functions. Since it is done substantially by estimating the distributions of the preimages of the values, the idea has been naturally carried over to the case of several variables.

Nishino [N-1, 2, 3, 4, 5] initiated the classification theory of entire functions of two variables from this viewpoint of analyzing their level sets and clarified a link between SCV and one complex variable. The key to Nishino's theory is the following rigidity theorem which was called "lemme fondamental" in [N-2].

Theorem 1.1. *Let π be a holomorphic submersion from a two dimensional Stein manifold D onto the unit disc $\mathbb{D}$ such that $\pi^{-1}(c)$ $(c \in \mathbb{D})$ are all biholomorphically equivalent to $\mathbb{C}$. Then $\pi : D \to \mathbb{D}$ is equivalent to the projection $pr_2 : \mathbb{C} \times \mathbb{D} \to \mathbb{D}$, i.e. there exists a biholomorphic map $\alpha : D \to \mathbb{C} \times \mathbb{D}$ satisfying $pr_2 \circ \alpha = \pi$.*

The purpose of the present article is to discuss a relation between the notions of holomorphic motion and locally pseudoconvex maps, which the author considers to be basic for the study of deformations of non-compact manifolds, from a viewpoint suggested by Theorem 1.1. For that, we shall first review the proof of Theorem 1.1 at first, basically following [N-2] but in a more concise form. The author's intention is to make the points more directly accessible to the readers of modern textbooks such as [G-R], [W], and [F], because he believes that Theorem 1.1 and its proof deserve to be better known.

Then, after giving some expository accounts on locally pseudoconvex maps and holomorphic motions, we shall discuss questions on the deformations of non-compact manifolds. Some results will be given in special cases.

2. Preparations for the proof of Theorem 1.1

Step 1 — Defining $\varphi : D \to \mathbb{C}$. By an analytic family we shall mean a holomorphic map f from an irreducible analytic space X onto a reduced analytic space T. We shall also say then that X is an analytic family over T and call f the projection. We say that an analytic family $f : X \to T$ is closable if X is densely contained in an analytic space $\overline{X}$ in such a way that f is holomorphically extendable to a proper map from $\overline{X}$ to T.

Let D and π be as in Theorem 1.1. Since the principal Aut$\mathbb{C}$ bundles over $\mathbb{D}$ are trivial, the existence of α will follow from the local triviality of the family. Hence we may assume in advance that the family has a section, i.e. there exists a holomorphic map $s : \mathbb{D} \to D$ satisfying $\pi \circ s = id_{\mathbb{D}}$.

Let us fix a holomorphic vector field ξ on a neighborhood of $s(\mathbb{D})$ which is nowhere zero and satisfies $\pi_* \xi = 0$. For each $c \in \mathbb{D}$, let $\varphi_c : \pi^{-1} \to \mathbb{C}$ be a biholomorphic map satisfying

$$\varphi_c(s(c)) = 0 \tag{2.1}$$

and

$$\xi(\varphi_c)(s(c)) = 1. \tag{2.2}$$

Since φ_c is uniquely defined by these conditions, a map $\varphi : D \to \mathbb{C}$ is defined by $\varphi|_{\pi^{-1}(c)} = \varphi_c$.

Step 2 — Continuity of φ. Let $p \in D$ be any point. We are going to show that φ is continuous at p. For that, we may assume that $\pi(p) = 0$ in advance. Since D is Stein, one has a holomorphic extension w of φ_0 satisfying

$$w \circ s = 0. \tag{2.3}$$

We shall denote $\pi^{-1}(c)$ by D_c for simplicity. Note that dw has no zeros on a neighborhood of D_0, say U. Hence, for any sequence $c_n \in \mathbb{D}$ $(n = 1, 2, \dots)$ converging to 0, there exists a sequence M_n of positive numbers such that $w_n := w|_{D_{c_n} \cap w^{-1}(M_n \cdot \mathbb{D})}$ are biholomorphic maps to $M_n \cdot \mathbb{D}$.

By (2.1) and (2.3),

$$\lim_{n\to\infty} \varphi_{c_n} \circ w_n^{-1}(0) = 0 \tag{2.4}$$

and by (2.2) and (2.3)

$$\lim_{n\to\infty} (\varphi_{c_n} \circ w_n^{-1})'(0) = 1. \tag{2.5}$$

Therefore, by the following lemma $\varphi_{c_n} \circ w_n^{-1}$ locally converges to $id_{\mathbb{C}}$ uniformly, so that we obtain

$$\lim_{q\to p} \varphi(q) = \varphi(p)$$

for any $p \in D$.

Lemma 2.1. *For any sequence M_n with $\lim_{n\to\infty} M_n = \infty$ and for any sequence f_n of holomorphic injections from $M_n \cdot \mathbb{D}$ to $\mathbb{C}$ satisfying $f_n(0) = 0$ and $f_n'(0) = 1$, f_n converges to $id_{\mathbb{C}}$ locally uniformly on $\mathbb{C}$.*

Proof. By Koebe's distorsion theorem (cf. [A], p.84) one has

$$\frac{|z|}{\left(1 + \frac{|z|}{M_n}\right)^2} \le |f_n(z)| \le \frac{|z|}{\left(1 - \frac{|z|}{M_n}\right)^2}.$$

Hence the desired conclusion follows from the Schwarz lemma.
□

Step 3 — A lemma for the analyticity of φ. It remains to show the analyticity of φ. Let D, π, U and w be as before. Then φ can be expanded in a neighborhood of $(z, w) = (0, 0)$ into a series

$$\varphi(z, w) = w + \sum_{k=2}^{\infty} a_k(z) w^k,$$

where $a_k(z)$ are all continuous. It suffices to show that there exists a neighborhood of 0 on which $a_k(z)$ are holomorphic, since the analyticity of φ will follow from its definition and continuity shown as above.

In order to prove the analyticity of $a_k(z)$, the following subtle preparation is necessary.

Lemma 2.2. (cf. [N-2, Lemme 6]) *Any continuous function f on the closed disc $\overline{\mathbb{D}}$ satisfying the following condition is holomorphic on $\mathbb{D}$.*
For any two positive numbers ϵ and η, one can find a finite set $\Gamma \subset \mathbb{D}$ and a holomorphic function $h_\Gamma(t)$ on the universal covering $\widetilde{\mathbb{D} \setminus \Gamma} = \mathbb{D} = \{t \in \mathbb{C}; |t| < 1\}$ such that

$$\sup_{|t|<1-\eta} |f(z(t)) - h_\Gamma(t)| < \epsilon. \tag{2.6}$$

Proof. The harmonicity of f will be shown at first. Since $z(t)$ is a bounded holomorphic function on $\mathbb{D}$, by a theorem of Fatou the radial limit of $z(\rho e^{i\theta})$ exists as $\rho \nearrow 1$ on $\partial\mathbb{D}$ almost everywhere (cf. [R], [A-O-S]). Moreover, it is clear from the definition of $z(t)$ that

$$|\lim_{\rho \nearrow 1} z(\rho e^{i\theta})| = 1 \qquad a.e. \tag{2.7}$$

Therefore,

$$\lim_{\rho \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(\sigma(z(\rho e^{i\theta}))d\theta = f(\sigma(z(0))) \tag{2.8}$$

holds for any $\sigma \in \mathrm{Aut}\mathbb{D}$. Hence f is harmonic.

Since f^2 and $(f+z)^2$ also satisfy the assumption, one has

$$\partial\overline{\partial}(f(z)^2) = 2\frac{\partial f}{\partial z}\frac{\partial f}{\partial \overline{z}} = 0$$

and

$$\partial\overline{\partial}(f(z) + z)^2 = \left(\frac{\partial f}{\partial z} + 1\right)\frac{\partial f}{\partial \overline{z}} = 0,$$

whence

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

3. Proof of Theorem 1.1

Let us prove the analyticity of $a_k(z)$ by using Lemma 2.2. First we note that the existence of α is clear if π is a closable projection, because the Riemann sphere is rigid. In this case $a_k(z)$ are obviously holomorphic. The general analytic family in question will be approximated by this special one. To realize a situation where Lemma 2.2 can be applied, we need some coordinates to describe the approximation.

Proposition 3.1. *Let $\pi : D \to \mathbb{D}$ be as in Theorem 1.1. Then there exists a holomorphic submersion $u : D \to \mathbb{C}$ with one dimensional fibers.*

Proof. Since D is a contractible Stein manifold, there exists a holomorphic 1-form, say ω, on D whose restriction to D_c (as a differential form) are all zero free. The map defined by integrating ω from $s(\mathbb{D})$ along the paths contained in the fibers satisfies the required property. □

Proposition 3.2. *For any $M > 0$, there exist a closable analytic family $\pi_M : D_M \to \mathbb{D}$, a holomorphic map $u_M : D_M \to \mathbb{C}$, and a biholomorphic map β_M from $\Omega_M := \varphi^{-1}(M \cdot \mathbb{D}) \cap \pi^{-1}(\frac{1}{2} \cdot \mathbb{D})$ into D_M satisfying the following properties.*

1) $\pi_M \circ \beta_M(\zeta, z) = z, \qquad (\zeta, z) \in M \cdot \mathbb{D} \times \frac{1}{2} \cdot \mathbb{D}.$

2) $\sup_{\Omega_M} |u_M \circ \beta_M - u| < \frac{1}{M}.$

Proof. Since D is a Stein manifold of dimension 2, there exist holomorphic functions $f_j : D \to \mathbb{C}$ $(1 \leq j \leq 4)$ such that the map

$$F = (f_1, f_2, f_3, f_4) : D \to \mathbb{C}^4$$

is a proper embedding (cf. [F], Theorem 8.2.4). Since the image of (F, u) is defined by three entire functions (cf. [F-R]), say g_1, g_2 and g_3, it can be approximated by algebraic sets on compact subsets of $\mathbb{C}^5$.

Hence one can approximate $(F, \pi, u)(D)$ by an algebraic set, which may be regarded as an approximation of F by a multivalued analytic function in (z, u) which depends algebraically in u.

Let D_M be the graph of such a map, let π_M and u_M be respetively the restrictions of the projections

$$z : \mathbb{C}^4 \times \mathbb{D} \times \mathbb{C} \to \mathbb{D}$$

and

$$u : \mathbb{C}^4 \times \mathbb{D} \times \mathbb{C} \to \mathbb{C}$$

to D_M, and let β_M be the inverse of the restriction of a holomorphic retraction to $(F, \pi, u)(D) \cong D$ defined on a tubular neighborhood of $(F, \pi, u)(D)$ in $\mathbb{C}^4 \times \mathbb{D} \times \mathbb{C}$. Then D_M is closable by the algebraicity property and, since Ω_M

is relatively compact in D, u_M and β_M satisfy the requirements by choosing the defining polynomials of D_M sufficiently close to g_1, g_2 and g_3. □

Proof of the analyticity of $a_k(z)$. By Proposition 3.2, for any $M_0 > 0$ and $r_0 < \frac{1}{2}$, the map (u_M, π_M) is a local homeomorphism from $\beta_M(\varphi^{-1}(M_0 \cdot \mathbb{D}) \cap \pi_M^{-1}(r_0 \cdot \mathbb{D}))$ to $\mathbb{C} \times \mathbb{D}$ for sufficiently large M. Hence, in this part, D_M is nonsingular and the restrictions of u_M to the fibers of π_M do not have ramification points.

Let $\nu_M : D_M^* \to D_M$ be the normalization of D_M and put $u_M^* = u_M \circ \nu_M$, $\pi_M^* = \pi_M \circ \nu_M$. Let $\Sigma_M \subset D_M^*$ be the set of points around which the map (u_M^*, π_M^*) is not a homeomorphism, and let Γ_M be the image by π_M^* of the set of points around which $\pi_M^*|_{\Sigma_M}$ is not a homeomorphism. Then, $\Gamma_M \cap r_0 \cdot \mathbb{D}$ is a finite set by the definition of D_M^*.

We restrict the analytic family D_M^* over $r_0 \cdot \mathbb{D} \setminus \Gamma_M$, let $\tilde{\pi}_M : \tilde{D}_M^* \to \mathbb{D}$ be its lift to the universal covering $\mathbb{D}$ of $r_0 \cdot \mathbb{D} \setminus \Gamma_M$, and let $\tilde{u}_M$ be the lift of u_M^* to $\tilde{D}_M^*$. Then a ramified covering space $(\tilde{\pi}_M, \tilde{u}_M) : \tilde{D}_M^* \to \mathbb{D} \times \mathbb{C}$ is obtained. This covering space can be modified by making a slit at first on each fiber of $\tilde{\pi}_M$ along curves eminating from the lift of $\Sigma_M \setminus \pi_M^{-1}(\Gamma_M)$, lying over the curves on the $\tilde{u}_M$ plane which reach ∞ by avoiding the ramification points, and then by gluing the sheets along the slits in such a way that the resulting analytic cover

$$(\check{\pi}_M, \check{u}_M) : \check{D}_M \to \mathbb{D} \times \mathbb{C}$$

induces a closable analytic family $\check{\pi}_M : \check{D}_M \to \mathbb{D}$ which has simply connected fibers.

This "surgery" can be operated without touching the part corresponding to $\beta_M(\varphi^{-1}(M_0 \cdot \mathbb{D}) \cap \pi_M^{-1}(r_0 \cdot \mathbb{D}))$, so that the properties of β_M and u_M described in 1) and 2) of Proposition 3.2 are carried over. $\check{D}_M$ is closable beause so is D_M. The Steiness is obvious because $(\check{\pi}_M, \check{u}_M)$ is proper.

For the family $\check{D}_M$, let φ_M be the map defined similarly as φ corresponding to $\beta_M \circ s$ and $(\beta_M)_*\xi$. Then φ_M is holommorphic because $\check{D}_M$ is closable. Moreover, as $M \to \infty$, $(\varphi, \pi) \circ \beta_M^{-1} \circ (\varphi_M, \check{\pi})^{-1}$ converges to the identity map uniformly on compact subsets of $M_0 \cdot \mathbb{D} \times \mathbb{D}$.

This implies that $a_k(z)$ can be approximated by holomorphic functions in the sense of Lemma 2.2 so that they are holomorphic. Hence one may put $\alpha = (\varphi, \pi)$. □

Remark 3.1. Combining the above proof with Osgood's separate analyticity theorem, it is clear that a holomorphic submersions from a Stein manifold with fibers $\cong \mathbb{C}$ are analytic fiber bundles. Since $\mathrm{Aut}\mathbb{C} = \{az + b \; ; \; a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}\}$, they are affine line bundles.

Remark 3.2. Proposition 3.1 can be generalized as a relative version of a theorem of Gunning and Narasimhan. (cf. [G-N] and [Nm]).

Remark 3.3. As a corollary of Theorem 1.1, one knows that locally Stein families of finite Riemann surfaces are closable if the Betti numbers of the fibers are locally bounded. It is likely that the latter property is also a consequence of the Steinness. For a related extension of Nishino's theory, see [Y], [C] and [Oh-3], where alternate proofs of Theorem 1.1 are given.

4. WEAKLY 1-COMPLETE MANIFOLDS AND LOCAL PSEUDOCONVEXITY

Theorem 1.1 and its proof suggest further studies on analytic families in a larger class of manifolds including Stein manifolds and compact manifolds, since compactification, modification and covering spaces naturally enter the arguments of Nishino. So, before going into specific questions lying in this direction, let us first give expository accounts on some basic notions.

A complex manifold X is said to be weakly 1-complete, or weakly pseudoconvex in the terminology of [D-1], if X admits a C^∞ plurisubharmonic exhaustion function, say $\Psi : X \to [0, \infty)$. X is said to be 1-complete if a strictly plurisubharmonic function can be chosen as Ψ. By a theorem of Grauert [G-1], X is 1-complete if and only if X is Stein.

A holomorphic map $\psi : X \to Y$ will be said to be locally pseudoconvex if every point $y \in Y$ has a neighborhood U such that $\psi^{-1}(U)$ is weakly 1-complete.In this situation, we shall also say that X is locally pseudoconvex over Y. To find a passage from local to global is the principal question in this context. As is well known, the following is the first decisive answer in this direction.

Theorem 4.1. (cf.[O-3,6]) *A complex manifold X is 1-complete if there exists a 1-complete manifold Y and a locally pseudoconvex submersion $\psi : X \to Y$ with 0-dimensional fibers.*

By Grauert's theorem it is easy to see from Theorem 1.1 that locally pseudoconvex maps with fibers $\cong \mathbb{C}$ are affine line bundles.

Theorem 4.1 is known to be a solution of the Levi problem. A generalized Levi problem asks conditions for X to be weakly 1-complete when a locally pseudoconvex map is given from X to a weakly 1-complete manifold Y. Let us recall some cases where affirmative results hold.

Theorem 4.2. (cf. [T], [S], [E]) *Let D be a relatively compact domain in a Kähler manifold with semipositive holomorphic bisectional curvature. Then* $-\log \operatorname{dist}(x, \partial D)$ *is* *a continuous plurisubharmonic exhaustion function on* D*. Here* $\operatorname{dist}(x, \partial D)$ *denotes the distance from* $x \in D$ *to the boundary* ∂D *of* D*.*

Theorem 4.3. (cf. [U]) *Let* Y *be a compact Kähler manifold and let* $f : X \to Y$ *be a locally pseudoconvex map whose fibers are* $\mathbb{C}$*. Suppose moreover that* f *is topologically equivalent to the projection* $\mathbb{C} \times Y \to Y$*. Then* X *is weakly 1-complete.*

The proof of Theorem 4.3 will be given in section 6.

Remark 4.1. Theorems 4.2 and 4.3 are false without the Kähler condition. Analytically nontrivial $\mathbb{D}$-bundles and $\mathbb{C}$-bundles over Hopf manifolds are counterexamples (cf. [D-F] and [Oh-5]).

Remark 4.2. An elementary proof of Theorem 4.2 is given in [Oh-3].

5. Holomorphic motions

From now on, let us consider analytic families $f : X \to T$ where X and T are nonsingular and f is differentiably locally trivial. In this situation, we shall say that f is a holomorphic motion if X is locally foliated by holomorphic sections. In other words, $f : X \to T$ is a holomorphic motion if every point $t \in T$ has a neighborhood U such that there exists a continuous retraction $\rho : f^{-1}(U) \to f^{-1}(t)$ whose preimages are the graphs of holomorphic sections over U. If ρ can be chosen to be of class C^k, f is called a C^k-holomorphic motion. By an abuse of language, we shall also say that f, as well as X, is a holomorphic motion of the fibers of f.

Theorem 5.1. (cf. [Ku]) $f : X \to T$ *is a real analytic holomorphic motion if* f *is proper.*

Corollary 5.1. *Every covering space of an analytic family of a compact Riemann surface is locally pseudoconvex.*

Theorem 5.2. (cf. [Sl]) *A holomorphic motion $f : X \to T$ is foliated by holomorphic sections over T if* $\dim X = 2$ *and* $T = \mathbb{D}$.

Corollary 5.2. *Holomorphic motions of Riemann surfaces over $\mathbb{D}$ are holomorphically convex.*

Remark 5.1. Corollary 5.1 and Corollary 5.2 have been noticed in [Oh-2] and [Oh-1], respectively.

Remark 5.2. Since X is Stein if $f : X \to T$ is a fiber bundle with Stein 1-dimensional fibers and Stein base T (cf. [M]), it seems natural to ask if a holomorphic motion is Stein under the same assumptions on the fibers and the base.

Remark 5.3. Combining Theorem 5.1 with an example of Nakamura in [N], one sees that holomorphic motions of a Stein manifold of dimension ≥ 3 are not necessarily locally pseudoconvex. For Stein surfaces, local pseudoconvexity of their holomorphic motions does not seem to be known.

6. BUNDLES OVER COMPACT KÄHLER MANIFOLDS

Let $f : X \to T$ be as before. From now on we shall restrict ourselves to the cases where the fibers of f are analytic fiber bundles over compact complex manifolds whose fibers are either $\mathbb{C}$ or the unit disc $\mathbb{D}$, since interesting weakly 1-complete manifolds arise in such forms.

First we recall when such bundles become weakly 1-complete. Since $\mathrm{Aut}(\mathbb{C})$ consists of polynomials of degree one, analytic $\mathbb{C}$-bundles are those fiber bundles whose transition functions are of the form $\zeta_j = a_{jk}(z)\zeta_k + b_{jk}(z)$ with respect to an open covering $M = \cup_j U_j$. Here ζ_j denotes the fiber coordinate over U_j and $z \in U_j \cap U_k$. We shall denote by L_0 the line bundle associated to the 1-cocyle a_{jk}.

Proposition 6.1. *Let M be a compact complex manifold and let $L \to M$ be an analytic fiber bundle whose fibers are $\mathbb{C}$. Then L_0 is weakly 1-complete if L_0 is seminegative.*

Here a line bundle is said to be seminegative if it admits a fiber metric h_j whose curvature form $-\partial\bar{\partial} \log h_j$ is seminegative. The proof of Proposition 6.1 is straighforward from the definition. Indeed, $|\zeta_j|^2 h_j$ is then a plurisubharmonic exhaustion function on L_0. As for the converse, Grauert [G-3] proved that the zero section of a holomorphic line bundle over a compact complex manifold admits a strongly pseudoconvex neighborhood system if and only if the bundle is negative. Although the seminegative case has not been discussed in the literature, the following result for the one-dimensional case is implicitly contained in [U].

Theorem 6.1. *If* $\dim M = 1$, *L is weakly 1-complete if and only if L_0 is seminegative.*

The proof of Theorem 6.1 can be generalized without difficulty to obtain Theorem 4.3, which can be stated in this context as follows.

Theorem 6.2. *Topologically trivial analytic affine line bundles over compact Kähler manifolds are weakly 1-complete.*

Proof. Let M be a compact Kähler manifold, let $L \to M$ be a topologically trivial analytic affine line bundle. Then, since M is Kähler, one can find an open covering $\{U_j\}$ of M and local trivializations of L such that the transition relations are of the form $\zeta_j = e^{\sqrt{-1}\theta_{jk}}\zeta_k + a_{jk}(z)$ for some $\theta_{jk} \in \mathbb{R}$ and $a_{jk} \in \mathcal{O}(U_j \cap U_k)$, where $\mathcal{O}(U)$ denotes the set of holomorphic functions on U (by a standard application of the $\partial\bar{\partial}$-lemma). Applying the Kähler condition again, by replacing $\{U_j\}$ by its refinement if necessary, one can find $a_j, b_j \in \mathcal{O}(U_j)$ such that

$$a_{jk} = a_j + \overline{b_j} - e^{\sqrt{-1}\theta_{jk}}(a_k + \overline{b_k})$$

holds on $U_j \cap U_k$. This is also a consequence of the $\partial\bar{\partial}$-lemma (cf. [D-Oh-1], Lemma 2). For simplicity we put $h_j = a_j + \overline{b_j}$. The system h_j is naturally identified with a pluriharmonic section of the bundle $L \to M$.

Then it is straightforward that the function

$$\Phi = |\zeta_j - h_j|^2$$

is a well defined plurisubharmonic exhaustion function on L. The plurisubharmonicity of Φ can be seen from

$$\begin{aligned}\partial\bar{\partial}\Phi &= d\zeta_j d\overline{\zeta}_j - d\zeta_j \bar{\partial}\overline{h}_j - d\overline{\zeta}_j \partial h_j + \partial h_j \bar{\partial}\overline{h}_j + \partial\overline{h}_j \bar{\partial} h_j \\ &\geq \partial\overline{h}_j \bar{\partial} h_j. \end{aligned}$$ □

Inspired by Theorems 6.1 and 6.2,the following was obtained in [D-Oh-2] .

Theorem 6.3. *Analytic $\mathbb{D}$-bundles over compact Kähler manifolds are weakly 1-complete.*

By these results, a similarity is apparent between the divisors with topologically trivial normal bundle and Levi flat hypersurfaces. In [Oh-5] and [Oh-4], Theorem 6.2 and Theorem 6.3 are complemented respectively by the following results where the similarity still persists.

Theorem 6.4. *Let X be a compact Kähler manifold and let Y be an effective divisor of X whose normal bundle is topologically trivial. Then the complement of the support of Y does not admit a C^∞plurisubharmonic exhaustion function whose Levi form has at least 3 positive eigenvalues everywhere outside a compact set.*

Theorem 6.5. *Let X be a compact Kähler manifold and let $\Sigma \subset X$ be a real analytic Levi flat hypersurface. Then $X \setminus \Sigma$ does not admit a C^∞ plurisubharmonic exhaustion function whose Levi form has at least 3 positive eigenvalues everywhere outside some compact subset of $X \setminus \Sigma$.*

For the families of affine line bundles, the following was proved in [Oh-7].

Theorem 6.6. *Let T be a complex manifold, let $p : S \to T$ be a proper holomorphic map with smooth one-dimensional fibers, and let $q : \mathcal{L} \to S$ be an analytic affine line bundle. Then $p \circ q : \mathcal{L} \to T$ is locally pseudoconvex if one of the following conditions is satisfied.*

(i) *Fibers L_t $(t \in T)$ of $p \circ q$ are of negative degrees over the fibers S_t of p.*

(ii) *L_t are topologically trivial over S_t and not analytically equivalent to line bundles.*

(iii) *$\mathcal{L} \to S$ is a $U(1)$-flat line bundle.*

For the families of $\mathbb{D}$-bundles, we have shown in advance to Theorem 6.6 its counterpart in the following form.

Theorem 6.7. (cf. [Oh-6]) *Let $p : S \to T$ be as in Theorem 6.6 and let $\mathcal{D}$ be an analytic $\mathbb{D}$-bundle over S. Then $\mathcal{D}$ is weakly 1-complete if T is Stein.*

The parallerism between $\mathbb{C}$-bundles and $\mathbb{D}$-bundles stops here because it turned out that one cannot drop the assumptions in Theorem 6.6. At this point, which will be discussed below, the so-called Ohsawa-Takegoshi theorem entered the argument unexpectedly.

7. A COUNTEREXAMPLE

Let A be a complex torus of dimension one (i.e. an elliptic curve), say $A = (\mathbb{C}\setminus\{0\}/\mathbb{Z})$, where the action of $\mathbb{Z}$ on $\mathbb{C}\setminus\{0\}$ is given by $z \mapsto e^m z$ for $m \in \mathbb{Z}$. Over the product $A \times \mathbb{C}$ as an analytic family of compact Riemann surfaces over $\mathbb{C}$, we define an affine line bundle $\mathcal{F} \to A \times \mathbb{C}$ as the quotient of the trivial bundle $((\mathbb{C}\setminus\{0\} \times \mathbb{C}) \times \mathbb{C}) \times \mathbb{C} \to (\mathbb{C}\setminus\{0\} \times \mathbb{C})$by the action of $\mathbb{Z}$ defined by $(z, t, \zeta) \mapsto (e^m z, t, \zeta + mt)$. Suppose that $\mathcal{F}$ is locally pseudoconvex with respect to the map

$\pi : \mathcal{F} \to \mathbb{C}$ induced by the projection to the second factor of $A \times \mathbb{C}$. Then there will exist a neighborhood $V \ni 0$ such that $\pi^{-1}(V)$ is weakly 1-complete. Then, since the canonical bundle of $\mathcal{F}$ is obviously trivial, holomorphic functions on $\pi^{-1}(t)$ must be holomorphically extendable by the L^2 extension theorem in [Oh-3]. (See also [Oh-T].) But this will mean that $\pi^{-1}(0)$ can be blown down to $\mathbb{C}$ in $\mathcal{F}$ because the other fibers of π are equivalent to $(\mathbb{C}\setminus\{0\})^2$. This contradicts that the normal bundle of the divisor $\pi^{-1}(0)$ is trivial.

Remark 7.1. The above counterexample can be generalized to any non-trivial analytic family of $\mathbb{C}$-bundles over a compact Riemann surface that deforms the trivial bundle, because one can apply the L^2 extension theorem in the same way on the total space of the anti-canonical bundle of the family.

8. Proof of Theorem 6.6

Since the conclusion is obvious when (i) or (iii) is the case, let us assume (ii). Then, as in the proof of Theorem 2.2, one can find a system of fiber coordinates ζ_j of the bundle over S and a system of C^∞ functions h_j which are harmonic on the fibers of $S \to T$ such that $|\zeta_j - h_j|$ is globally defined on $\mathcal{L}$. Note that h_j are nonconstant on the fibers of p by assumption. Then it is easy to see that, for any Stein open set $V \subset T$, there exist a C^∞ positive plurisubharmonic exhaustion function ψ on V and a positive C^∞ function φ on $q^{-1}(p^{-1}(V))$ satisfying $\varphi \le \psi \circ p \circ q$ such that $\varphi + |\zeta_j - h_j|^2$ is strictly plurisubharmonic on each fiber of $p \circ q$. Then it is easy to verify by a direct computation that one can find a convex increasing function λ on $\mathbb{R}$ such that

$$(1 + \varphi + |\zeta_j - h_j|^2) \cdot \lambda \circ \psi \circ p \circ q$$

is strictly plurisubharmonic everywhere.

References

[A-O-S] Aleman, A., Olsen, J.-F. and Saksman, E., *Fatou and brothers Riesz theorems in the infinite-dimensional polydisc,* arXiv:1512.01509v2[math.CV] 19Oct 2017.

[A-V] Andreotti, A. and Vesentini, E. *Sopra un teorema di Kodaira,* Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 283-309.

[C-G] Carlson, J. and Griffiths, P., *A defect relation for equidimensional holomorphic mappings between algebraic varieties,* Ann. of Math. (2) **95** (1972), 557-584.

[C-1] Cartan, H., *Idéaux et modules de fonctions analytiques de variables complexes,* Bull. Soc. Math. France **78** (1950), 29-64.

[C-2] ——, *Variétés analytiques complexes et cohomologie,* Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 41-55. Georges Thone, Liège; Masson & Cie, Paris, 1953.

[C-3] ——, *Quotients of complex analytic spaces,* 1960 Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960) pp. 1-15 Tata Institute of Fundamental Research, Bombay

[Ck] Cirka, E. M.,*Holomorphic motions and the uniformization of holomorphic families of Riemann surfaces,* (Russian) Uspekhi Mat. Nauk **67** (2012), no. 6(408), 125-202; translation in Russian Math. Surveys **67** (2012), no. 6, 1091-1165.

[D-1] Demailly, J.-P.,*Analytic Methods in Algebraic Geometry,* Beijing: Higher Education Press, 2010.

[D-2] ——,*Towards the Green-Griffiths-Lang conjecture,* Analysis and geometry, 141159, Springer Proc. Math. Stat., **127**, Springer, Cham, 2015.

[D-F] Diederich, K. and Fornaess, J.-E., *A smooth pseudoconvex domain without pseudoconvex exhaustion,* Manuscripta Math. **39** (1982), 119-123.

[D-Oh-1] Diederich, K. and Ohsawa, T., *A Levi problem on two-dimensional complex manifolds,* Math. Ann. **261** (1982), no. 2, 255-261.

[D-Oh-2] ——, *Harmonic mappings and disc bundles over compact Kähler manifolds,* Publ. Res. Inst. Math. Sci. **21** (1985), 819833.

[E] Elencwajg, G., *Pseudo-convexité locale dans les variétés kählériennes,* Ann. Inst. Fourier (Grenoble) **25** (1975), 295-314.

[F-R] Forster, O. and Ramspott, K. J., *Analytische Modulgarben und Endromisbndel,*. Invent. Math. **2** (1966), 145-170.

[F] Forstnerič, F., *Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis,* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **56**. Springer, Heidelberg, 2011. xii+489 pp.

[G-1] Grauert, H., *On Levi's problem and the imbedding of real-analytic manifolds,*. Ann. of Math. (2) **68** (1958), 460-472.

[G-2] ——, *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen,* Inst. Hautes Études Sci. Publ. Math. **5** (1960) 64 pp.

[G-3] ——, *Über Modifikationen und exzeptionelle analytische Mengen,* Math. Ann. **146** (1962), 331-368.

[G-R] Grauert, H. and Remmert, R., *Theorie der Steinschen Räume,* Grundlehren der Mathematischen Wissenschaften **227**. Springer-Verlag, Berlin-New York, 1977. xx+249 pp. (*Theory of Stein spaces,* Translated from the German by Alan Huckleberry. Grundlehren der Mathematischen Wissenschaften **236**. Springer-Verlag, Berlin-New York, 1979. xxi+249 pp.

[G-H] Griffiths, P. and Harris, J., *Principles of algebraic geometry,* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978. xii+813 pp.

[G-N] Gunning, R. C. and Narasimhan, R., *Immersion of open Riemann surfaces,* Math. Ann. **174** (1967), 103-108.

[H] Hopf, H., *Zur Topologie der komplexen Mannigfaltigkeiten,* Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, pp. 167-185.

[Hö] Hörmander, L., *L^2 estimates and existence theorems for the $\bar\partial$ operator,* Acta Math. **113** (1965), 89-152.

[K-O] Kobayashi, S. and Ochiai, T., *Mappings into compact manifolds with negative first Chern class,* J. Math. Soc. Japan **23** (1971), 137-148.

[K-1] Kodaira, K., *On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties),*. Ann. of Math. (2) **60**, (1954). 28-48.

[K-2] ——, *Nevanlinna Theory* Springer Briefs in Mathematics , to appear.

[K-N] Kohn, J. and Nirenberg,L., *Non-coercive boundary value problems,* Comm. Pure Appl. Math. **18** (1965), 443-492.

[Ku] Kuranishi, M., *Deformations of compact complex manifolds,* Séminaire de Mathématiques Supérieures, No. 39 (Été 1969). Les Presses de l'Université de Montréal, Montreal, Que., 1971. 99 pp.

[M] Mok, N., *Le problème de Serre pour les surfaces de Riemann,* C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 4, A179-A180.

[Na] Nakamura, I., *On complex parallelisable manifolds and their*

small deformations, Proc. Japan Acad. **48** (1972), 447-449.
[Nm] Nishimura, Y., *Immersion analytique d'une famille de surfaces de Riemann ouvertes,* Publ. Res. Inst. Math. Sci.**14** (1978),643-654.
[N-1] Nishino, T., *Nouvelles recherches sur les fonctions entières de plusieurs variables complexes. I,* J. Math. Kyoto Univ. **8** (1968), 49-100.
[N-2] ——, *Nouvelles recherches sur les fonctions entières de plusieurs variables complexes. II, Fonctions entières qui se réduisent à celles d'une variable,* J. Math. Kyoto Univ. **9** (1969), 221-274.
[N-3] ——, *Nouvelles recherches sur les fonctions entières de plusiers variables complexes. III. Sur quelques propriétés topologiques des surfaces premières,* J. Math. Kyoto Univ. **10** (1970), 245-271.
[N-4] ——, *Nouvelles recherches sur les fonctions entières de plusieurs variables complexes. IV. Types de surfaces premières,* J. Math. Kyoto Univ. **13** (1973), 217-272.
[N-5] ——, *Nouvelles recherches sur les fonctions entières de plusieurs variables complexes. V. Fonctions qui se réduisent aux polynômes,* J. Math. Kyoto Univ. **15** (1975), no. 3, 527-553.
[Oh-1] Ohsawa, T., *A note on the variation of Riemann surfaces,* Nagoya Math. J. **142** (1996), 1-4.
[Oh-2] ——, *On the stability of pseudoconvexity for certain covering spaces,* Nagoya Math. J. **147** (1997), 107-112.
[Oh-3] ——, *Pseudoconvex domains in P^n : a question on the 1-convex boundary points,* Analysis and geometry in several complex variables (Katata, 1997), 239-252, Trends Math., Birkhäuser Boston, Boston, MA, 1999.
[Oh-4]——, *On the complement of Levi-flats in Kähler manifolds of dimension ≥3,* Nagoya Math. J. **185** (2007), 161-169.
[Oh-5]——, *A remark on pseudoconvex domains with analytic complements in compact Kähler manifolds,* J. Math. Kyoto Univ.**47** (2007), 115-119.
[Oh-6]——, *Stability of pseudoconvexity of disc bundles over compact Riemann surfaces and application to a family of Galois coverings,* Interat. J. Math. **26** (2015), no. 4, 1540003, 7pp.
[Oh-7]——, *On the local pseudoconvexity of certain families of* $\mathbb{C}$, preprint.
[Oh-8] ——, *L^2 proof of Nishino's rigidity theorem,* preprint.
[Oh-T] Ohsawa, T. and Takegoshi, K., *On the extension of L^2 holomorphic functions,* Math. Z. **195** (1987), no. 2, 197-204.
[O-1] Oka, K., *Sur les fonctions analytiques de plusieurs variables. Domaines convexes par rapport aux fonctions rationelles,* J. Sci. Hiroshima Univ. **6** (1936), 245-255.

[O-2] ——, *Sur les fonctions analytiques de plusieurs variables. III. Deuxième problème de Cousin,* J. Sci. Hiroshima Univ. **9** (1939), 7-19.
[O-3] ——, *Sur les fonctions analytiques de plusieurs variables. VI. Domaines pseudoconvexes,* Tôhoku Math. J. **49**, (1942). 15-52.
[O-4] ——, *Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques,* Bull. Soc. Math. France **78**, (1950), 1-27.
[O-5] ——, *Sur les fonctions analytiques de plusieurs variables. VIII. Lemme fondamental,* J. Math. Soc. Japan **3**, (1951), 204-214, 259-278.
[O-6] ——, *Sur les fonctions analytiques de plusieurs variables. IX. Domaines finis sans point critique intérieur,* Jap. J. Math. **23** (1953), 97-155 (1954), J. Math. Soc. Japan.
[R] Rudin, W., *Function theory in polydiscs,* W. A. Benjamin, Inc., New York-Amsterdam,1969.
[S] Siegel, C., *Topics in complex function theory. Vol. III. Abelian functions and modular functions of several variables,* Translated from the German by E. Gottschling and M. Tretkoff. John Wiley & Sons, Inc., New York, 1989.
[Sl] Slodkowski, Z., *Holomorphic motions and polynomial hulls,* Proc. Amer. Math. Soc. **111** (1991), no. 2, 347-355.
[Su] Suzuki, O., *Pseudoconvex domains on a Kähler manifold with positive holomorphic bisectional curvature,* Publ. Res. Inst. Math. Sci. **12** (1976/77), 191-214.
[T] Takeuchi, A., *Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif,* J. Math. Soc. Japan **16** (1964), 159-181.
[U] Ueda, T., *On the neighborhood of a compact complex curve with topologically trivial normal bundle,* J. Math. Kyoto Univ. **22** (1982/83), 583-607.
[W] Wells, R. O., Jr. *Differential analysis on complex manifolds,* Third edition. With a new appendix by Oscar Garcia-Prada. Graduate Texts in Mathematics, **65**. Springer, New York, 2008.
[Y] Yamaguchi, H., *Parabolicité d'une fonction entière,* J. Math. Kyoto Univ. **16** (1976), no. 1, 71-92.

TAKEO OHSAWA, GRADUATE SCHOOL OF MATHEMATICS NAGOYA UNIVERSITY 464-8602 CHIKUSAKU FUROCHO NAGOYA JAPAN
E-mail address: ohsawa@math.nagoya-u.ac.jp