SYMMETRY AND INTERPOLATION OF ESTIMATES FOR THE COMPLEX GREEN OPERATOR

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ABSTRACT. This note is a summary of a lecture on the results of [8] about estimates for the complex Green operator, given by the author in the occasion of the conference: "Topology of pseudoconvex domains and analysis of reproducing kernels" on November 20-22nd, 2017, in RIMS, Kyoto, Japan. The results of this note are contained in [7] and [8].

1. INTRODUCTION

A CR-manifold M of $\mathbb{C}^n$ is of hypersurface type if the real codimension of the complex tangent space inside the real tangent space is one. We will assume that M is compact, closed, and orientable. A particular case of such CR-manifold is the boundary of pseudoconvex domains in $\mathbb{C}^n$. As such, a well-behaved L^2-theory holds for the tangential Cauchy–Riemann operator and the L^2-Sobolev theory of its associated complex Green operator – inverse of the Kohn Laplacian – may then be compared to that of the $\bar{\partial}$-Neumann operator on pseudoconvex domains. For a survey on the sufficient conditions for compactness estimates and Sobolev estimates for the complex Green operator, we refer to [7]. However, compactness estimates for the $\bar{\partial}$-Neumann operator failed to hold simutaneously at symmetric bidegrees (see [12]), while the compactness estimates for the complex Green operator are known to hold simultaneously at symmetric bidegrees (p,q) and $(p,m-1-q)$ ([18, 16, 7]), where $m-1$ is the CR-dimension of M. The first result presented in this note is the fact that Sobolev estimates for the complex Green operator also hold simultaneously at symmetric bidegrees (p,q) and $(m-p,m-q-1)$ (Theorem 1). On the other hand, while the compactness estimates for the $\bar{\partial}$-Neumann operator percolate up to the $\bar{\partial}$-complex on pseudoconvex domains, i.e. if the compactness estimates hold for (p,q)-forms then the compactness estimates hold for $(p,q+1)$-forms, the compactness estimates for the complex Green operator do not. One of the main theorems, presented in this note is to give an alternative to the percolation for the complex Green operator on M, an interpolation result (Theorem 2). A similar result was proved recently in [15] when M is an actual hypersurface.

The purpose of this note is to give a survey of the joint work with E. Straube [8]. After recalling some fundamental properties on the tangential $\bar{\partial}$-operator and the definition of the complex Green operator, we give a short review of the properties of the complex Green operator such as compactness estimates in Section 3. In Section 4, we state the Sobolev

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estimates for forms of symmetric bidegrees and give the idea of the proof. In Section 5, we state the interpolation of compactness estimates for the complex Green operator that relies on the microlocalization introduced and used in [17, 23]. We end this note by mentionning a series of interesting remarks, including an open question related to the note.

2. Preliminaries and Notations

We keep the notations from [8]; it is fairly standard. Let M be a smooth compact CR-submanifold in $\mathbb{C}^n$, without boundary. Define m via $dim_{\mathbb{C}}T^{\mathbb{C}}M = (m-1)$, where $T^{\mathbb{C}}_P M$ denotes the complex tangent space at P, i.e. $T_P M \cap JT_P M$, where $T_P M$ is the real tangent space to M and J the complex structure map on $\mathbb{C}^n$ (i.e. multiplication by i). This dimension, called the CR-dimension of M is independent of P.

M is said to be of *hypersurface type* if, at each point $P \in M$, $T^{\mathbb{C}}_P M$ has real codimension one in $T_P M$. Note that the real dimension of M is then $2m-1$. Indeed, CR-submanifolds of hypersurface type can be represented locally as a graph over an actual hypersurface in $\mathbb{C}^m$, $m \leq n$. We refer to [7] for sketches.

A vector field $X(z) = \sum_{j=1}^{n} a_j(z)\partial/\partial z_j$ (on an open set of $\mathbb{C}^n$ or of M) is called of type $(1,0)$, while a field $Y(z) = \sum_{j=1}^{n} b_j(z)\partial/\partial\overline{z_j}$ is of type $(0,1)$, as usual. X is tangential to M if and only if $(a_1(z),\dots,a_n(z)) \in T^{\mathbb{C}}_z M$, for all z; similarly, $Y(z)$ is tangential if and only if $(\overline{b_1(z)},\dots,\overline{b_n(z)}) \in T^{\mathbb{C}}_z M$, for all z. We say that $X \in T^{1,0}M$, $Y \in T^{0,1}M$ ($T^{1,0}M$ and $T^{\mathbb{C}}M$ are thus naturally isomorphic). For detailed information on CR-(sub)manifolds, the reader may consult [10, 5].

We assume from now that M is orientable, then there exits a purely imaginary vector field T on M of unit length that is orthogonal to $T^{\mathbb{C}}M$ at all points. Let η be the form dual to T, that is $\eta(T) \equiv 1$, and $\eta \equiv 0$ on $T^{1,0}M \oplus T^{0,1}M$. Denote by L_m the vector field $L_m := (1/\sqrt{2})(T - iJT)$ defined on M; L_m is of type $(1,0)$ and has length one. Near a point $P \in M$, choose an orthonormal basis $\{L_1,\dots,L_{(m-1)}\}$ of $T^{1,0}M$. Choose $(1,0)$-forms $\{\omega_1,\dots,\omega_m\}$ that at each point vanish on $\{L_1,\dots,L_m\}^{\perp}$ and so that $\omega_k(L_j) = \delta_{kj}$, where δ_{kj} is the Kronecker δ. These are the usual local frames.

An important point that plays a role in Section 3 is the following: when we restrict ω_m to M as a form, this restriction does not equal η; rather, we have $\omega_m|_M = (1/\sqrt{2})\eta$ (see for example [25], ch. III.3 for a discussion of the Hermitian structure on $\mathbb{C}^n$ that pays attention to norms of the dz_j, etc.).

The space of (p,q)–forms on M at P, $\Lambda^{p,q}T^*_P M$, is defined as those forms in $\Lambda^{p,q}T^*_P\mathbb{C}^n$ that have the form

$$(1) \qquad u = {\sum_{|I|=p,|J|=q}}' u_{IJ}\omega_I(P) \wedge \overline{\omega_J}(P),\ I \subseteq \{1,\dots,m\},\ J \subseteq \{1,\dots,m-1\}.$$

The notation $\sum'$ indicates summation over strictly increasing multi-indices. This definition is independent of the choice of orthonormal basis $\{L_1,\dots,L_{(m-1)}\}$ of $T^{1,0}M$ near P (L_m is defined globally, and therefore, so is ω_m).

We remind the extrinsic tangential Cauchy–Riemann operator, defined in the usual way. Locally, we represent a (p,q)–form as in (1). Extend u coefficientwise to a form $\widetilde{u}$ defined in a full neighborhood in $\mathbb{C}^n$ (note that the local frame 'lives' in such a full neighborhood). Then from the $\bar{\partial}$-operator in $\mathbb{C}^n$, we define

$$\bar{\partial}_M u = (\bar{\partial}\tilde{u})_{t_M} \tag{2}$$

where $t_M : \Lambda^{p,q}T_P^*\mathbb{C}^n \to \Lambda^{p,q}T_P^*M$ is the orthogonal projection, for $P \in M$ (that is t_M gives the tangential part of a form). This definition is independent of the local frame and/or the extension chosen, so that $\overline{\partial}_M$ is well defined by (2). The tangential $\overline{\partial}_M$-operator inherits from the $\bar{\partial}$-operator on $\mathbb{C}^n$, the property to be a complex. It is useful to have the following expression for $\overline{\partial}_M$ in a local frame:

$$\bar{\partial}_M u = \sum_{k=1}^{m-1} \sum_{|I|=p,|J|=q}{}' \bar{L}_k(u_{IJ})\bar{\omega}_k \wedge \omega_I \wedge \overline{\omega_J} + \text{terms of order zero}\,. \tag{3}$$

Here, terms of order zero' means terms where the coefficients of u are not differentiated. We refer to [10, 7] for more details.

The pointwise inner product between (p,q)–forms at $P \in M$,

$$< u, v > \; = \sum_{|I|=p,|J|=q}{}' u_{IJ}\overline{v_{IJ}} \tag{4}$$

is independent of the choice of the local othonormal frame. It provides an L^2-inner product on M by integrating against the (Euclidean) volume element on M, as usual:

$$(u,v)_{L^2_{(p,q)}(M)} = \int_M < u(z), v(z) > dV_M(z)\,. \tag{5}$$

$L^2_{(p,q)}(M)$, $0 \le p \le m, 0 \le q \le (m-1)$ denotes the completion of $\Lambda^{p,q}T^*M$ under the norm induced by this inner product, that we also denote by $\|.\|$ for short.

The tangential $\bar{\partial}$-operator $\bar{\partial}_M : L^2_{(p,q)}(M) \to L^2_{(p,q+1)}(M)$ extends to an unbounded operator on $L^2_{(p,q)}(M)$ acting in the sense of distributions, with the maximal domain of definition $dom(\overline{\partial}_M) = \{u \in L^2_{(p,q)}(M) \mid \bar{\partial}_M u \in L^2_{(p,q+1)}(M)\}$, where $\overline{\partial}_M$ acts in a local frame as in (3).

As a closed and densely defined operator on $L^2_{(p,q)}(M)$, $1 \le q \le m-1$, $\bar{\partial}_M$ has a Hilbert space adjoint, denoted by $\bar{\partial}_M^*$. In a local frame, integration by parts gives

$$\bar{\partial}_M^* u = -\sum_{j=1}^{m-1} \sum_{|I|=p,|K|=q-1}{}' L_j(u_{IjK})\omega_I \wedge \overline{\omega_K} + \text{ terms of order zero}\,. \tag{6}$$

We say that a CR-submanifold of hypersurface type is *pseudoconvex* if the Levi form λ that appears in the commutator between two vector fields $X, Y \in T_P^{1,0}(M)$ at a point $P \in M$,

$$[X, \bar{Y}]_P = \lambda_P(X, \bar{Y})T_P \mod T_P^{1,0}M \oplus T_P^{0,1}M,$$

is positive semi-definite at each point P of M. Because T is chosen to be purely imaginary, λ is a Hermitian form.

Orientable, smooth, compact and pseudoconvex CR-submanifolds M of hypersurface type of $\mathbb{C}^n$ can be considered as a natural generalization of boundary of pseudoconvex domains. Indeed, Baracco [1] proved that M has one-sided complexification to a complex submanifold

$\widehat{M}$ of $\mathbb{C}^n$, called a "strip" so that M is the connected pseudoconvex component of the boundary of $\widehat{M}$. He then proved that M bounds a complex manifold in the $\mathcal{C}^\infty$ sense [2, 3]. We also refer to [30].

The crucial property is that $\bar\partial_M$, hence $\bar\partial^*_M$, have closed range in $L^2_{(0,q)}(M)$, $0 \leq q \leq m-1$. This was first proved by Nicoara [23] for $m \geq 2$ by microlocal methods and improved by Baracco [3] for $m \geq 1$[1]. From a well-behaved L^2-theory for the $\bar\partial_M$-operator and its associated Hodge decomposition, we get the corresponding L^2-estimate:

$$(7) \qquad \|u\|^2_{L^2_{(p,q)}(M)} \lesssim \|\bar\partial_M u\|^2_{L^2_{(p,q+1)}(M)} + \|\bar\partial^*_M u\|^2_{L^2_{(p,q-1)}(M)} + \|H_{p,q}u\|^2_{L^2_{(p,q)}(M)} ,$$

where $H_{p,q} : L^2_{(p,q)}(M) \to \mathcal{H}_{p,q}(M) := \ker(\bar\partial_M) \cap \ker(\overline{\partial}^*_M)$ is the orthogonal projection.

Let $1 \leq q \leq (m-2)$. The complex Laplacian on $L^2_{(p,q)}(M)$, denoted by $\Box_{(p,q)}$, is defined as $\bar\partial_M\bar\partial^*_M + \bar\partial^*_M\bar\partial_M$; its domain $dom(\Box_{(p,q)})$ is understood to be the set of forms where this expression makes sense. This operator is the unique self-adjoint operator associated to the quadratic form $Q_{p,q}(u,u) = (\bar\partial_M u, \bar\partial_M u)_{L^2_{(p,q+1)}(M)} + (\bar\partial^*_M u, \bar\partial^*_M u)_{L^2_{(p,q-1)}(M)}$, via

$$(8) \qquad Q_{p,q}(u,u) = (\Box_{p,q}u, u)_{L^2_{(p,q)}(M)} \ , \ u \in dom(\Box_{p,q}) \ .$$

We denote $ker(\Box_{(p,q)}) = \mathcal{H}_{(p,q)}(M)$, the harmonic (p,q)-forms on M with L^2-coefficients. The dimension of $\mathcal{H}_{(p,q)}(M)$ is known to be finite when $1 \leq q \leq (m-2)$ ([23, 13]). It is reflected in a version of the basic L^2–estimate of (7) where the norm of the harmonic component of a form u is replaced by $\|u\|_{W^{-1}}$, the dual of the L^2-Sobolev space W^1 (see [30], estimate 7,[7], Lemma 5):

$$(9) \quad \|u\|^2_{L^2_{(p,q)}(M)} \lesssim \|\overline{\partial}_M u\|^2_{L^2_{(p,q+1)}(M)} + \|\overline{\partial}^*_M u\|^2_{L^2_{(p,q-1)}(M)} + \|u\|^2_{W^{-1}_{(p,q)}(M)} ,$$
$$u \in dom(\overline{\partial}_M) \cap dom(\overline{\partial}^*_M) \ , \ 0 \leq p \leq m \ , \ 1 \leq q \leq (m-2) \ .$$

Because the range of $\overline{\partial}_M$ is closed, so is that of $\Box$. Also, $\Box_{(p,q)}$ maps $\mathcal{H}_{(p,q)}(M)^\perp$ onto itself.

The *complex Green operator*, $G_{p,q}$ is defined to be the inverse operator of the restriction of $\Box_{(p,q)}$ to $\mathcal{H}_{(p,q)}(M)^\perp$. It is convenient to extend it to all of $L^2_{(p,q)}(M)$ by setting it equal to zero on $\mathcal{H}_{(p,q)}(M)$. $G_{p.q}$ is a bounded self–adjoint operator. A detailed discussion of these matters may be found in [7, 11].

3. Estimates for the complex Green operator

We keep the previous notations and M is a smooth compact orientable and pseudoconvex CR-submanifold of hypersurface type in $\mathbb{C}^n$ and of CR-dimension $m-1$.

Like the $\bar\partial$-Neumann operator, the complex Green operator verifies a couple of properties that makes its study very interesting. For example, the complex Green operator gives the minimal solution (also called the Kohn solution) to the inhomogeneous tangential Cauchy-Riemann equation. Indeed if f is a (p,q)-form $\bar\partial_M$-closed and orthogonal to $\mathcal{H}_{p,q}(M)$, then

$$f = \bar\partial_M\bar\partial^*_M G_{p,q}f + \bar\partial^*_M\bar\partial_M G_{p,q}f = \bar\partial_M(\bar\partial^*_M G_{p,q}f),$$

[1]This property holds in $L^2_{(p,q)}(M)$ since the holomorphic part is not getting involved in the proofs.

since $\bar{\partial}_M^*\bar{\partial}_M G_{p,q}f \in ker(\bar{\partial}_M)$ by default and $\bar{\partial}_M^*\bar{\partial}_M G_{p,q}f \in \mathcal{H}_{p,q}^{\perp}(M)$. Hence, $u = \bar{\partial}_M^* G_{p,q}f \in \mathcal{H}_{p,q-1}(M)^{\perp}$ and verifies $\bar{\partial}_M u = f$.

Let $j_{p,q} : \mathcal{H}_{p,q}(M)^{\perp} \cap dom(\bar{\partial}_M) \cap dom(\bar{\partial}_M^*) \hookrightarrow \mathcal{H}_{p,q}(M)^{\perp}$ be the imbedding and we set

$$Q_{p,q}(u,u) = \|\bar{\partial}_M u\|^2_{L^2_{(p,q+1)}(M)} + \|\bar{\partial}_M^* u\|^2_{L^2_{(p,q-1)}(M)},$$

also called the *graph* norm. With respect this norm, $\mathcal{H}_{p,q}(M)^{\perp} \cap dom(\bar{\partial}_M) \cap dom(\bar{\partial}_M^*)$ is a Hilbert space that makes $j_{p,q}$ continuous and so its adjoint $j_{p,q}^*$. When we study the compactness estimates for $G_{p,q}$, the following expression is very useful. We refer to [7], Lemma 4 for a proof.

Lemma 1.

$$G_{p,q}|_{\mathcal{H}_{p,q}(M)^{\perp}} = j_{p,q} \circ (j_{p,q})^*.$$

The interest of the compactness of the complex Green operator is as important as the compactness of the $\bar{\partial}$-Neumann operator. From compactness estimates, we recover a well-behaved L^2-theory for $\bar{\partial}_M$. Moreover, compactness estimates imply Sobolev estimates. The following Lemma gives useful characterizations of compactness of $G_{p,q}$. We refer to [7], Lemma 6 for a proof.

Lemma 2. *Let $1 \le q \le (m-2)$. The following properties are equivalent:*

a) The complex Green operator $G_{p,q}$ is compact.

b) $j_{p,q}$ is compact.

c) $\bar{\partial}_M$ verifies the following compactness estimate: for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|u\|^2_{L^2_{(p,q)}(M)} \le \varepsilon \left(\|\bar{\partial}_M u\|^2_{L^2_{(p,q+1)}(M)} + \|\bar{\partial}_M^* u\|^2_{L^2_{(p,q-1)}(M)}\right) + C_\varepsilon \|u\|^2_{W^{-1}_{(p,q)}(M)} \tag{10}$$
$$u \in dom(\overline{\partial}_M) \cap dom(\overline{\partial}_M^*).$$

The Lemma 2 holds when M is not orientable since compactness estimates are local, as proved by [28] using the fact that locally M is CR-equivalent to a hypersurface.

The property *c)* is also referred to as *compactness estimates* for $G_{p,q}$. Note that by fixing ε in (10), we have (9) and we get the finite dimension of the harmonic space $\mathcal{H}_{p,q}(M)$ for $1 \le q \le m-2$. As mentioned previously, the compactness estimates imply the L^2-theory discussed in Section 2. In particular, $\bar{\partial}_M$ and $\bar{\partial}_M^*$ have closed range.

Since M has no boundary, subelliptic and compactness estimates for the complex Green operator hold at symmetric levels i.e, $G_{0,q}$ is compact if and only if $G_{0,m-q-1}$ is compact. Compactness estimates are local, it is then enough to work on smooth forms supported in a local coordinate chart via a partition of unity. The idea, from Koenig [16], is to construct an operator denoted T_q that acts on a $(0,q)$-form u as follows

$$T_q\Big({\sum_{|J|=q}}' u_J \overline{w_J}\Big) = {\sum_{|J|=q,|K|=(m-1-q)}}' \varepsilon^{JK}_{(1,\dots,m-1)} u_J \overline{w_K}\,, \tag{11}$$

where $\varepsilon^{JK}_{(1,\dots,m-1)}$ is the Kronecker symbol. Then,

$$T_{(m-1-q)}T_q u = (-1)^{q(m-1-q)}u$$

(12) $$\bar{\partial}_M T_q u = (-1)^q T_{(q-1)}(\bar{\partial}^*_M u) + \text{terms of order zero}\ ,$$

and

(13) $$\bar{\partial}^*_M T_q u = (-1)^{q+1} T_{(q+1)}(\bar{\partial}_M u) + \text{terms of order zero}\ .$$

This operator T_q intertwines $\bar{\partial}_M$ and $\bar{\partial}^*_M$ up to terms of order zero. Those terms are absorbed in the compactness estimates. For more details, we refer to [16].

As we mentioned in the introduction, the compactness estimates for the $\bar{\partial}$-Neumann operator do not hold at symmetric levels on pseudoconvex domains. Let M be the boundary of a smooth bounded convex domain in $\mathbb{C}^n$. It was proved by [12] that the $\bar{\partial}$-Neumann operator $N_{0,q}$ is compact on the domain if and only if there is no q-dimensional complex variety on M nor higher dimensional variety. If $n \geq 5$ and M contains an analytic disc, then $N_{0,1}$ is not compact, but $N_{0,n-1}$ is.

4. Symmetry

Let M be a smooth compact orientable pseudoconvex CR-submanifold of hypersurface type in $\mathbb{C}^n$ of CR-dimension $m-1$.

Koenig's operator (11) does not work anymore to obtain Sobolev estimates at symmetric levels, since the terms of order zero in (12) and (13) cannot be absorbed in Sobolev norms. We wish then to construct an operator that intertwines $\bar{\partial}_M$ and $\bar{\partial}^*_M$ without terms of order 0, similar to the Hodge-$\star$ operator that maps a (p,q)-form into $(m-p, m-q-1)$-form. However, the operator that we build is slightly different from the Hodge-$\star$ operator since the pointwise inner product (4) between two forms in $\Lambda^{p,q}T^*_P M$ does not necessarily agree with the inner product of their restrictions to M at P. This is due to the orthonormal frame in Section 2: the unit form $\omega_m(P) \in \Lambda^{1,0}T^*_P M$ restricts to $(1/\sqrt{2})\eta \in \mathbb{C}T^*_P M$, a form of norm $(1/\sqrt{2})$.

In order to rectify this situation, we change the metric on $\mathbb{C}TM$, hence on $\mathbb{C}T^*M$ by declaring, at each point $P \in M$, $\{\omega_1, \dots, \omega_{(m-1)}, \overline{\omega_1}, \dots, \overline{\omega_{(m-1)}}, (1/\sqrt{2})\eta\}$ to be an orthonormal basis. In other words, we rescale in the direction of η by a factor of $\sqrt{2}$ (equivalently, by a factor of $1/\sqrt{2}$ in the direction of T). When we equip M with this new Riemannian structure, the restriction of forms in $\Lambda^{p,q}T^*_P M$ to M (restriction as forms) becomes an isometry (at the point P). We use $\widetilde{\star}$, $<\ ,\ >_\sim$, and $d\widetilde{V}$ to denote, respectively, the Hodge-$\star$ operator, the pointwise inner product on forms, and the volume element on M with respect to this new Riemannian structure. All properties of the Hodge-$\star$ operator that we will use can be found in [25], section III.3.4 and/or in [21], section 4.1 (c).

This operator, denoted $A_{p,q}$ is a conjugate linear operator $A_{p,q} : L^2_{(p,q)}(M) \to L^2_{(m-p,m-1-q)}(M)$, defined via

$$(14)\quad (v, A_{p,q}u) := \sqrt{2}\int_M u \wedge v\ , \quad u \in L^2_{(p,q)}(M),\ v \in L^2_{(m-p,m-1-q)}(M)\ ,$$
$$0 \le p \le m\ ,\ 0 \le q \le (m-1)\ .$$

This definition is analogous to the one in the appendix of [24]. It will be convenient to express $A_{p,q}$ with the help of $\widetilde{\star}$. We have

$$(15)\quad (v, A_{p,q}u) = \sqrt{2}\int_M u \wedge v = \sqrt{2}\int_M (\widetilde{\star}\widetilde{\star}(u|_M)) \wedge v = \sqrt{2}\int \overline{\widetilde{\star}\left(\overline{\widetilde{\star}(u|_M)}\right)} \wedge v$$
$$= \sqrt{2}\int_M v \wedge \overline{\widetilde{\star}\left(\overline{\widetilde{\star}(u|_M)}\right)} = \sqrt{2}\int_M < v|_M, \overline{\widetilde{\star}(u|_M)} >_\sim d\widetilde{V} = \int_M < v, \overline{\widetilde{\star}(u|_M)}) > dV\ .$$

Therefore,

$$(16)\qquad A_{p,q}u = \widetilde{\star}\left(\overline{u|_M}\right)\ ,\ u \in L^2_{(p,q)}(M)\ ,$$

in the sense that $A_{p,q}u$ equals the unique form in $L^2_{(m-p,m-1-q)}(M)$ whose restriction to M equals $\widetilde{\star}(\overline{u|_M})$ (that is, η is replaced by ω_m). We are then able to prove

Theorem 1. *Let M be a smooth compact pseudoconvex orientable CR-submanifold of $\mathbb{C}^n$ of hypersurface type, of CR-dimension $m-1$. Let $0 \le p \le m$, $1 \le q \le (m-2)$. Then $G_{p,q}$ is regular in Sobolev norms (respectively globally regular) if and only if $G_{m-p,m-1-q}$ is.*

Note that by regular in Sobolev spaces, we mean $G_{p,q}$ satisfy the Sobolev estimates $\|G_{p,q}u\|_s \le C_s\|u\|_s$, where $\|\cdot\|_s$ denotes the L^2–Sobolev norm of order $s > 0$. We say that $G_{p,q}$ is globally regular if it maps (C^∞) smooth forms to smooth forms.

Proof. The expression of $A_{p,q}$ in terms of the Hodge-$\widetilde{\star}$ operator shows that $A_{p,q}$ is continuous in L^2 but also in Sobolev norms. Hence, it is enough to prove that $A_{p,q}$ commutes with $G_{p,q}$. We resume the properties of this operator in the following proposition:

Proposition 1. *Let $0 \le p \le m$, $0 \le q \le (m-1)$. Then*

$$(17)\qquad A_{p,q} : L^2_{(p,q)}(M) \to L^2_{(m-p,m-1-q)}(M)\ \textit{is an isometry}\ ,$$

$$(18)\qquad A_{m-p,m-q-1}A_{p,q}u = u, \quad \forall u \in L^2_{(p,q)}(M)\ .$$

Let $0 \le p \le m$, $1 \le q \le (m-2)$. Then

$$(19)\qquad \bar\partial_M A_{p,q}u = (-1)^{p+q}A_{p,q-1}\bar\partial^*_M u, \quad \forall u \in dom(\bar\partial^*_M) \subseteq L^2_{(p,q)}(M)\ ,$$

$$(20)\qquad A_{p,q}\bar\partial_M u = (-1)^{p+q}\bar\partial^*_M A_{p,q-1}u, \quad \forall u \in dom(\bar\partial_M) \subseteq L^2_{(p,q-1)}(M)\ ,$$

$$(21)\qquad A_{p,q}\Box_{p,q}u = \Box_{m-p,m-1-q}A_{p,q}u, \quad \forall u \in dom(\Box_{p,q}) \subseteq L^2_{(p,q)}(M)\ ,$$

$$(22)\qquad A_{p,q}G_{p,q} = G_{m-p,m-1-q}A_{p,q}\ ,$$

$$A_{p,q}(\mathcal{H}_{(p,q)}(M)) = \mathcal{H}_{(m-p,m-1-q)}(M)\,. \tag{23}$$

Proof of the Proposition. It suffices to prove all the statements for smooth forms; they are dense in $L^2_{(p,q)}(M)$ and in the graph norms of both $\overline{\partial}_M$ and $\overline{\partial}^*_M$. We remind that there is no boundary conditions when we integrate by parts that makes things work perfectly. (17) and (18) are immediate from (16) and the fact that $\tilde{\star}$ is an isometry in the modified metric on M, and that $\tilde{\star}\tilde{\star}u = u$ (there is a factor $(-1)^{(p+q)(2m-1-p-q)}$; however, $(p+q)(2m-1-p-q) \equiv 0 \, mod\, 2)$.

To verify the crucial intertwining properties (19), (20), let us look at (20). The computation is as follows. Let $u \in \Lambda^{p,q-1}T^*M$, $v \in \Lambda^{m-p,m-1-q}T^*M$. Note that

$$\int_M \overline{\partial}_M u \wedge v = \int_M (\partial_M + \overline{\partial}_M)u \wedge v = \int_M du \wedge v \tag{24}$$

($\int_M \partial_M u \wedge v = 0$, because at least one of the ω_j, $1 \le j \le m$, will appear twice, or there will be an ω_j with $j > m$; in either case, the integral over M vanishes). Integration by parts therefore gives

$$\begin{aligned}(v, A_{p,q}\overline{\partial}_M u)_{L^2_{(m-p,m-1-q)}(M)} &= \sqrt{2}\int_M \overline{\partial}_M u \wedge v = (-1)^{p+q}\sqrt{2}\int_M u \wedge \overline{\partial}_M v \\ &= (-1)^{p+q}\sqrt{2}\int_M \tilde{\star}(\overline{\tilde{\star}u|_M}) \wedge \overline{\partial}_M v = (-1)^{p+q}\sqrt{2}\int_M \overline{\partial}_M v \wedge \tilde{\star}(\overline{\tilde{\star}u|_M})\,.\end{aligned} \tag{25}$$

We have also used that $\tilde{\star}$ is real, so that $\tilde{\star}(\overline{\tilde{\star}u|_M}) = u|_M$. Using $\tilde{\star}$ to mediate between wedge products and inner products gives

$$\begin{aligned}\sqrt{2}\int_M \overline{\partial}_M v \wedge \tilde{\star}(\overline{\tilde{\star}u|_M}) &= \sqrt{2}\int_M < \overline{\partial}_M v|_M, \overline{\tilde{\star}u|_M} >_\sim d\tilde{V} \\ &= (\overline{\partial}_M v, \overline{\tilde{\star}u|_M})_{L^2_{(m-p,m-q)}(M)} = (v, \overline{\partial}^*_M A_{p,q-1}u)_{L^2_{(m-p,m-1-q)}(M)}\,.\end{aligned} \tag{26}$$

In the second equality, we use that $\sqrt{2} <,>_\sim d\tilde{V} = <,> dV$, as well as (16). (25) and (26) now imply (20).

The remaining properties easily follow from (20), (17) and (18). $\square$

5. Interpolation

Let M be as in Section 4. We mentionned previously that subelliptic and compactness estimates for the $\overline{\partial}$-Neumann operator percolate up the $\overline{\partial}$-complex on pseudoconvex domains in $\mathbb{C}^n$: if these estimates hold for (p,q)-forms then they hold for $(p,q+1)$-forms. See for example [27] Proposition 4.5. The question of what happens for the complex Green operator on the boundary of those domains arises naturally. Since compactness estimates for $G_{p,q}$ hold at symmetric levels, if such percolation property holds, then the compactness of $G_{p,q}$ would imply the compactness estimates for all forms. That is, unfortunately, too good to be true as the following example witnesses: Similar to the property of the $\overline{\partial}$-Neumann operator on convex domains, in [24], Theorem 1.5, the authors proved that if M is the boundary of a smooth bounded convex domain in $\mathbb{C}^n$, then $G_{p,q}$ is compact if and only if M has no

q-dimensional nor $(n-1-q)$-dimensional complex varieties. Again, if $n \geq 5$ and M contains an analytic disc but no higher dimensional complex variety, then by this property, all the complex Green operators $G_{0,2}, \dots G_{0,n-3}$ are compact but $G_{0,1}$ and $G_{0,n-2}$ are not. There is no percolation of compactness estimates from $(0, n-3)$-forms to $(0, n-2)$-forms.

However, we obtain a substitute of the percolation, that is analogous to an interpolation:

Theorem 2. *Let M be a smooth compact pseudoconvex orientable CR-submanifold of $\mathbb{C}^n$ of hypersurface type, of CR-dimension $m-1$, let $0 \leq p \leq m$ and $1 \leq q_1 \leq q_2 \leq (m-2)$. If G_{p,q_1} and G_{p,q_2} are compact, then so is $G_{p,r}$ for $q_1 \leq r \leq q_2$.*

To prove this result, we use the microlocalization from [17, 19] and an idea from [22]. Since compactness estimates are local, we choose a good cover of M by local coordinate charts and work with smooth forms supported in such chart. The microlocalization allows to decompose the complex Green operators into 3 pseudodifferential operators $\mathcal{P}^{\pm}$ and $\mathcal{P}^0$ supported in 3 different cones: $G_{p,q}u = \mathcal{P}^+ G_{p,q}u + \mathcal{P}^0 G_{p,q}u + \mathcal{P}^- G_{p,q}u$. To prove the compactness of $G_{p,q}$ is then equivalent to prove the compactness of $\mathcal{P}^{\pm} G_{p,q}$ and $\mathcal{P}^0 G_{p,q}$. But $\mathcal{P}^j$, $j \in \{-,+,0\}$ are bounded in $L^2_{(p,q)}(M)$ and in particular, $\mathcal{P}^0 G_{p,q}$ is always compact because of elliptic estimates for $\overline{\partial}_M \oplus \overline{\partial}^*_M$ on that part of the microlocalization, so only $\mathcal{P}^+ G_{p,q}$ and $\mathcal{P}^- G_{p,q}$ are relevant for the question of compactness of $G_{p,q}$. The key result is the fact that $\mathcal{P}^+ G_{p,q}$ and $\mathcal{P}^- G_{p,q}$ do percolate. However, while for $\mathcal{P}^+ G_{p,q}$, percolation is indeed *up* the $\overline{\partial}_M$–complex, for $\mathcal{P}^- G_{p,q}$ it is *down* the complex. Theorem 2 is then just a corollary: if $G_{p,q}$ is compact at two levels (p, q_1) and (p, q_2), $q_1 \leq q_2$, then both $\mathcal{P}^+ G_{p,q_j}$ and $\mathcal{P}^- G_{p,q_j}$ are compact, $j = 1, 2$, and percolation (up from $\mathcal{P}^+ G_{p,q_1}$, down from $\mathcal{P}^- G_{p,q_2}$) implies that at the intermediate form levels (p, r), $q_1 \leq r \leq q_2$, both $\mathcal{P}^+ G_{p,r}$ and $\mathcal{P}^- G_{p,r}$ are compact. Hence so is $G_{p,r}$.

The meaning of compactness of those operators is given by the following Lemma. We refer to [8], Lemma 1 for the proof.

Lemma 3. *Let $0 \leq p \leq m$, $1 \leq q \leq (m-2)$, $k \in \{+,-,0\}$. Then the following are equivalent:*

(i) $\mathcal{P}^k G_{p,q}$ is compact.

(ii) $\mathcal{P}^k j_{p,q} : dom(\overline{\partial}_M) \cap dom(\overline{\partial}^*_M) \cap \mathcal{H}_{(p,q)}(M)^{\perp} \to L^2_{(p,q)}(M)$ is compact.

(iii) For all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$(27)\quad \|\mathcal{P}^k u\|^2 \leq \varepsilon \left(\|\overline{\partial}_M u\|^2 + \|\overline{\partial}^*_M u\|^2 \right) + C_\varepsilon \|u\|^2_{W^{-1}} \quad , u \in dom(\overline{\partial}_M) \cap dom(\overline{\partial}^*_M) \cap \mathcal{H}_{(p,q)}(M)^{\perp} \,.$$

(iii) For all $\varepsilon > 0$, there is a constant C_ε such that*

$$(28)\qquad \|\mathcal{P}^k u\|^2 \leq \varepsilon \left(\|\overline{\partial}_M u\|^2 + \|\overline{\partial}^*_M u\|^2 \right) + C_\varepsilon \|u\|^2_{W^{-1}} \quad , u \in dom(\overline{\partial}_M) \cap dom(\overline{\partial}^*_M) \,.$$

Note that the dimension $m \geq 3$ is required since we invoke the finite dimension of $\mathcal{H}_{p,q}(M)$ to obtain the equivalence between *(iii)* and *(iii)**. The property *(iii)** is the one we will use.

So the most important result of this section is the following:

Theorem 3. *Let M be a smooth compact pseudoconvex orientable CR-submanifold of $\mathbb{C}^n$ of hypersurface type, of CR-dimension $m-1$, let $0 \leq p \leq m$. We have:*

(i) if $\mathcal{P}^+G_{p,q}$ is compact, then so is $\mathcal{P}^+G_{p,q+1}$, $1 \leq q \leq (m-3)$.
(ii) if $\mathcal{P}^-G_{p,q}$ is compact, then so is $\mathcal{P}^-G_{p,q-1}$, $2 \leq q \leq (m-2)$.

We give below the idea of the proof for (i), the proof of (ii) working with similar arguments. We refer to [8] for the details.

Proof. Since smooth forms are dense in $dom(\overline{\partial}_M) \cap dom(\overline{\partial}_M^*)$ by the Friedrichs Lemma, via a partition of unity argument, it is enough to work with smooth forms supported in a special boundary chart. We define $\mathcal{P}^j$ by use of the microlocalization from [17, 19]. Let $\chi, \chi^+, \chi^-, \chi^0$ be functions with compact support in a neighborhood of such chart and define

$$\mathcal{P}^+u = \chi\mathcal{F}^{-1}\chi^+\hat{u}\ , \quad \mathcal{P}^-u = \chi\mathcal{F}^{-1}\chi^-\hat{u}\ , \quad \mathcal{P}^0u = \chi\mathcal{F}^{-1}\chi^0\hat{u}\ , \tag{29}$$

where $\hat{u} = \mathcal{F}u$ is the Fourier transform on $\mathbb{R}^{2m-1}$ and the operators act coefficientwise with respect to a fixed (chosen) frame $\{\omega_1, \dots, \omega_m\}$. Then $\mathcal{P}^\pm$ and $\mathcal{P}^0$ also act coefficientwise, as pseudo-differential operators of order zero. Note that $\mathcal{P}^+u + \mathcal{P}^-u + \mathcal{P}^0u = u$.

By assumption, for any $v \in L^2_{(p,q)}(M)$, $\mathcal{P}^+G_{p,q}v$ is compact. We want to obtain the compactness estimates $(iii)^*$ for a form $(p,q+1)$-form u in $dom(\overline{\partial}_M) \cap dom(\overline{\partial}_M^*)$. The idea is to build, from any smooth $(p,q+1)$-form u, a set (p,q)-forms v_k whose norms control that of u, for which the compactness estimate $(iii)^*$ are satisfied, in such away that the resulting right hand sides can be estimated by the corresponding right hand side for u. The first part is standard (but see [27], proof of Proposition 4.5).

$$v_k = \sum_{|I|=p,|K|=q}{}^{\prime} u_{IkK}\omega_I \wedge \overline{\omega_K}.$$

Since $\mathcal{P}^+$ acts coefficientwise, observe that

$$\|\mathcal{P}^+u\|^2 = \frac{1}{q+1}\sum_{k=1}^{m-1}\|\mathcal{P}^+v_k\|^2, \tag{30}$$

where $\|.\|$ holds for $\|.\|_{L^2(M)}$. The norms being equivalent, the idea is to get $(iii)^*$ in terms of u by estimating $\|\mathcal{P}^+v_k\|$. However, there is here a difference from [27], applying directly the compactness estimates $(iii)^*$ on v_k will make appear $\overline{\partial}_Mv_k$ and $\overline{\partial}_M^*v_k$. While $\overline{\partial}_M^*v_k$ is easily related to $\overline{\partial}_M^*u$, the same is not true for $\overline{\partial}_Mv_k$ and $\overline{\partial}_Mu$. To address this difficulty, we note that $\mathcal{P}^+$ is essentially a projection, so $\|(\mathcal{P}^+)^2v_k - \mathcal{P}^+v_k\| \lesssim \|(\mathcal{P}^+)^2u - \mathcal{P}^+u\|$ and then we can invoke the microlocal ellipticity of $\overline{\partial}_M \oplus \overline{\partial}_M^*$ on the support of $(\chi^+)^2 - \chi^+$ (since this support stays away from the direction dual to the "bad", or T direction). We get

$$\|(\mathcal{P}^+)^2u - \mathcal{P}^+u\| \lesssim \varepsilon\left(\|\overline{\partial}_Mu\| + \|\overline{\partial}_M^*u\|\right) + C_\varepsilon\|u\|_{W^{-1}}\ . \tag{31}$$

More needs to be precised and some care is required here but we refer to [8] p. 10 for the details. Hence, by using (31), we obtain

$$\begin{aligned}\|\mathcal{P}^+v_k\|^2 &\lesssim \|(\mathcal{P}^+)^2v_k - \mathcal{P}^+v_k\|^2 + \|(\mathcal{P}^+)^2v_k\|^2 \\ &\lesssim \varepsilon\left(\|\overline{\partial}_Mu\|^2 + \|\overline{\partial}_M^*u\|^2\right) + C_\varepsilon\|u\|^2_{W^{-1}} + \|(\mathcal{P}^+)^2v_k\|^2.\end{aligned} \tag{32}$$

Only the last term of (32) is left to estimate. We then invoke the assumption that $\mathcal{P}^+$ is compact on (p,q)-forms $\mathcal{P}^+ v_k$ and we obtain from $(iii)^*$ of Lemma 3,

$$\begin{aligned}
\|(\mathcal{P}^+)^2 v_k\|^2 &\leq \varepsilon \left(\|\overline{\partial}_M(\mathcal{P}^+ v_k)\|^2 + \|\overline{\partial}^*_M(\mathcal{P}^+ v_k)\|^2\right) + C_\varepsilon \|\mathcal{P}^+ v_k\|^2_{W^{-1}} \\
&\leq \varepsilon \left(\|\overline{\partial}_M(\mathcal{P}^+ v_k)\|^2 + \|\mathcal{P}^+\overline{\partial}^*_M v_k\|^2 + \|v_k\|^2\right) + C_\varepsilon \|\mathcal{P}^+ v_k\|^2_{W^{-1}} \qquad (33)\\
&\lesssim \varepsilon \left(\|\bar{\partial}_M(\mathcal{P}^+ v_k)\|^2 + \|\bar{\partial}^*_M u\|^2\right) + C_\varepsilon \|u\|^2_{W^{-1}} . \qquad (34)
\end{aligned}$$

In (33), we have commuted $\mathcal{P}^+$ with $\overline{\partial}^*_M$, and used that the commutator is an operator of order zero. As we mentionned earlier, we have used in (34) that $\overline{\partial}^*_M v_k$ is related to $\overline{\partial}^*_M u$ like it is done in [27], p.79–80 (see also [8]). Note that $\mathcal{P}^+$ is of order zero, the last term in (33) is not a problem: $\|\mathcal{P}^+ v_k\|_{W^{-1}} \lesssim \|v_k\|_{W^{-1}} \lesssim \|u\|_{W^{-1}}$. It suffices to estimate $\|\bar{\partial}_M(\mathcal{P}^+ v_k)\|^2$ in terms of u.

To do that, we use the local expression (3) of $\bar{\partial}$ by noting that via a partition of unity, we can work on a good boundary chart where $L_1, \cdots, L_{m-1}$ are defined. We get

$$\|\bar{\partial}_M(\mathcal{P}^+ v_k)\|^2 \lesssim \sum_{j=1}^{m-1} \sideset{}{'}\sum_{|I|=p,|J|=q+1} \|\overline{L_j}\mathcal{P}^+(u_{IJ})\|^2 + \|\mathcal{P}^+ u\|^2. \qquad (35)$$

Because of the presence of $\mathcal{P}^+$, it turns out that the $\overline{L_j}$–derivatives of $\mathcal{P}^+ u$ on the right-hand side of (35) can be estimated by $\|\overline{\partial}_M(\mathcal{P}^+ u)\| + \|\overline{\partial}^*_M(\mathcal{P}^+ u)\|$ (plus the benign term $\|\mathcal{P}^+ u\|$); thanks to the usual formula, obtained from integration by parts (see for example the proof of Theorem 8.3.5 in [11]),

$$\begin{aligned}
(36)\quad & \|\bar{\partial}_M(\mathcal{P}^+ u)\|^2 + \|\bar{\partial}^*_M(\mathcal{P}^+ u)\|^2 \\
&= \sum_{j=1}^{m-1} \sideset{}{'}\sum_{|I|=p,|J|=q+1} \|\overline{L_j}(\mathcal{P}^+ u_{IJ})\|^2 + \sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j,k=1}^{m-1} \left([L_j, \overline{L_k}]\mathcal{P}^+ u_{IjK}, \mathcal{P}^+ u_{IkK}\right)_{L^2_{(p,q+1)}(M)} \\
&\qquad + \mathcal{O}\left(\|\mathcal{P}^+ u\| \left(\|\bar{L}(\mathcal{P}^+ u)\| + \|L(\mathcal{P}^+ u)\|\right) + \|\mathcal{P}^+ u\|^2\right) ,
\end{aligned}$$

where $\|\overline{L}(\mathcal{P}^+ u)\|^2 = \sum_{j=1}^{m-1} \sideset{}{'}\sum_{|I|=p,|J|=q+1} \|\overline{L_j}(\mathcal{P}^+ u_{IJ})\|^2$. Note that

$$[L_j, \overline{L_k}] = c_{jk} T \mod T^{1,0}M \oplus T^{0,1}M,$$

where (c_{jk}) the matrix of the Levi form in the basis $L_1, \ldots, L_{m-1}$.

The presence of $\mathcal{P}^+$ in the second term of the right-hand term of (36) and the pseudo-convexity allow to conclude: commuting T with $\mathcal{F}^{-1}$ (from $\mathcal{P}^+$) makes appear the positivity required to apply Gårding's inequality (see for example [19], Lemma 2.5, [20], Theorems 3.1, 3.2) on this term:

$$\operatorname{Re}\left(\sideset{}{'}\sum_{|I|=p,|K|=q} \sum_{j,k=1}^{m-1} (c_{jk} T(\mathcal{P}^+ u_{IjK}), \mathcal{P}^+ u_{IkK})\right) \gtrsim -\|u\|^2 . \qquad (37)$$

Combining (37) with the real part of (36), we obtain

$$(38)\qquad \sum_{j=1}^{m-1} \sum_{|I|=p,|J|=q+1}' \|\overline{L_j}(\mathcal{P}^+ u_{IJ})\|^2 \lesssim \|\bar{\partial}_M u\|^2 + \|\bar{\partial}_M^* u\|^2 + \|u\|^2.$$

Finally, combining (38) with (35), inserting the result together with (34) into (32) gives the desired estimate for $\|\mathcal{P}^+ v_k\|^2$ as in $(iii)^*$ of Lemma 3.

The proof of (ii) is similar in terms of arguments, however instead of building a (p,q)-form from a $(p,q+1)$-form, we build a (p,q)-form from $(p,q-1)$-form u as follows:

$$(39)\qquad v_k = \sum_{|I|=p,|J|=q-1}' u_{IJ}\,\overline{\omega_k} \wedge \omega_I \wedge \overline{\omega_J} = \overline{\omega_k} \wedge u\ .$$

Now, contrary to (i), $\overline{\partial}_M v_k$ is easily related to $\overline{\partial}_M u$, but relating $\overline{\partial}_M^* v_k$ to $\overline{\partial}_M^* u$ plus benign terms requires work. The arguments are analogous, but with an additional twist. $\overline{\partial}_M^*$ produces L_j terms, rather than $\overline{L_j}$ terms, and one has to first integrate by parts to convert these to barred terms. This has the effect that instead of the Levi matrix (c_{jk}) in (37), the matrix $(c_{jk} - \frac{1}{q-1}\delta_{jk} tr(c_{rd}))$ appears, where $tr(c_{rd})$ denotes the trace of the Levi matrix. This matrix is no longer positive semi definite, but it still has the property that the sum of any $(q-1)$ eigenvalues is nonnegative. This suffices to make the argument with Gårding's inequality work. We refer to [8] p. 13-14 for the details. □

6. Further curiosities

In this section, we present an open problem, that is not new but which is related to the theory of foliations and fits with the general interest of the conference. We will end with a result obtained by Haslinger [14] that shows that the interpolation of compactness estimates holds when the percolations fails for the $\bar{\partial}$-Neumann operator.

In [30], Straube and Zeytuncu give a sufficient condition for Sobolev estimates for the complex Green operator on a smooth compact orientable and pseudoconvex CR-submanifold of hypersurface type of $\mathbb{C}^n$ in terms of the negative Lie derivative of the form η, introduced in Preliminaries, in the direction of T that is denoted by $\alpha := -\mathcal{L}_T\eta$. Note that this form α acts on the null space of the Levi form, denoted $\mathcal{N}_z$ at a point $z \in M$ as follows:

$$-\mathcal{L}_T\eta(\bar{L}) = \eta([T,\bar{L}]), \quad \forall L \in \mathcal{N}_z,$$

since $\eta(\bar{L}) = 0$. α is then on $\mathcal{N}_z$ the T-component of the commutators $[T,\bar{L}]$ mod $T_z^{1,0}M \oplus T_z^{0,1}M$, which needs to be controlled to obtain Sobolev estimates. We refer to [30] for the details and to [7] for the idea. This form α is closed on the null space, i.e, $d\alpha_{|\mathcal{N}_z} = 0$ for any $z \in M$. In particular, if S is a complex submanifold of M, $[\alpha_{|S}]$ defines a De Rham class of cohomology in $H^1(M)$. To obtain Sobolev estimates, the sufficient condition given in [30] is the exactness of α on $\mathcal{N}_z$, $z \in M$, i.e,

$$\exists h \in \mathcal{C}^\infty(M), \quad dh(L)(z) = \alpha(L)(z), \quad \forall L \in \mathcal{N}_z,\ z \in M.$$

It was proved in the same paper, that this condition happens if M is given by a set of defining plurisubharmonic functions or if M is strictly pseudoconvex except on a smooth complex submanifold S such that $[\alpha_{|S}] = 0$. Note that $[\alpha_{|S}] = 0$ if S is simply connected. However, $[\alpha_{|S}]$ may vanish when S is not simply connected, for example in an annuli in the boundaries of certain Hartogs domains in $\mathbb{C}^2$ (see [9]). In such domains that are in particular "nowhere wormlike", the Bergman projection is regular. Here is then the open question:

Open Problem. *Assume that the complex Green operator $G_{0,1}$ satisfies Sobolev estimates or is globally regular on a smooth orientable compact pseudoconvex CR-submanifold M of hypersurface type of $\mathbb{C}^n$ that is strictly pseudoconvex except on a smooth complex submanifold S of M. Do we have $[\alpha_{|S}] = 0$?*

If S is a flat piece that is foliated by complex submanifolds, then the question is equivalent to proving that α is d-exact on each leaf $\mathcal{L}$ of the foliation, i.e, $\alpha_{|\mathcal{L}} = dh_{|\mathcal{L}}$. This is also equivalent to saying that the foliation is globally defined by a closed one form, which is a crucial matter in the theory of foliation. The same problem occurs for the $\bar{\partial}$-Neumann operator on pseudoconvex domains and is still open. We refer to [29], Proposition 2 for more on this connection, also to [27], Section 5.11 and [4], Section 3.6.

In [14], Haslinger studies the sufficient assumption for the compactness of the $\bar{\partial}$-Neumann operator in the weighted L^2-space $L^2(\mathbb{C}^n, \varphi)$, that is on the plurisubharmonic weight function φ. However, Berger and Haslinger in [6] show that something remarkable happens when we consider a particular weight, said "decoupled weight" that is of the form $\varphi(z_1, \dots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$ where each φ_j, $j \in \{1, \dots, n\}$ is smooth and subharmonic on $\mathbb{C}$ and $\Delta\varphi_j$ is a nontrivial doubling measure (see Definition p.4 in [6]): the $\bar{\partial}$-Neumann operator $N^{\varphi}_{0,q}$ fails to be compact for $0 \leq q \leq n-1$ but $N^{\varphi}_{0,n}$ is compact if and only if $\lim_{z \to +\infty} \int_{B_1(z)} tr((i\partial\bar{\partial}\varphi(z)))d\lambda = +\infty$ where $B_1(z)$ is the unit ball in $\mathbb{C}^n$. The percolation of compactness estimates between $N^{\varphi}_{0,n-1}$ and $N^{\varphi}_{0,n}$ fails. The decoupled weights are an obstruction to the compactness. However, they pointed out that for a variation of decoupled weights such as,

$$\varphi_q(z_1, \dots, z_n) = |(z_1, \dots, z_{q-1})|^4 + |(z_q, \dots, z_n)|^4, \quad q > \frac{n}{2},$$

the weighted $\bar{\partial}$-Neumann operator $N^{\varphi_q}_{0,k}$ is compact for $q \leq k \leq n$. When percolation fails, an interpolation result holds.

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