Weighted Bergman spaces of domains with Levi-flat boundary

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Abstract. This note is an announcement of the author's recent works [1, 2, 3] on Levi-flat real hypersurfaces, which are motivated by the non-existence conjecture of smooth Levi-flat real hypersurface in the complex projective plane.

1 An approach to the non-existence conjecture of Levi-flats in $\mathbb{CP}^2$

This work is motivated by the following

Conjecture. *There is no C^∞-smooth closed Levi-flat real hypersurface in the complex projective plane $\mathbb{CP}^2$.*

A smooth real hypersurface M in a complex manifold X is said to be *Levi-flat* if M has a smooth foliation, called the *Levi foliation* of M, by complex hypersurfaces of X. Typical examples are invariant real hypersurface of (possibly singluar) holomorphic foliations of codimension one on X, and they can be regarded as holomorphic counterpart of limit cycles of trajectories of vector fields. One of the origin of this conjecture is the celebrated Poincaré–Bendixson's theorem and its holomorphic counterpart (cf. [10], [11]), and this conjecture is an important part of *exceptional minimal set conjecture*: any leaf of a singular holomorphic foliation $\mathcal{F}$ of $\mathbb{CP}^2$ must have an accumulation point in the singular locus of $\mathcal{F}$.

In higher dimensional setting, we have affirmative answers on the non-existence of Levi-flats or exceptional minimal sets. In particular, the non-existence of closed Levi-flat real hypersurface in $\mathbb{CP}^n$, $n \geq 3$, was settled by Lins Neto [22] for real analytic ones, and by Siu [25] for C^∞-smooth ones. Since these works depend on vanishing of certain Dolbeault cohomology at $(0,2)$ or the fact that singular locus of holomorphic foliation of $\mathbb{CP}^n$ has positive dimension, the proof for higher dimensional setting does not work at all in two-dimensional setting (cf. [9]).

Although there have been several published papers and preprints claiming affirmative solutions to this conjecture, up to 2018, the gaps in their proofs have not been fixed yet and the conjecture is still open. We refer the interested readers to [19] for the status of papers published before 2008. As far as the author knows, at the moment, we do not have any promising strategy to attack this conjecture, nor strong evidence that the conjecture must be true in two-dimensional case.

Some years ago, Brinkshulte and the author gave a supporting evidence for the conjecture:

Theorem 1.1 (A.-Brinkschulte [5]). *Suppose that there exists hypothetical C^∞-smooth closed Levi-flat real hypersurface M in $\mathbb{CP}^2$. We equip the Fubini–Study metric on $\mathbb{CP}^2$ and restrict it to M. Denote by ν the unit normal vector field of $M \subset \mathbb{CP}^2$ and rotate it to a totally real vector field $\xi := -J\nu$ on M using the complex structure J of $\mathbb{CP}^2$. Then, there exists $p \in M$ with $\mathrm{Ric}_M(\xi_p, \xi_p) \leq -4$.*

A weaker restriction to the hypothetical Levi-flat, the existence of $p \in M$ with $\mathrm{Ric}_M(\xi_p, \xi_p) \leq 0$, was previously shown by Bejancu and Deshmukh [6] using differential geometric techniques. We improved the curvature restriction by looking at the finite/infinite dimensionality of the Bergman space of the complement of the hypothetical Levi-flat M.

Sketch of the proof of Theorem 1.1. The hypothetical Levi-flat M divides $\mathbb{CP}^2$ into two domains Ω and Ω'. These domains are Stein from Takeuchi's theorem [26]. Moreover, Ohsawa and Sibony [23] showed that the distance function δ to M with respect to the Fubini–Study metric induces a bounded strictly plurisubharmonic function $-\delta^\eta$ on $\Omega \cup \Omega'$ for some $\eta \in (0,1)$. Namely, the Diederich–Fornæss index of Ω and Ω' is positve. Once we know this positivity, a Donnelly–Fefferman type L^2-estimate for the $\bar\partial$-equation (cf. [7]) yields the infinite dimensionality of

$$A^2(\Omega, K_{\mathbb{CP}^2}) := \{f\text{: holomorphic 2-form on } \Omega \mid \int_\Omega f \wedge \overline{f} < \infty\}.$$

Now, we assume $\mathrm{Ric}_M(\xi, \xi) > -4$ everywhere on M and try to deduce a contradiction. By Gauss–Codazzi equations and the adjunction formula $K_{\mathbb{CP}^2}|M \otimes N_{\mathcal{F}} = K_{\mathcal{F}}$, where $N_{\mathcal{F}}$ and $K_{\mathcal{F}}$ denote the normal and canonical bundles of the Levi foliation $\mathcal{F}$ respectively, this curvature condition can be rephrased with D'Angelo $(1,0)$-form α,

$$|\alpha|^2 = \frac{1}{4}(H(T_{\mathcal{F}}, T_{\mathcal{F}}^\perp) - \mathrm{Ric}_M(\xi, \xi)) < \frac{1}{4}(2 - (-4)) = \frac{3}{2} = -\frac{1}{2}\deg K_{\mathbb{CP}^2} \quad \text{on } M.$$

Applying an integral formula of Griffiths or Lelong–Jensen formula of Demailly, this boundary inequality yields the finite dimensionality of $A^2(\Omega, K_{\mathbb{CP}^2})$. Contradiction. □

Iordan [20] asked the author whether we could improve the curvature restriction by looking at the weighted Bergman space instead of $A^2(\Omega, K_{\mathbb{CP}^2})$.

Definition 1.2. Let L be a holomorphic line bundle over a complex manifold X with smooth hermitian metric h, and let $\Omega \Subset X$ be a smoothly bounded domain with smooth defining function $-\delta$. For each $\eta > -1$, the L-valued *weighted Bergman space* of Ω of order η is defined by

$$A^2_\eta(\Omega, L) := \{f \in H^0(\Omega, L) \mid \|f\|^2_\eta := \frac{1}{\Gamma(1+\eta)} \int_\Omega |f|^2_h \delta^\eta dV < \infty\}$$

where dV is a volume form of X and Γ denotes the gamma function.

In fact, it is possible to obtain another curvature restriction by modifying the proof of Theorem 1.1 to the case $A^2_\eta(\Omega, \mathcal{O}_{\mathbb{CP}^2}(d))$. However, we are too far from the conjecture since we have severe technical limitation in producing holomorphic functions belonging to weighted Bergman space of order close to -1. For instance, take a look at $A^2_\eta(\Omega, K_{\mathbb{CP}^2})$ for $\eta > -1$. The same argument as in the proof of Theorem 1.1 implies infinite dimensionality of weighted Bergman spaces too:

Proposition 1.3. *Let X be complex manifold and $\Omega \Subset X$ be a smoothly bounded domain with smooth defining function $-\delta$. Suppose that this defining function has the Diederich–Fornæss exponent $\eta \in (0,1)$ in the strongest sense, namely, $-\delta^\eta$ is strictly plurisubharmonic on Ω. Then, $A^2_{-\eta}(\Omega, K_X)$ is infinite dimensional.*

This is a beautiful application of the L^2-estimate of $\overline{\partial}$-equation. However, Proposition 1.3 is not powerful enough to approach the conjecture. When the boundary of domain Ω is Levi-flat, the Diederich–Fornæss exponent of any defining function $-\delta$ cannot exceed $1/\dim_{\mathbb{C}} X$ ([4], [14]. cf. [12]). Hence, Proposition 1.3 cannot tell us infinite dimensionality of $A^2_\eta(\Omega, K_{\mathbb{CP}^2})$ for $\eta \in (-1, -1/2)$, which is not satisfactory in view of our main theorem:

Main Theorem (A. [1, 2, 3])**.** *There is an explicit example of smoothly bounded domain Ω with Levi-flat boundary in compact complex surface X such that*

- *There exists a smooth defining function $-\delta$ of Ω having the Diederich–Fornæss exponent $1/2$, which is the best possible value.*
- *The weighted Bergman space $A^2_\eta(\Omega)$ is infinite dimensional not only for $\eta \geq -1/2$ but also for $\eta \in (-1, -1/2)$.*
- *The Hardy space $A^2_{-1}(\Omega)$ is one-dimensional, namely, consists of constant functions only.*

Definition 1.4. Let $(L, h) \to X$, $\Omega \Subset X$ and $\|\cdot\|_\eta$ as in Definition 1.2. For $\eta = -1$, the L-valued *Hardy space* of Ω is defined by

$$A^2_{-1}(\Omega, L) := \{f \in H^0(\Omega, L) \mid \|f\|_{-1} := \lim_{\eta \searrow -1} \|f\|_\eta < \infty\}.$$

In §2, we explain the construction of the example and show that the example has the Diederich–Fornæss index 1/2. In §3, we give a sketch of proof for the infinite dimensionality of weighted Bergman spaces, which is done by explicit construction since it is beyond the scope of Proposition 1.3. If we could develop the L^2-theory of $\overline{\partial}$-equation deeply enough so that it recovers this result, we might have a chance to reach the conjecture via our approach. In §4, we give a sketch of proof for a slightly weaker finite dimensionality statement: *the space of bounded holomorphic functions $A^\infty(\Omega)$ and $A^\infty(\Omega')$ consist of constant functions.* Although this is an immediate consequence of Hopf's ergodicity theorem, we explain another proof in the spirit of the proof of Theorem 1.1, which suggests that our finite dimensionality argument in Theorem 1.1 should be"sharp".

2 The Example

Let Σ be a compact Riemann surface of genus > 1. By Koebe–Poincaré uniformization theorem, the universal covering of Σ is isomorphic to the unit disk $\mathbb{D}$ and we may regard Σ as the quotient space of $\mathbb{D}$ by a Fuchsian group Γ. Since each element of Γ is a linear fractional transformation, Γ also acts on the Riemann sphere $\mathbb{CP}^1$.

We define a ruled surface over Σ by $X := \mathbb{D} \times \mathbb{CP}^1/\Gamma$ where Γ acts on $\mathbb{D} \times \mathbb{CP}^1$ diagonally. The ruling map is given by the first projection $\pi \colon X \to \mathbb{D}/\Gamma = \Sigma$. Since the action of Γ on $\mathbb{CP}^1$ preserves the decomposition $\mathbb{CP}^1 = \mathbb{D} \cup S^1 \cup (\mathbb{CP}^1 \setminus \overline{\mathbb{D}})$, we can define domains $\Omega := \mathbb{D} \times \mathbb{D}/\Gamma$ and $\Omega' := \mathbb{D} \times (\mathbb{CP}^1 \setminus \overline{\mathbb{D}})/\Gamma$ in X which share the common boundary $M := \mathbb{D} \times S^1/\Gamma$. This M is Levi-flat since the product foliation $\{\mathbb{D} \times \{t\}\}_{t \in S^1}$ induces a foliation on M. These Ω and Ω' are the examples claimed in our main theorem.

Proposition 2.1. *The domain Ω' admits a smooth defining function $-\delta'$ whose Diederich–Fornæss exponent is 1/2 in the strongest sense. On the other hand, the domain Ω admits a smooth defining function $-\delta$ whose Diederich–Fornæss exponent is 1/2 in a weak sense, namely, $-\sqrt{\delta}$ is strictly plurisubharmonic on Ω away from a compact subset.*

Proof. Following Diederich and Ohsawa [13], we define $\delta\colon \mathbb{D} \times \overline{\mathbb{D}} \to [0, 1]$ by

$$\delta(z, w) = 1 - \left| \frac{w - z}{1 - \overline{z}w} \right|^2 .$$

Then, we can easily check that δ induces a well-defined function on a neighborhood of $\overline{\Omega}$ and $-\delta$ is a defining function of Ω. By direct computation, we see that $-\sqrt{\delta}$ is strictly plurisubharmonic on $\Omega \setminus D$ where $D := \{(z, z) \mid z \in \mathbb{D}\}/\Gamma$ is a divisor isomorphic to Σ. Hence, $-\delta$ has the Diederich–Fornæss exponent $1/2$ in the weak sense.

To explain the construction of δ', we first remark that Ω' is isomorphic to the quotient of $\mathbb{D} \times \mathbb{D}$ by "conjugated" action of Γ:

$$\Gamma \times (\mathbb{D} \times \mathbb{D}) \to \mathbb{D} \times \mathbb{D}, \quad \gamma \cdot (z, w') = (\gamma \cdot z, \overline{\gamma \cdot \overline{w'}}).$$

The isomorphism is induced from $\mathbb{D} \times (\mathbb{CP}^1 \setminus \overline{\mathbb{D}}) \to \mathbb{D} \times \mathbb{D}$, $(z, w) \mapsto (z, w') = (z, 1/w)$. Using this (z, w')-coordinate, we define $\delta'\colon \mathbb{D} \times \overline{\mathbb{D}} \to [0, 1]$ by

$$\delta'(z, w') = 1 - \left| \frac{w' - \overline{z}}{1 - zw'} \right|^2 .$$

Then, we can easily check that $-\delta'$ gives a defining function of Ω' as before. By direct computation, we see that $-\sqrt{\delta'}$ is strictly plurisubharmonic on entire Ω'. Hence, $-\delta'$ has the Diederich–Fornæss exponent $1/2$ in the strongest sense. □

As the proof of Proposition 2.1 explains, Ω and Ω' are similar but not biholomorphic. The domain Ω contains the divisor D, but Ω' is Stein and does not contain any divisor. One might say that Ω and Ω' are "half"-biholomorphic, or they are in a "mirror image". We remark that the "mirror image" of D is a totally real subset $D' \subset \Omega'$ given by the quotient of "anti-diagonal" subset $\{(z, \overline{z}) \mid z \in \mathbb{D}\} \subset \mathbb{D} \times \mathbb{D}$ by the conjugated action of Γ. Since D' is real analytically isomorphic to Σ, we can regard Ω' as a complexification of Σ, in fact, Ω' is the Grauert tube of hyperbolic surface Σ in the sense of Guillemin and Stenzel [16] and Lempert and Szőke [21].

Remark 2.2. The canonical bundle K_X of X can be described as $K_X = -2[D]$ since meromorphic 2-form $dz \wedge dw/(z - w)^2$ on $\mathbb{D} \times \mathbb{CP}^1$ induces a meromorphic 2-form on X that trivializes K_X over $X \setminus D$ and has second order pole along D. Hence, our main theorem can be read as a claim for K_X-valued weighted Bergman space $A^2_\eta(\Omega, K_X)$ or K_X-valued Hardy space $A^2_{-1}(\Omega, K_X)$.

3 Infinite dimensionality of the weighted Bergman spaces

In this section, we explain how to construct holomorphic functions of slow growth without appealing to Proposition 1.3 to show the infinite dimensionality of the weighted Bergman spaces. The proof consists of direct computations on the weighted L^2 norm of holomorphic functions to be constructed. For a detailed account, we refer the reader to [1, 3].

Theorem 3.1 (A. [1]). *For $\Omega \Subset X$ constructed in §2, $\dim_{\mathbb{C}} A^2_\eta(\Omega) = \infty$ for any $\eta > -1$.*

The idea is to extend given jet ψ of holomorphic function along the divisor $D \subset \Omega$ to a holomorphic function $I(\psi)$ on Ω with smallest possible L^2 norm, and to check that $I(\psi)$ has finite weighted L^2 norm of any order $\eta > -1$. There have been numerous number of studies dealing with this kind of problem since the discovery of Ohsawa–Takegoshi L^2-extension theorem [24]. Recently, optimal versions of L^2-extension theorem were established by Błocki [8] and Guan and Zhou [15], and an optimal jet L^2-extension theorem was shown by Hosono [18]. The author does not know whether there is an optimal jet L^2-extension theorem that is applicable to this setting and enables us to estimate the weighted L^2-norm of the optimal extension. So, at the moment, we try to construct the optimal estimate in a direct way, by power series.

Sketch of the proof of Theorem 3.1. We use non-holomorphic coordinate $(z,t) \in \mathbb{D}\times\mathbb{D}$ of the universal covering space of Ω given by

$$t = \frac{w-z}{1-\bar{z}w}$$

where (z,w) is the original holomorphic coordinate of $\mathbb{D}\times\mathbb{D}$. Notice that the divisor $D \subset \Omega$ is expressed as $\mathbb{D}\times\{0\} \subset \mathbb{D}\times\mathbb{D}$ in this (z,t)-coordinate. We try to construct a holomorphic function f on Ω, that is, holomorphic function on $\mathbb{D}\times\mathbb{D}$ invariant under Γ, by determining coefficients $\{f_n\}_{n=k}^\infty \subset C^\infty(\mathbb{D})$ in the power series expansion in t

$$f(z,t) = \sum_{n=k}^{\infty} f_n(z) t^n$$

when the first coefficient $f_k(z)$, corresponding to a k-jet of holomorphic function along D, is given. From Γ-invariance of f, we see that each coefficient $f_n(z)$ should correspond to a n-differential $\psi_n \in C^\infty(\Sigma, K_\Sigma^{\otimes n})$ on Σ in a way that

$$\pi^*\psi_n = f_n(z)\left(\frac{\sqrt{2}dz}{1-|z|^2}\right)^{\otimes n}$$

where $\pi\colon \mathbb{D}\times\mathbb{D} \to \mathbb{D}\times\mathbb{D}/\Gamma = \Omega$ is the covering map. In particular, the given jet data $f_k(z)$ should be identified with a holomorphic k-differential $\psi_k \in H^0(\Sigma, K_\Sigma^{\otimes k})$.

Since f should be holomorphic in (z,w), the coefficients $\{f_n\}_{n=k}^\infty$ should enjoy

$$\frac{\partial f_n}{\partial \bar{z}} + \frac{nz}{1-|z|^2} f_n + \frac{n-1}{1-|z|^2} f_{n-1} = 0.$$

We can inductively solve this $\overline{\partial}$-equation so that the coefficients f_n identified with ψ_n have smallest possible L^2-norm in $C^\infty(\Sigma, K_\Sigma^{\otimes n})$ at each step. Then, it turned out, after direct computations, that the resulting power series $I(\psi_k)$ converges to a holomorphic function on Ω and we can describe its expression by an integral transform

$$I(\psi_k)(z,w) = \frac{1}{B(k,k)} \int_z^w \left(\frac{(w-\tau)(\tau - z)}{(w-\tau)d\tau} \right)^{\otimes(k-1)} \pi^*\psi_k(\tau)(d\tau)^{\otimes k}$$

where $B(p,q)$ is the beta function and the integration is performed along any path from z to w in $\mathbb{D}$.

We can compute the weighted L^2-norm of $I(\psi_k)$ on Ω for suitably chosen volume form and defining function as

$$\|I(\psi_k)\|_\eta^2 = \pi \|\psi_k\|^2_{H^0(\Sigma, K_\Sigma^{\otimes k})} {}_3F_2 \left(\begin{matrix} k+1, k, k \\ 2k, k+2+\eta \end{matrix} ; 1 \right),$$

which is finite for any $\eta > -1$. Here ${}_3F_2$ denotes the generalized hypergeometric function.

In summary, we have constructed a linear map

$$I \colon \bigoplus_{k=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes k}) \to \bigcap_{\eta > -1} A^2_\eta(\Omega) \subset \mathcal{O}(\Omega),$$

which is easily seen to be injective and have a dense image in the compact open topology of $\mathcal{O}(\Omega)$. Hence, $A^2_\eta(\Omega)$ is infinite dimensional for any $\eta > -1$. $\square$

For the "mirror" side Ω', the same strategy works. :

Theorem 3.2 (A. [3]). *For $\Omega' \Subset X$ constructed in §2, $\dim_{\mathbb{C}} A^2_\eta(\Omega') = \infty$ for any $\eta > -1$.*

Rough sketch of the proof. We begin with the totally real set $\Sigma \simeq D' \subset \Omega'$ instead of the divisor $D \subset \Omega$. It is a classical fact that any eigenfunction f_λ of the hyperbolic laplacian $\triangle$ on Σ is analytically continued to a holomorphic function $I'(f_\lambda)$ on Ω'. We can reprove this fact by the same method as in the proof of Theorem 3.1 having an advantage that we have the power series expression of $I'(f_\lambda)$. We can confirm that the weighted L^2-norm $\|f_\lambda\|_\eta$ is finite thanks to its explicit expression in terms of $\|f_\lambda\|_{L^2(\Sigma)}$ and ${}_3F_2$ as in Theorem 3.1.

In summary, we have an injective linear map

$$I' \colon \bigoplus_{k=0}^{\infty} \mathrm{Ker}(\triangle - \lambda_k I) \to \bigcap_{\eta > -1} A^2_\eta(\Omega') \subset \mathcal{O}(\Omega'),$$

having a dense image in the compact open topology of $\mathcal{O}(\Omega')$, where $\{\lambda_k\}$ is the eigenvalues of the hyperbolic laplacian $\triangle$. Hence, $A^2_\eta(\Omega')$ is infinite dimensional for any $\eta > -1$. $\square$

4 Finite dimensionality of the Hardy space

In this section, we first see that the finite dimensionality of the space of bounded holomorphic functions $A^\infty(\Omega)$ and $A^\infty(\Omega')$ is a corollary of Hopf's ergodicity theorem:

Theorem 4.1 (Hopf [17]). *Let $\Sigma = \mathbb{D}/\Gamma$ be a Riemann surface of finite hyperbolic area. Then, the diagonal action of Γ on $S^1 \times S^1$ is ergodic with respect to its Lebesgue measure. Namely, for any Lebesgue measurable subset $E \subset S^1 \times S^1$ invariant under the diagonal action of Γ has Lebesgue measure zero or full Lebesgue measure.*

Corollary 4.2. *For $\Omega, \Omega' \Subset X$ constructed in §2, $A^\infty(\Omega)$ and $A^\infty(\Omega')$ consist of constant functions only.*

Sketch of the proof. Let f be a bounded holomorphic function on $\Omega = \mathbb{D} \times \mathbb{D}/\Gamma$. We identify this f with a Γ-invariant holomorphic function on $\mathbb{D} \times \mathbb{D}$. Then, a Fatou type theorem (cf. [27, Theorem IV.13]) yields the boundary value function of f, which is a Γ-invariant bounded measurable function on $S^1 \times S^1$. From Theorem 4.1, this boundary value function must be constant almost everywhere in the Lebesgue measure, and it follows from the Fatou type theorem that f itself is constant on $\mathbb{D} \times \mathbb{D}$.

The same argument applies to bounded holomorphic functions on Ω'. $\square$

We give another proof of Corollary 4.2 in the spirit of the finite dimensionality part in the proof of Theorem 1.1. For a detailed account of the proof, we refer the reader to [2].

Sketch of another proof for Corollary 4.2. We explain it for $A^\infty(\Omega')$. Define a potential function $\varphi\colon \Omega' \to [0, \pi/2)$ by $\varphi := \arccos\sqrt{\delta'}$ where δ' is the one given in Proposition 2.1. Then, by direct computation, we see that φ^2 is a smooth strictly plurisubharmonic function on Ω' and $(i\partial\bar\partial\varphi)^2 = 0$ on $\Omega' \setminus D'$. This, in particular, means that Ω' is the Grauert tube of hyperbolic surface Σ of maximal radius.

We combine this potential function φ with the integral formula used in the proof of Theorem 1.1. For any holomorphic function f on Ω', we have

$$\int_{\varphi^{-1}(s,t)} i\partial\bar\partial|f|^2 \wedge d\varphi \wedge d^c\varphi = \int_{\varphi^{-1}(t)} |f|^2 d^c\varphi \wedge i\partial\bar\partial\varphi - \int_{\varphi^{-1}(s)} |f|^2 d^c\varphi \wedge i\partial\bar\partial\varphi \tag{4.1}$$

thanks to $(i\partial\bar\partial\varphi)^2 = 0$. Substituting $\varphi = \arccos\sqrt{\delta'}$ yields

$$\begin{aligned}&\int_{\varphi^{-1}(s,t)} i\partial\bar\partial|f|^2 \wedge \frac{d\delta' \wedge d^c\delta'}{\delta'(1-\delta')}\\ &= \frac{1}{\sin^2 t}\int_{\varphi^{-1}(t)} |f|^2 d^c(-\delta') \wedge \frac{i\partial\bar\partial(-\delta')}{\delta'} - \frac{1}{\sin^2 s}\int_{\varphi^{-1}(s)} |f|^2 d^c(-\delta') \wedge \frac{i\partial\bar\partial(-\delta')}{\delta'}.\end{aligned}$$

When $f \in A^\infty(\Omega')$, we can show that RHS remains bounded as $t \nearrow \pi/2$ since the boundary M of Ω' in X is Levi-flat. On the other hand, since

$$(\text{LHS}) = \int_{\cos^2 t}^{\cos^2 s} \frac{dx}{x(1-x)} \int_{\varphi^{-1}(x)} i\partial f \wedge \bar\partial f \wedge d^c(-\delta'),$$

the integrability requires that

$$\lim_{x \nearrow \pi/2} \int_{\varphi^{-1}(x)} i\partial f \wedge \bar\partial f \wedge d^c(-\delta') = 0$$

if this limit exists.

We can compute this limit in two other ways. One is

$$\lim_{x\nearrow\pi/2}\int_{\varphi^{-1}(x)} i\partial f\wedge\overline{\partial}f\wedge d^c(-\delta') = \int_M i\partial f\wedge\overline{\partial}f\wedge d^c(-\delta')$$

where f in the RHS should be understood as the boundary value function. To explain the other way, we embed Ω' to $\widetilde{X} := \mathbb{CP}^1\times\mathbb{D}/\Gamma$, where the action of Γ is the conjugated one, and attach Ω' another boundary $\widetilde{M}\subset\widetilde{X}$. Then, we can check

$$\lim_{x\nearrow\pi/2}\int_{\varphi^{-1}(x)} i\partial f\wedge\overline{\partial}f\wedge d^c(-\delta') = \int_{\widetilde{M}} i\partial f\wedge\overline{\partial}f\wedge d^c(-\delta').$$

Now we identify f with a holomorphic function on $\mathbb{D}\times\mathbb{D}$ which is invariant under the conjugated action of Γ. Since these integrals are zero, the boundary value function of f enjoys

$$\frac{\partial f}{\partial z}(z,w)=0\quad\text{a.e. on }\mathbb{D}\times S^1,\qquad \frac{\partial f}{\partial w}(z,w)=0\quad\text{a.e. on }S^1\times\mathbb{D}.$$

This implies that the boundary value of f, which is a CR function, is independent of z on $\mathbb{D}\times S^1$ and of w on $S^1\times\mathbb{D}$. Thanks to the Fatou type theorem, it follows that f should be constant function on $\mathbb{D}\times\mathbb{D}$.

This proof also applies to $A^\infty(\Omega)$ since the essential point in the proof was the fact that the defining function chosen has the Diederich–Fornæss exponent $1/2$. One may use $\varphi = -\sqrt{\delta'}$ instead of the good choice of φ above. Then, $(i\partial\overline{\partial}\varphi)^2\neq 0$ but it modifies the integral formula (4.1) only up to a bounded term, hence, the rest of the proof still works. □

Remark 4.3. In [1], the finite dimensionality of $A^2_{-1}(\Omega)$ was shown via the topological basis of $\mathcal{O}(\Omega)$ constructed in the proof of Theorem 3.1. This method also works for $A^2_{-1}(\Omega')$.

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