Monotone increment processes, classical Markov processes and Loewner chains

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Abstract

This is a summary of the paper [FHS20]. The main result is the construction of bijections of the three objects: non-commutative stochastic processes with monotonically independent increments; certain decreasing Loewner chains in the upper half-plane; a special class of real-valued Markov processes.

1 Non-commutative probability

In non-commutative probability theory, operators on Hilbert spaces are regarded as random variables.

Definition 1.1. A *non-commutative probability space* (H, ξ) consists of a Hilbert space H and a unit vector $\xi \in H$, which defines the vector state $\Phi_\xi : B(H) \to \mathbb{C}$ via

$$\Phi_\xi(X) = \langle \xi, X\xi \rangle.$$

Here, $B(H)$ denotes the space of all bounded linear operators on H and we use inner products which are linear in the second argument. A densely defined closed operator X on H is called a *normal random variable* if $XX^* = X^*X$. In particular:

(1) if X is a self-adjoint operator then we call it a *self-adjoint random variable*;

(2) if X is a unitary operator, then we call it a *unitary random variable*.

Our random variables X will be possibly unbounded operators, and so the domain of X may not contain ξ. The resolvents $1/(z - X)$ are useful in such cases.

Definition 1.2. Let X be a self-adjoint random variable on a non-commutative probability space (H, ξ). The distribution of X is the unique probability measure μ on $\mathbb{R}$ such that

$$\Phi_\xi((z - X)^{-1}) = \int_{\mathbb{R}} \frac{1}{z - x}\, \mu(\mathrm{d}x) =: G_\mu(z), \qquad z \in \mathbb{C}^+ := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}. \tag{1.1}$$

The function $G_\mu(z)$, which will also be denoted by G_X, is called the *Cauchy transform* of μ or of X. The F-transform (or reciprocal Cauchy transform) of μ or of X (denoted by F_μ or F_X) is defined to be the inverse of the Cauchy transform, i.e. as the mapping

$$F_\mu \colon \mathbb{C}^+ \to \mathbb{C}^+, \quad F_\mu(z) = \frac{1}{G_\mu(z)}.$$

The reciprocal Cauchy transforms have an analytic characterization.

Proposition 1.3. *Let $F \colon \mathbb{C}^+ \to \mathbb{C}^+$ be analytic. The following statements are equivalent.*

(1) *$F = F_\mu$ for some probability measure μ on $\mathbb{R}$;*

(2) $\lim_{y\to\infty} \frac{F(iy)}{iy} = 1$*;*

(3) $F(z) = z(1 + o(z))$ *as* $z \to \infty$ *non-tangentially.*

In what follows we denote by $C_b(S)$ the set of all continuous and bounded functions $f : S \to \mathbb{C}$, where S is a topological space. If X is self-adjoint, we can define $f(X)$ by functional calculus for $f \in C_b(\mathbb{R})$. Then (1.1) can be generalized to

$$\Phi_\xi(f(X)) = \int_{\mathbb{R}} f(x)\,\mu(\mathrm{d}x), \qquad f \in C_b(\mathbb{R}).$$

A basic (and crucial in this work) example of random variables arises from the classical probability theory.

Example 1.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. Then $(L^2(\Omega, \mathcal{F}, \mathbb{P}), \mathbf{1}_\Omega)$ is a concrete non-commutative probability space, where $\mathbf{1}_\Omega$ is the constant function on Ω taking the value 1. If $f : \Omega \to \mathbb{R}$ is a real-valued $\mathcal{F}$-measurable function, then the multiplication operator $X : h \mapsto fh$ defined for h in the dense domain

$$\{h \in L^2(\Omega, \mathcal{F}, \mathbb{P}) : fh \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$$

is a self-adjoint random variable. The distribution of X in the sense of Definition 1.2 is exactly the usual distribution of f, that is, $\mathbb{P} \circ f^{-1}$.

For further basics of non-commutative probability we refer the reader to introductions such as [Att, FS16, Mey93].

2 Monotone independence

Independence is a central notion in probability theory. It can be formulated for σ-subfields, which correspond to $*$-subalgebras of $B(H)$ in non-commutative probability. A striking feature of non-commutative probability theory is that several notions of independence have been defined in the literature. Muraki has shown in [Mur03] that the tensor, Boolean, free, monotone, and anti-monotone independences are the only five possible universal notions of independence in non-commutative probability theory. We study monotone independence in this paper. This independence was introduced by Muraki [Mur00, Mur01a, Mur01b] based on earlier work on monotone Fock spaces [Mur96, Mur97, dGL97, Lu97].

Definition 2.1. Let (H, ξ) be a non-commutative probability space.

(1) A family of *-subalgebras $(\mathcal{A}_\iota)_{\iota \in I}$ of $B(H)$ indexed by a linearly ordered set I is called *monotonically independent* if the following two conditions are satisfied.

(i) For any $r, s \in \mathbb{N} \cup \{0\}$, $i_1, \dots, i_r, j, k_1 \dots, k_s \in I$ with

$$i_1 > \cdots > i_r > j < k_s < \cdots < k_1{}^1$$

and for any $X_1 \in \mathcal{A}_{i_1}, \dots, X_r \in \mathcal{A}_{i_r}$, $Y \in \mathcal{A}_j$, $Z_1 \in \mathcal{A}_{k_1}, \dots, Z_s \in \mathcal{A}_{k_s}$, we have

$$\Phi_\xi(X_1 \cdots X_r Y Z_s \cdots Z_1) = \Phi_\xi(X_1) \cdots \Phi_\xi(X_r)\Phi_\xi(Y)\Phi_\xi(Z_s) \cdots \Phi_\xi(Z_1).$$

(ii) For any $i, j, k \in I$ with $i < j > k$ and any $X \in \mathcal{A}_i$, $Y \in \mathcal{A}_j$, $Z \in \mathcal{A}_k$ we have

$$XYZ = \Phi_\xi(Y)XZ.$$

(2) A family $(X_\iota)_{\iota \in I}$ of normal random variables indexed by a linearly ordered set I is called *monotonically independent* if the family $(\mathcal{A}_\iota)_{\iota \in I}$ of *-algebras is monotonically independent, where

$$\mathcal{A}_\iota = \{f(X_\iota) \mid f \in C_b(\mathbb{C}), f(0) = 0\}.$$

Remark 2.2. The following definition of monotone independence is also commonly used in the literature:

[1] If $r = 0$, then we just assume $j < k_s < \cdots < k_1$, and similarly for the case $s = 0$.

(iii) For any $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$ and any $X_1 \in \mathcal{A}_{i_1}, \dots, X_n \in \mathcal{A}_{i_n}$, we have

$$\Phi_\xi(X_1 \cdots X_n) = \Phi_\xi(X_p)\Phi_\xi(X_1 \cdots X_{p-1}X_{p+1} \cdots X_n)$$

whenever p is such that $i_{p-1} < i_p > i_{p+1}$, where the first or the last inequality is eliminated if $p = 1$ or $p = n$ respectively.

It can be checked that (i) and (ii) imply (iii). We prefer (i) and (ii) since our operator model satisfies these stronger conditions. As noted in [Fra09a, Remark 3.2 (c)], the condition (iii) is equivalent to (i) and (ii) if the vacuum vector ξ is cyclic regarding the algebra generated by $\mathcal{A}_i, i \in I$.

Remark 2.3. Monotone (and anti-monotone) independence of two random variables is defined for ordered pairs (X, Y), while tensor, free and Boolean independences do not need an order. Indeed, it is easy to see that (X, I) is monotonically independent for all random variables X, where $I \in B(H)$ denotes the identity. However, if (I, X) is monotonically independent for $X \in B(H)$, then we have $X = IXI = \Phi_\xi(X)I$, i.e. X is a multiple of the identity. This also explains why we take functions $f \in C_b(\mathbb{C})$ such that $f(0) = 0$ in ((2)). If we remove the condition $f(0) = 0$, then we can take $f \equiv 1$ and so X_ι must be multiples of the identity for all but the maximal index.

Once a notion of independence of random variables is defined, one can introduce many concepts similar to those in probability theory: convolution of probability measures, central limit theorems, non-commutative stochastic processes with independent increments, and non-commutative stochastic differential equations. For non-commutative independent increment processes, see the two books [ABKL05, BN+al06]. We also refer to [Oba17], where the author shows how independences in non-commutative probability theory can be applied to the analysis of graphs. The different notions of independence appear in connection with certain products for graphs.

Assume that (X, Y) is a pair of monotonically independent self-adjoint random variables on a non-commutative probability space (H, ξ) such that $X + Y$ is essentially self-adjoint. If μ and ν denote the distributions of X and Y respectively, then it can be shown that the distribution λ of $X+Y$ can be computed by

$$F_\lambda = F_\mu \circ F_\nu. \tag{2.1}$$

Conversely, given two probability measures μ and ν on $\mathbb{R}$, one can always find monotonically independent self-adjoint operators X and Y with the distributions μ and μ, respectively (e.g. use the operators in [Fra09a, Proposition 3.9]).

Thus the formula (2.1) defines the binary operation $\mu \rhd \nu := \lambda$, called the *(additive) monotone convolution* of probability measures μ and ν on $\mathbb{R}$.

Remark 2.4. Monotone convolution was originally defined by Muraki in [Mur00]. He first derived formula (2.1) for compactly supported probability measures by computing the moments of $(X + Y)^n$ when X and Y are monotonically independent bounded self-adjoint random variables [Mur00, Theorem 3.1]. Then he extended the definition of monotone convolution to arbitrary probability measures via complex analysis [Mur00, Theorem 3.5]. Franz [Fra09a] constructed an unbounded self-adjoint operator model for monotone convolution of arbitrary probability measures as mentioned above.

A non-commutative stochastic process is simply a family $(X_t)_{t\geq 0}$ of random variables. In this work we study the following monotone increment processes, which is an analogue of additive processes in probability theory [Sat13].

Definition 2.5. Let (H, ξ) be a non-commutative probability space and $(X_t)_{t\geq 0}$ a family of essentially self-adjoint operators on H with $X_0 = 0$. We call (X_t) a *self-adjoint additive monotone increment process (SAIP)* if the following conditions are satisfied.

(a) The increment $X_t - X_s$ with domain $\mathrm{Dom}(X_t) \cap \mathrm{Dom}(X_s)$ is essentially self-adjoint for every $0 \leq s \leq t$.

(b) $\mathrm{Dom}(X_s) \cap \mathrm{Dom}(X_t) \cap \mathrm{Dom}(X_u)$ is dense in H and is a core for the increment $X_u - X_s$ for every $0 \leq s \leq t \leq u$.

(c) The mapping $(s,t) \mapsto \mu_{st}$ is continuous w.r.t. weak convergence, where μ_{st} denotes the distribution of the increment $X_t - X_s$.

(d) The tuple
$$(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$$
is monotonically independent for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in \mathbb{R}$ s.t. $0 \le t_1 \le t_2 \le \cdots \le t_n$.

Furthermore if $X_t - X_s$ has the same distribution as X_{t-s} for all $0 \le s \le t$ (the condition of *stationary increments*), then $(X_t)_{t\ge 0}$ is called a *monotone Lévy process.*

Note that the conditions (a) and (b) are rather technical; they are not necessary if $(X_t)_{t\ge 0}$ consists of bounded operators. We introduce an equivalence relation for two processes.

Definition 2.6. Let (H,ξ) and (H',ξ') be non-commutative probability spaces and let $(X_t)_{t\ge 0}$ and $(X'_t)_{t\ge 0}$ be families of essentially self-adjoint operators on (H,ξ) and (H',ξ') respectively. Then (X_t) and (X'_t) are equivalent if the finite dimensional distributions are equal, namely

$$\langle \xi, f_1(X_{t_1}) \cdots f_n(X_{t_n})\xi\rangle_H = \langle \xi', f_1(X'_{t_1}) \cdots f_n(X'_{t_n})\xi'\rangle_{H'} \tag{2.2}$$

for all $n \in \mathbb{N}, t_1, \ldots, t_n \ge 0, f_1, \ldots, f_n \in C_b(\mathbb{R})$.

Proposition 2.7. *Two SAIPs are equivalent if and only if their distributions of increments coincide.*

3 Markov processes

We give basic concepts on Markov processes. The main references are [RY99] and [Kal02]. In this section S denotes a locally compact space with countable basis, and $\mathcal{S}$ denotes the Borel σ-field.

A *probability kernel* k on $(S,\mathcal{S})$ is a map $k : S \times \mathcal{S} \to [0,1]$ such that

(i) $k(x,\,\cdot\,) : \mathcal{S} \ni B \mapsto k(x,B)$ is a probability measure for each $x \in S$;

(ii) $k(\,\cdot\,,B) : S \ni x \mapsto k(x,B)$ is a measurable function for each $B \in \mathcal{S}$.

For two probability kernels k and l we define its composition

$$(k \star l)(x,B) = \int_S k(x,\mathrm{d}y)l(y,B) \qquad \text{for } x \in S, B \in \mathcal{S}.$$

A family $(k_{st})_{0\le s\le t}$ of probability kernels is called *transition kernels* if it satisfies

$$k_{su} = k_{st} \star k_{tu} \ \text{ and } \ k_{ss}(x,\,\cdot\,) = \delta_x(\cdot) \tag{3.1}$$

for all $0 \le s \le t \le u$ and $x \in S$. The former relation is called the Chapman-Kolmogorov relation.

Definition 3.1. Let $(k_{st})_{0\le s\le t}$ be a family of transition kernels. A stochastic process $(M_t)_{t\ge 0}$ on $(S,\mathcal{S})$ adapted to a filtration $(\mathcal{F}_t)$ is called a *Markov process with transition kernels* $(k_{st})_{0\le s\le t}$ if for each $0 \le s \le t$ and $B \in \mathcal{S}$ we have

$$\mathbb{P}[M_t \in B|\mathcal{F}_s] = k_{st}(M_s,B) \ \ a.s. \tag{3.2}$$

The distribution $\mathbb{P} \circ M_0^{-1}$ on $(S,\mathcal{S})$ is called the *initial distribution.* When we simply say a *Markov process*, it is a Markov process with some transition kernels and some filtration. A Markov process is said to be *stationary* if its transition kernels satisfy $k_{st} = k_{0,t-s}$. Then we simply denote k_{0t} by k_t and call $(k_t)_{t\ge 0}$ the transition kernels as well. In this case the Chapman-Kolmogorov relation reads $k_s \star k_t = k_{s+t}$ for $s,t \ge 0$.

The equation (3.2) is called the *Markov property*. It is equivalent to

$$\mathbb{E}[f(M_t)|\mathcal{F}_s] = \int_S f(x) k_{st}(M_s, \mathrm{d}x) \quad a.s. \tag{3.3}$$

for all bounded measurable functions $f\colon S \to \mathbb{C}$.

It is known that for a distribution μ on $(S,\mathcal{S})$ and a family of transition kernels $(k_{st})_{0\le s\le t}$ on $(S,\mathcal{S})$ satisfying (3.1), there exists a Markov process $(M_t)_{t\ge 0}$ with initial distribution μ and $(k_{st})_{0\le s\le t}$ as transition kernels. Moreover, the Markov process is unique up to finite dimensional distributions, namely, with respect to the following equivalence.

Definition 3.2. Two stochastic processes $(M_t)_{t\ge 0}$ and $(N_t)_{t\ge 0}$ are equivalent if

$$\mathbb{P}[(M_{t_1},\dots,M_{t_n}) \in B] = \mathbb{P}[(N_{t_1},\dots,N_{t_n}) \in B] \tag{3.4}$$

for all times $t_1,\dots,t_n \ge 0$, all $n \in \mathbb{N}$ and all $B \in \mathcal{S}^n$.

Suppose that two Markov processes $(M_t)_{t\ge 0}$ and $(N_t)_{t\ge 0}$ have the same transition kernels $(k_{st})_{0\le s\le t}$ and initial distribution μ. Then they are equivalent, and actually the above common value (3.4) is given by

$$\int_{S^{n+1}} \mathbf{1}_B(x_1,\dots,x_n)\mu(\mathrm{d}x_0)k_{0t_1}(x_0,\mathrm{d}x_1)\cdots k_{t_{n-1}t_n}(x_{n-1},\mathrm{d}x_n).$$

In this paper, we fix the initial distribution to be a delta measure. Then an equivalence class of Markov processes is determined by (k_{st}), on which we mainly focus.

Investigation of SAIPs gives rise to the notion of monotonically homogeneous probability kernels.

Definition 3.3. We say that a probability kernel k is *monotonically homogeneous* ($\rhd$-homogeneous, for short) if

$$\delta_x \rhd k(y,\cdot) = k(x+y,\cdot), \qquad x,y \in \mathbb{R}$$

and that a Markov process $(M_t)_{t\ge 0}$ on $\mathbb{R}$ with transition kernels $(k_{st})_{0\le s\le t}$ is $\rhd$*-homogeneous* if each k_{st} is $\rhd$-homogeneous and the mapping $(s,t)\mapsto k_{st}(x,\cdot)$ is continuous w.r.t. weak convergence for every $x \in \mathbb{R}$.

A probabilistic interpretation of this $\rhd$-homogeneity is still missing, while this notion naturally appears in the study of SAIPs.

4 Loewner chains

The distributions of processes with monotonically independent increments will lead us to certain families of holomorphic mappings. These families turn out to be decreasing *Loewner chains*.

Definition 4.1. (1) Let $(f_{st})_{0\le s\le t}$ be a family of holomorphic mappings $f_{st} : \mathbb{C}^+ \to \mathbb{C}^+$ satisfying

(TM1) $f_{ss}(z) = z$ for all $z \in \mathbb{C}^+$ and $s \ge 0$,

(TM2) $f_{su} = f_{st} \circ f_{tu}$ for all $0 \le s \le t \le u$,

(TM3) $(s,t) \mapsto f_{st}$ is continuous with respect to locally uniform convergence.

The family $(f_t)_{t\ge 0} := (f_{0t})_{t\ge 0}$ is called a *(decreasing) Loewner chain on* $\mathbb{C}^+$. We will call the mappings f_{st} the *transition mappings* of the Loewner chain.

(2) We call a Loewner chain $(f_t)_{t\ge 0}$ an *additive Loewner chain* if $\lim_{y\to\infty} f_{st}(iy)/(iy) = 1$, or equivalently

$$f_{st} = F_{\mu_{st}}$$

for all $0 \le s \le t$, where each μ_{st} is a probability measure on $\mathbb{R}$.

In case of an additive Loewner chain, condition (TM3) is equivalent to

$$(s,t) \mapsto \mu_{st} \text{ is continuous with respect to weak convergence.}$$

For a Loewner chain $(f_t)_{t\geq 0}$, its range $f_t(\mathbb{C}^+)$ is decreasing in time as a consequence of (TM2). Such Loewner chains are called decreasing. Loewner chains appearing in SLE are also decreasing. In the context of complex analysis, however, it is more common to work on increasing Loewner chains, where (TM2) is replaced by $f_{su} = f_{tu} \circ f_{st}$ for all $0 \leq s \leq t \leq u$.

A basic property of Loewner chains is the univalence.

Theorem 4.2. *All transition mappings f_{st} of a Loewner chain are univalent.*

5 Main results

5.1 Bijections of SAIPs, Markov processes and Loewner chains

The first goal is to establish one-to-one correspondences between SAIPs, $\triangleright$-homogeneous Markov processes, and additive Loewner chains, motivated by or extending the past works [Bia98, FM05, Fra09a, LM00, Sch17].

Theorem 5.1. *We establish one-to-one correspondences between the following objects:*

(1) *SAIPs $(X_t)_{t\geq 0}$ up to equivalence,*

(2) *additive Loewner chains $(F_t)_{t\geq 0}$ in $\mathbb{C}^+$,*

(3) *real-valued $\triangleright$-homogeneous Markov processes $(M_t)_{t\geq 0}$ with $M_0 = 0$ up to equivalence.*

The details of the correspondences in Theorem 5.1 are as follows. If $(X_t)_{t\geq 0}$ is a SAIP, then the reciprocal Cauchy transforms $(F_{X_t})_{t\geq 0}$ form an additive Loewner chain in $\mathbb{C}^+$ by (2.1). Given an additive Loewner chain $(F_t)_{t\geq 0}$ in $\mathbb{C}^+$, the Markov transition kernels $(k_{st})_{0\leq s\leq t}$ defined by the identity

$$\int_{y\in\mathbb{R}} \frac{1}{z-y} k_{st}(x, \mathrm{d}y) = \frac{1}{F_s^{-1} \circ F_t(z) - x}, \quad z \in \mathbb{C}^+, x \in \mathbb{R},$$

determine a unique real-valued $\triangleright$-homogeneous Markov process $(M_t)_{t\geq 0}$.

Finally, if $(M_t)_{t\geq 0}$ is a real-valued $\triangleright$-homogeneous Markov process on $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$, then the non-commutative stochastic process $(X_t)_{t\geq 0}$ defined by

$$X_t h = \mathbb{E}[M_t h | \mathcal{F}_t]$$

with dense domain $D(X_t) = D(M_t) = \{h \in L^2(\Omega, \mathcal{F}, \mathbb{P}) : M_t h \in L^2(\Omega, \mathcal{F}, \mathbb{P})\}$ is a SAIP. Note that the conditional expectation $P_t = \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ is viewed as the orthogonal projection from $L^2(\Omega, \mathcal{F}, \mathbb{P})$ onto the subspace $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$. Thus, with a slight abuse of notation, we may write $X_t = P_t M_t$ by viewing the function M_t as a multiplication operator on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ in the sense of Example 1.4. Note that $\mu_{X_t} = k_{0,t}(0, \cdot)$ holds.

Remark 5.2. In the literature constructions of SAIPs have been limited to the case of bounded operators. In [Mur97], Muraki constructed a monotone Brownian motion, i.e. a SAIP $(X_t)_{t\geq 0}$ where the distribution of $X_t - X_s$ is the arcsine distribution with mean 0 and variance $t - s$. More generally, monotone Lévy processes consisting of bounded self-adjoint operators have been constructed in [FM05, Theorem 4.1]. Jekel [Jek20, Theorem 6.25] constructed (operator-valued) bounded monotone increment processes on a monotone Fock space. Our construction based on classical Markov processes is different from all of them and has the advantage that we can include any unbounded processes.

5.2 Stationary $\triangleright$-homogeneous Markov processes

The class of stationary $\triangleright$-homogeneous Markov processes may be of particular interest. It is not hard to show that they are Feller processes. In this section we compute the generators of those processes and their relation to the Berkson-Porta formula [BP78] for one-parameter semigroups of holomorphic self-mappings.

The stationarity means that $k_{s,t} = k_{0,t-s}$, and so let us simply denote by $k_t = k_{0,t}$ and call $(k_t)_{t\geq 0}$ the transition kernels of the Markov process. The associated Loewner chain has the so-called *monotone Lévy-Khintchine representation* proved by Muraki [Mur00] in the finite variance case. The general case follows from Berkson and Porta's result [BP78].

Theorem 5.3. (1) *Let $(M_t)_{t\geq 0}$ be a $\rhd$-homogeneous Markov process with transition kernels $(k_t)_{t\geq 0}$, and let $F_t = F_{k_t(0,\cdot)}$. Then $F_{s+t} = F_s \circ F_t$ for all $s, t \geq 0$, the right derivative $A(z) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} F_t(z)$ exists locally uniformly (and hence is holomorphic), and F_t satisfies the equation*

$$F_t(z) = z + \int_0^t A(F_s(z))\,ds, \quad F_0(z) = z, \qquad z \in \mathbb{C}^+, t \geq 0. \tag{5.1}$$

Moreover, the analytic function A is of the form

$$-A(z) = \gamma + \int_{\mathbb{R}} \frac{1+zx}{z-x}\rho(\mathrm{d}x), \qquad z \in \mathbb{C}^+, \tag{5.2}$$

where $\gamma \in \mathbb{R}$ and ρ is a finite non-negative measure on $\mathbb{R}$. The pair (γ, ρ) is unique and is called the generating pair.

(2) *Conversely, given a pair (γ, ρ) of a real number and a finite non-negative measure, define a function A by* (5.2). *Then the solution to the equation* (5.1) *uniquely exists and defines a flow $(F_t)_{t\geq 0}$ on $\mathbb{C}^+$, and then there exists a $\rhd$-homogeneous Markov process $(M_t)_{t\geq 0}$ such that $F_t = F_{k_t(0,\cdot)}$ for all $t \geq 0$.*

The above theorem is stated from the viewpoint of one-parameter semigroups of holomorphic self-mappings. Here we relate it to the generator of stationary $\rhd$-homogeneous Markov processes, extending the formula in [FM05]. We start by introducing the free difference quotient $\partial\colon C^1(\mathbb{R}) \to C(\mathbb{R}^2)$ by

$$(\partial f)(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y}, & x \neq y, \\ f'(x), & x = y. \end{cases} \tag{5.3}$$

Then for $f \in C^2(\mathbb{R})$ we have

$$(\partial_x \partial f)(x,y) = \begin{cases} \frac{f(y)-f(x)-(y-x)f'(x)}{(y-x)^2}, & x \neq y, \\ \frac{1}{2}f''(x), & x = y. \end{cases} \tag{5.4}$$

Let $\mathcal{B}_b(\mathbb{R})$ be the set of all bounded Borel measurable functions $f\colon \mathbb{R} \to \mathbb{C}$.

Theorem 5.4. *Let $(M_t)_{t\geq 0}$ be a stationary $\rhd$-homogeneous Markov process with transition kernels $(k_t)_{t\geq 0}$. Let $T_t\colon \mathcal{B}_b(\mathbb{R}) \to \mathcal{B}_b(\mathbb{R})$ be its transition semigroup*

$$(T_t f)(x) = \int_{\mathbb{R}} f(y)\, k_t(x, \mathrm{d}y), \qquad f \in \mathcal{B}_b(\mathbb{R}),$$

which satisfies $T_s T_t = T_{s+t}$ for $s, t \geq 0$. The generator of the transition semigroup is then given by

$$\begin{aligned}(Gf)(x) &:= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (T_t f)(x) \\ &= \gamma f'(x) + \int_{\mathbb{R}} \left\{(1+y^2)(\partial_x \partial f)(x,y) + y f'(x)\right\} \rho(\mathrm{d}y)\end{aligned}$$

for $f \in C_b(\mathbb{R}) \cap C^2(\mathbb{R})$ and $x \in \mathbb{R}$, where (γ, ρ) is the pair in (5.2).

Example 5.5. If $\gamma = 0$ and $\rho = \delta_0$, then the generator is

$$(Gf)(x) = \begin{cases} \frac{f(0)-f(x)+xf'(x)}{x^2}, & x \neq 0, \\ \frac{1}{2}f''(0), & x = 0, \end{cases}$$

and $k_t(0,\cdot)$ is the arcsine law with mean 0 and variance t. The infinitesimal generator (see Theorem 5.3) for the compositional semigroup $(F_t)_{t\geq 0}$ is given by $A(z) = -\frac{1}{z}$, and hence $F_{k_t(0,\cdot)}(z) = \sqrt{z^2-2t}$. The transition probability of the associated stationary Markov process is

$$k_t(x, \mathrm{d}y) = \frac{\sqrt{2t-y^2}}{\pi(x^2-y^2+2t)} \mathbf{1}_{(-\sqrt{2t},\sqrt{2t})}(y)\,\mathrm{d}y + \frac{|x|}{\sqrt{x^2+2t}}\delta_{\mathrm{sign}(x)\sqrt{x^2+2t}}(\mathrm{d}y).$$

This Markov process, when started at 0, is known as the Aźema martingale (see e.g. [Eme89]).

5.3 Marginal distributions of SAIPs

It is another remarkable fact that a probability measure μ can occur as a marginal distribution of an SAIP iff its Cauchy transform $G_\mu = 1/F_\mu$ is univalent.

Theorem 5.6. *Let μ be a probability measure on $\mathbb{R}$. The following statements are equivalent.*

(1) *F_μ is univalent.*

(2) *There exists a SAIP $(X_t)_{t\geq 0}$ such that the distribution of X_1 is μ.*

(3) *There exists an additive Loewner chain $(F_t)_{t\geq 0}$ in $\mathbb{C}^+$ such that $F_1 = F_\mu$.*

(4) *There exists a $\rhd$-homogeneous Markov process $(M_t)_{t\geq 0}$ such that $M_0 = 0$ and $\mathbb{P}\circ M_1^{-1} = \mu$.*

The idea of the proof of Theorem 5.6 is as follows. The equivalence between (2)–(4) is a part of Theorem 5.1. Under suitable Cayley transforms and a suitable time change, the Loewner chain (F_t) can be transformed into a Loewner chain on the unit disk that is differentiable regarding t almost everywhere and satisfies Loewner's partial differential equation. Then we can use recent work on Loewner chains [CDMG14] to prove the equivalence between (1) and (3).

Remark 5.7. The equivalence between (1) and (2) of Theorem 5.6 is to be compared with classical probability (see [Sat13, Theorems 7.10 and 9.1]): given a stochastic process $(Y_t)_{t\geq 0}$ with independent (not necessarily stationary) increments, $Y_0 = 0$ and suitable continuity properties, the distribution of Y_1 is infinitely divisible; conversely any infinitely divisible distribution can be realized as such. The same statement is true if we consider Lévy processes (namely, if we assume stationary increments). However, there exists a probability measure μ which is not monotonically infinitely divisible but F_μ is univalent. Therefore there exists a gap between the laws of SAIPs and those of monotone Lévy processes.

Remark 5.8. R. Bauer has studied univalent Cauchy transforms in [Bau05] and he has also regarded Loewner's differential equation from a non-commutative probabilistic point of view, see [Bau03] and [Bau04]. The relation to monotone independence is also discussed in [Sch17].

In view of Theorem 5.6, it is interesting to investigate univalent F-transforms from a (non-commutative) probabilistic viewpoint. On the other hand, *geometric function theory* investigates univalent functions via the geometry of their image domains.

The fact that a probability measure μ on $\mathbb{R}$ is uniquely determined by its F-transform (due to the Stieltjes-Perron inversion formula) poses a question about how properties of μ and analytic/geometric properties of F_μ are related. For results in this direction, the reader is referred to [FHS20].

Finally, one can translate most of the notions and results we discussed into unitary operators, which leads to bijections of unitary monotone increment processes, Markov processes on the unit circle and radial Loewner chains in the unit disk. In the unitary case, we can further introduce a bijection to additive processes on the unit circle, which was recently established in [HH].

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