Freezing Laguerre ensemble in the hard edge

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Abstract

The fluctuation structure of the freezing limit in finite-dimensional random matrix ensembles, in which the inverse temperature parameter β tends to infinity, has been a topic with several recent developments. It is known that in this regime, the joint probability density of the eigenvalues obeys a multivariate Gaussian distribution. Recently, it was found that the covariance matrix involved in this distribution shows a surprisingly regular structure, and a complete description of its eigenvalues and eigenvectors was given by Andraus, Hermann and Voit for the Hermite (or Gaussian), Laguerre (or Wishart) and Jacobi ensembles of random matrices. In this paper, we showcase an application of these results to the hard edge statistics of the Laguerre ensemble. We show that the eigenvalue variance in the hard edge region is given asymptotically by a specific integral involving Bessel functions, which is itself derived from asymptotics of the covariance eigenvector matrix.

1 Introduction

Historically, random matrix ensembles were defined as real symmetric, complex Hermitian or quaternion self-dual, a process which is summarized as Dyson's threefold way [1, 2]. For each of these symmetries, it was found that the eigenvalue distribution had the form of the partition function of a one-dimensional system of log-interacting charged particles at an inverse temperature β equal to 1, 2 or 4, corresponding to the matrix symmetry in question. Decades later, Dumitriu and Edelman [3] introduced tridiagonal random matrix ensembles with eigenvalue distributions similar to those of the threefold-way, with the important distinction that β could be chosen as a positive real parameter.

Then, it became meaningful to study the limiting cases of these eigenvalue distributions, and in particular a description of the freezing (large-β) asymptotics of the Hermite and Laguerre ensembles was given in [4]. There, it was shown that the freezing asymptotics of the eigenvalue distributions follow a multivariate Gaussian distribution, and an elaborate form of the covariance matrices was found. More recently, a study of the freezing regime was carried out in [5, 6], where the properties of the *inverse* covariance matrix were clarified for the Hermite, Laguerre and Jacobi cases. Specifically, the eigenvalue structure for all three cases was derived, and the corresponding eigenvectors were identified as being generated by a finite family of orthogonal polynomials with respect to discrete measures localized at the zeros of Hermite, Laguerre and Jacobi polynomials, respectively.

The most recent development on this topic concerns the concrete form of the inverse covariance matrix eigenvectors and of the covariance matrix itself. It was shown by Andraus, Hermann and Voit [7] that the polynomials which generate the covariance matrix eigenvectors are in fact the de Boor-Saff dual polynomials of the Hermite, Laguerre and Jacobi polynomials [8, 9]. Moreover, with the explicit form of the eigenvectors, significantly simpler new forms for the covariance matrix in the Hermite and Laguerre case were found, while the covariance matrix in the Jacobi case was previously unknown.

In this paper, we make use of the results in [7] to investigate the variances in the hard edge for the Laguerre case. Consider the β-Laguerre ensemble in N dimensions; after an appropriate rescaling by a factor of $\sqrt{\beta}$, its eigenvalue density $p_{N,\beta,\alpha}(y)$ is given by

$$p_{N,\beta,\alpha}(y) = c_{N,\beta,\alpha} e^{-\beta\|y\|^2/2} \cdot \prod_{i<j} (y_j^2 - y_i^2)^\beta \cdot \prod_{i=1}^{N} y_i^{\beta(\alpha+1)}, \tag{1}$$

where $y \in \mathbb{W}_{B,N} := \{x \in \mathbb{R}^N : 0 \leq x_1 \leq \cdots \leq x_N\}$ (the closed Weyl chamber of type B_N), $\beta > 0$, $\alpha > -1$, $c_{N,\beta,\alpha}$ is a normalization constant that is obtained from a Selberg integral [2], and $\|\cdot\|$ is the L^2 norm. Consider a random variable $Y_{N,\beta,\alpha}$ whose distribution is $p_{N,\beta,\alpha}$, and denote by $z_{N,\alpha} = (z_{1,N,\alpha}, \ldots, z_{N,N,\alpha})^T \in \mathbb{W}_{B,N}$ the ordered vector of zeros of the Nth Laguerre polynomial

$$L_N^{(\alpha)}(x) := \sum_{j=0}^{N} (-1)^j \binom{N+\alpha}{N-j} \frac{x^j}{j!}. \tag{2}$$

Lemma 1.1 (Andraus, Voit [5]). *In the freezing limit,*

$$\frac{Y_{N,\beta,\alpha}/\sqrt{\beta} - z_{N,\alpha}}{1/\sqrt{\beta}} \overset{\beta\to\infty}{\longrightarrow} Y_{N,\alpha} \tag{3}$$

weakly. $Y_{N,\alpha}$ *follows a centered multivariate Gaussian distribution with inverse covariance matrix given by*

$$[S^{(\alpha+1)}]_{ii} = \\ 1 + (\alpha+1)(z_{i,N,\alpha})^{-2} + \sum_{l\neq i} \Big[(z_{i,N,\alpha} - z_{l,N,\alpha})^{-2} + (z_{i,N,\alpha} + z_{l,N,\alpha})^{-2} \Big], \tag{4}$$

and

$$[S^{(\alpha+1)}]_{ij} = (z_{i,N,\alpha} + z_{j,N,\alpha})^{-2} - (z_{i,N,\alpha} - z_{j,N,\alpha})^{-2} \tag{5}$$

with $i \neq j$ *and* $1 \leq i, j \leq N$.

The covariance matrix $\Sigma^{(\alpha+1)} = [S^{(\alpha+1)}]^{-1}$ can be characterized as follows.

Theorem 1.2 (Andraus, Hermann, Voit [7]). *The matrix* $S^{(\alpha+1)}$ *has eigenvalues* $\lambda_j = 2j$, $j = 1, \ldots, N$, *and eigenvectors*

$$v_{j,\alpha} = \left(\sqrt{\frac{z_{1,N,\alpha}}{N(N+\alpha)}} \frac{\tilde{L}_{N-j}^{(\alpha)}(z_{1,N,\alpha})}{\tilde{L}_{N-1}^{(\alpha)}(z_{1,N,\alpha})}, \ldots, \sqrt{\frac{z_{N,N,\alpha}}{N(N+\alpha)}} \frac{\tilde{L}_{N-j}^{(\alpha)}(z_{N,N,\alpha})}{\tilde{L}_{N-1}^{(\alpha)}(z_{N,N,\alpha})} \right)^T, \tag{6}$$

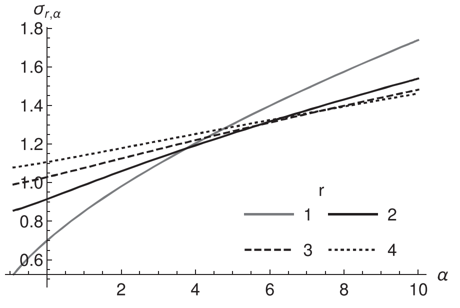

Figure 1: Plot of $\sigma_{r,\alpha}$ for $r = 1$ (gray), 2 (black), 3 (dashed), and 4 (dotted). Note that while each of the curves is monotonically increasing in α, their steepness is strongly dependent on r.

where $\{\tilde{L}_j^{(\alpha)}(x)\}_{j=0}^{\infty}$ *denotes the orthonormal Laguerre polynomials, and* $1 \leq j \leq N$*. Then, the covariance matrix reads*

$$[\Sigma^{(\alpha)}]_{ij} = \frac{\sqrt{z_{i,N,\alpha} z_{j,N,\alpha}}}{N(N+\alpha)} \sum_{k=1}^{N} \frac{1}{2k} \frac{\tilde{L}_{N-k}^{(\alpha)}(z_{i,N,\alpha}) \tilde{L}_{N-k}^{(\alpha)}(z_{j,N,\alpha})}{\tilde{L}_{N-1}^{(\alpha)}(z_{i,N,\alpha}) \tilde{L}_{N-1}^{(\alpha)}(z_{j,N,\alpha})}. \tag{7}$$

We emphasize here that similar results for the Hermite and Jacobi cases are given in [7], as well as a detailed study of the soft edge statistics on both the Hermite and Laguerre cases. The main objective of this paper is to prove the following statement, where we denote by $J_\alpha(x)$ the Bessel function of the first kind, and by $j_{\alpha,r}$ its rth zero.

Theorem 1.3. *Consider the* r*th smallest eigenvalue of the Laguerre ensemble in the freezing regime,* $Y_{r,N,\alpha}$*, which obeys a centered Gaussian distribution. Its variance*

$$\sigma_{r,N,\alpha} := [\Sigma^{(\alpha)}]_{rr} \tag{8}$$

follows the asymptotic limit

$$\sigma_{r,\alpha} := \lim_{N\to\infty} N\sigma_{r,N,\alpha} = \int_0^1 \frac{2J_\alpha(j_{\alpha,r}\sqrt{1-y})^2}{y[J_{\alpha-1}(j_{\alpha,r}) - J_{\alpha+1}(j_{\alpha,r})]^2}\,\mathrm{d}y. \tag{9}$$

We show a plot of $\sigma_{r,\alpha}$ as a function of α for various values of r in Fig. 1. There, it can be seen that these variances increase with α, which is consistent with the intuition that as α grows, the eigenvalues are pushed away from the origin and hence have more space to fluctuate.

As is apparent from Thm. 1.3, the proof of this statement is based on the asymptotics of

$$\sqrt{\frac{z_{r,N,\alpha}}{N+\alpha}} \frac{\tilde{L}_{N-k}^{(\alpha)}(z_{r,N,\alpha})}{\tilde{L}_{N-1}^{(\alpha)}(z_{r,N,\alpha})} \tag{10}$$

as $N \to \infty$. We first introduce a cubic spline to interpolate each of the points given by this quantity in Sec. 2. Then, we show that the spline converges uniformly to the Bessel function of the first kind by using the recurrence relation of Laguerre polynomials in Sec. 3. We complete the proof by showing that the sum in (7) for $i = j = r$ converges to the integral given in Thm. 1.3 in Sec. 4. We end the paper with a few concluding remarks in Sec. 5.

2 Interpolation by cubic splines

Let us introduce the cubic spline $\varphi_N(y) \in \mathcal{C}^2([0,1])$ following [10]. For every $j = 0, \ldots, N$, we impose

$$\varphi_N(j/N) = \sqrt{\frac{z_{r,N,\alpha}}{N+\alpha}} \frac{\tilde{L}^{(\alpha)}_{N-j-1}(z_{r,N,\alpha})}{\tilde{L}^{(\alpha)}_{N-1}(z_{r,N,\alpha})} =: F_j, \ j = 0, \ldots, N. \tag{11}$$

where we have abbreviated the parameter α and introduced the notation F_j for brevity. φ_N interpolates between j/N and $(j+1)/N$ via a third-order polynomial such that φ_N and its first two derivatives are continuous for all $y \in (0,1)$. This means that when $j/N \leq y \leq (j+1)/N$, the second derivative of $\varphi_N(y)$ must be a linear function, and introducing the notation $\varphi''_N(j/N) = M_j$, we can write

$$\begin{aligned} \varphi''_N(y) &= N(M_{j+1} - M_j)(y - j/N) + M_j \\ &= N(M_{j+1}(y - j/N) + M_j((j+1)/N - y)), \ j/N \leq y \leq (j+1)/N. \end{aligned}$$

Integrating twice and imposing $\varphi_N(j/N) = F_j$, $\varphi_N((j+1)/N) = F_{j+1}$ allows us to write

$$\begin{aligned} \varphi_N(y) =& \frac{NM_{j+1}}{6}(y - j/N)^3 + \frac{NM_j}{6}((j+1)/N - y)^3 \\ &+ (NF_{j+1} - M_{j+1}/6N)(y - j/N) \\ &+ (NF_j - M_j/6N)((j+1)/N - y). \end{aligned} \tag{12}$$

The most important equation is obtained from the requirement of continuous derivatives at every j/N, as it allows us to compute all of the $\{M_j\}_{j=0}^{N-1}$ given the $\{F_j\}_{j=0}^{N}$ and two of the M's.

$$N^2(F_{j+1} - 2F_j + F_{j-1}) = (M_{j+1} + 4M_j + M_{j-1})/6, j = 1, \ldots, N-2. \tag{13}$$

There exist several ways to choose two of the M's for solving this system of equations. Here, we will only assume that $M_0 = M_1 = 0$, that is, the first section of the spline is a linear function. Then, $\varphi_N(y)$ is determined uniquely.

3 Recurrence and convergence to the Bessel differential equation

Our strategy is predicated in understanding the behavior of φ_N as $n \to \infty$. In order to do this, we look at the recurrence relation obeyed by the de Boor-Saff dual polynomials of $\{\tilde{L}^{(\alpha)}\}_{j=0}^{N-1}$ [9]. It was shown in [7] that the dual polynomials $\{\tilde{Q}^{(\alpha)}_j\}_{j=0}^{N-1}$ satisfy

$$\tilde{Q}^{(\alpha)}_j(z_{r,N,\alpha}) = \frac{\tilde{L}^{(\alpha)}_{N-j-1}(z_{r,N,\alpha})}{\tilde{L}^{(\alpha)}_{N-1}(z_{r,N,\alpha})}. \tag{14}$$

Moreover, it can be shown that the recurrence relation for

$$\tilde{L}_j^{(\alpha)}(x) = \sqrt{\frac{\Gamma(\alpha+1)j!}{\Gamma(j+\alpha+1)}} L_j^{(\alpha)}(x) \tag{15}$$

is

$$x\tilde{L}_j^{(\alpha)}(x) = -\sqrt{(j+1)(j+1+\alpha)}\tilde{L}_{j+1}^{(\alpha)}(x) + (2j+\alpha+1)\tilde{L}_j^{(\alpha)}(x) - \sqrt{j(j+\alpha)}\tilde{L}_{j-1}^{(\alpha)}(x). \tag{16}$$

It turns out that the defining characteristic of the dual polynomials is that the coefficients in their recurrence relation have the same form, with some indexes reversed. Specifically, $j+1$ and j in the first and third coefficients are replaced by $N-j+1$ and $N-j$ respectively, while j in the second coefficient is replaced by $N-j+1$. The result is

$$\begin{aligned} x\tilde{Q}_j^{(\alpha)}(x) = &- \sqrt{(N-j-1)(N-j-1+\alpha)}\tilde{Q}_{j+1}^{(\alpha)}(x) \\ &+ (2(N-j)+\alpha-1)\tilde{Q}_j^{(\alpha)}(x) \\ &- \sqrt{(N-j)(N-j+\alpha)}\tilde{Q}_{j-1}^{(\alpha)}(x), \; j=0,\dots,N-2. \end{aligned} \tag{17}$$

By convention, we set $\tilde{Q}_{-1}^{(\alpha)}(x) = 0$ and $\tilde{Q}_0^{(\alpha)}(x) = 1$. By (11) and (14), we can transform (17) into

$$\begin{aligned} z_{r,N,\alpha}F_j = &- \sqrt{(N-j-1)(N-j-1+\alpha)}F_{j+1} \\ &+ (2(N-j)+\alpha-1)F_j \\ &- \sqrt{(N-j)(N-j+\alpha)}F_{j-1}, \; j=0,\dots,N-2. \end{aligned} \tag{18}$$

We are now ready to prove the following claim.

Lemma 3.1. *The spline $\varphi_N(y)$ converges uniformly to the solution of*

$$(1-y)^2 f''(y) - (1-y)f'(y) + \frac{1}{4}(j_{\alpha,r}^2(1-y) - \alpha^2)f(y) = 0 \tag{19}$$

with boundary conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = j_{\alpha,r}/2. \tag{20}$$

Proof. We begin by finding asymptotics for the difference equation (18). We recall the asymptotics of $z_{r,N,\alpha}$ due to [11],

$$z_{r,N,\alpha} = \frac{j_{\alpha,r}^2}{4N+2(\alpha+1)} + O(N^{-2}), \tag{21}$$

as well as the expansions

$$\begin{aligned} \sqrt{(1-x)(1+(\alpha-1)x)} &= 1 + \frac{\alpha-2}{2}x - \frac{\alpha^2}{8}x^2 + O(x^3), \quad \text{and} \\ \sqrt{1+\alpha x} &= 1 + \frac{\alpha}{2}x - \frac{\alpha^2}{8}x^2 + O(x^3). \end{aligned} \tag{22}$$

Let us now fix $y \in (0,1)$. Then, for every N, we can find a $k(y,N)$ such that $k(y,N)-1/2 < yN < k(y,N)+1/2$, and it is clear that $\lim_{N\to\infty} k(y,N)/N = y$. In (18), we fix $j = k(y,N)$ while writing $k = k(y,N)$ for brevity, and use the expansions to obtain

$$\begin{aligned}&\frac{j_{\alpha,r}^2(N-k)}{4N}F_k + (N-k)^2\Big[1+\frac{\alpha-2}{2(N-k)}-\frac{\alpha^2}{8(N-k)^2}\Big]F_{k+1}\\ &-(N-k)^2\Big[2+\frac{\alpha-1}{N-k}\Big]F_k + (N-k)^2\Big[1+\frac{\alpha}{2(N-k)}-\frac{\alpha^2}{8(N-k)^2}\Big]F_{k-1}\\ &=O(N^{-1}).\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}&(N-k)^2\Big[F_{k+1}-2F_k+F_{k-1}\Big]\\ &+(N-k)\Big[\frac{\alpha-2}{2}(F_{k+1}-F_k)-\frac{\alpha}{2}(F_k-F_{k-1})\Big]\\ &+\frac{1}{4}\Big[j_{\alpha,r}^2(1-k/N)F_k-\alpha^2\frac{F_{k+1}+F_{k-1}}{2}\Big]=O(N^{-1}).\end{aligned}\tag{23}$$

Now we estimate $\varphi_N(y)$ by making use of the fact that $|y-k/N|$ is of order N^{-1}. We obtain

$$\begin{aligned}\varphi_N(y) =& F_k + O(N^{-1}) = \frac{F_{k+1}+F_{k-1}}{2} + O(N^{-1}), \quad \text{and}\\ \varphi_N'(y) =& N(F_{k+1}-F_k) + O(N^{-1}) = N(F_k - F_{k-1}) + O(N^{-1}).\end{aligned}$$

Now, note that the first line is just the lhs of (13), which itself can be rewritten as

$$\begin{aligned}N^2[F_{k+1}-2F_k+F_{k-1}] &= \frac{1}{3}\Big[\frac{M_{k+1}-M_k}{2}+M_k+\frac{M_k-M_{k-1}}{2}\Big]\\ &=\frac{1}{3}[\varphi_N''\Big(\frac{k+1/2}{N}\Big)+\varphi_N''(k/N)+\varphi_N''\Big(\frac{k-1/2}{N}\Big)].\end{aligned}$$

This is an equally weighted average of the second derivative for $k-1/2<y<k+1/2$, and this value can deviate from $\varphi_N''(y)$ at most a quantity of order N^{-1} because the variation of $\varphi_N''(y)$ at each interpolation segment is linear. Therefore we can write

$$\varphi_N''(y) = N^2[F_{k+1}-2F_k+F_{k-1}] + O(N^{-1}).\tag{24}$$

We combine all of these results together with the fact that $k(y,N)/N = y + O(N^{-1})$ to write

$$(1-y)^2\varphi_N''(y)-(1-y)\varphi_N'(y)+\frac{1}{4}[j_{\alpha,r}^2(1-y)-\alpha^2]\varphi_N(y) = g(y,N),\tag{25}$$

where $g(y,N)$ is a function of order N^{-1} for every $y\in(0,1)$. Taking the limit proves pointwise convergence to the differential equation, so it remains to show that the boundary conditions are correct. First, we can write

$$\lim_{N\to\infty}\varphi_N(0) = \lim_{N\to\infty}\sqrt{\frac{z_{r,N,\alpha}}{N+\alpha}} = \lim_{N\to\infty}\sqrt{\frac{j_{\alpha,r}^2}{(N+\alpha)(4N+2(\alpha+1))}} = 0.\tag{26}$$

Secondly, using (18) with $j = 0$, we obtain

$$\lim_{N\to\infty} \varphi_N'(0) = \lim_{N\to\infty} N\sqrt{\frac{z_{r,N,\alpha}}{N+\alpha}}\Big[\frac{2N+\alpha-1-z_{r,N,\alpha}}{\sqrt{(N-1)(N-1+\alpha)}} - 1\Big] = \frac{j_{\alpha,r}}{2}, \qquad (27)$$

because we set the first segment of the spline so that it would be linear. Finally, we make use of Thm. 1 of [12], which guarantees the uniform convergence of $\varphi_N(y)$ now that we have shown that it converges pointwise to the solution of (19). □

The solution of (19) can be found immediately.

Corollary 3.2. *The solution $f(y)$ of* (19) *is given by*

$$f(y) = \frac{2J_\alpha(j_{\alpha,r}\sqrt{1-y})}{J_{\alpha+1}(j_{\alpha,r}) - J_{\alpha-1}(j_{\alpha,r})}. \qquad (28)$$

Proof. Performing the variable substitution $u = j_{\alpha,r}\sqrt{1-y}$ in (19) yields the Bessel differential equation of parameter α. Because the solution must vanish at $y = 0$, the solution of the Bessel differential equation must vanish at $u = j_{\alpha,r}$, which rules out the Bessel function of second kind. Finally, computing the derivative at zero completes the calculation. □

4 Asymptotic behavior of the variance

In this section we give the proof of Thm. 1.3. The main strategy will be to argue that the summand is positive and bounded by a quantity independent of N or y in order to make use of the dominated convergence theorem.

Proof of Thm. 1.3. We begin by rewriting the expression of the rth variance in terms of φ_N.

$$\sigma_{r,N,\alpha} = \sum_{k=0}^{N-1} \frac{1}{2(k+1)}\frac{1}{N}\varphi_{N,r}^2(k/N). \qquad (29)$$

We have included the subscripts r and α to make explicit the dependence of the sum on these parameters. One can immediately rewrite this object in terms of a Lebesgue integral.

$$N\sigma_{r,N,\alpha} = \sum_{k=0}^{N-1} \frac{N}{2(k+1)}\frac{1}{N}\varphi_{N,r}^2(k/N) = \frac{1}{2}\int_0^1 \frac{N}{\lfloor yN\rfloor + 1}\varphi_{N,r}^2(\lfloor yN\rfloor/N)\,\mathrm{d}y. \qquad (30)$$

By construction, $\varphi_{N,r}(y)$ is a continuous function. Moreover, because the matrix $(v_1, \ldots, v_N)$ is orthogonal, both its row and its column vectors are mutually orthonormal. This means that the L^2 norm of the rth row reads

$$\sum_{k=0}^{N-1} \frac{1}{N}\varphi_{N,r}^2(k/N) = \int_0^1 \varphi_{N,r}^2(\lfloor yN\rfloor/N)\,\mathrm{d}y = 1,$$

and by the Hölder inequality,

$$\Big|\int_0^1 \varphi_{N,r}(\lfloor yN\rfloor/N)\,\mathrm{d}y\Big| = \Big|\sum_{k=0}^{N-1}\frac{1}{N}\varphi_{N,r}^2(k/N)\Big| \le \sqrt{\Big(\sum_{k=0}^{N-1}\frac{1}{N}\Big)\Big(\sum_{k=0}^{N-1}\frac{1}{N}\varphi_{N,r}^2(k/N)\Big)} \le 1,$$

which means that the spline $\varphi_{N,r}(y)$ is an integrable function on $(0,1)$. Consequently, $\varphi_{N,r}(y)$ is a continuous and bounded function for every N that converges uniformly to a function proportional to $J_\alpha(j_{\alpha,r}\sqrt{1-y})$. Therefore, the only problem lies in the behavior of $N/(\lfloor yN\rfloor+1)$ close to $y=0$, as it is clearly bounded elsewhere in the limit and it converges pointwise to y^{-1}. First note that for $0\le y<1/N$,

$$\begin{aligned} 0 &< \frac{N}{\lfloor yN\rfloor+1}\varphi_{N,r}^2(\lfloor yN\rfloor/N) \le N\varphi_{N,r}^2(1/N) \\ &= N\frac{z_{r,N,\alpha}}{N+\alpha}\Big[\frac{2N+\alpha-1-z_{r,N,\alpha}}{\sqrt{(N-1)(N-1+\alpha)}}\Big]^2 = \frac{j_{\alpha,r}^2}{N}+O(N^{-1}), \end{aligned} \tag{31}$$

which means that for N sufficiently large and $0\le y<1/N$,

$$\frac{N}{\lfloor yN\rfloor+1}\varphi_{N,r}^2(\lfloor yN\rfloor/N) \le j_{\alpha,r}^2.$$

This bound is independent of N, which means that there exists a constant C which bounds the integrand of (30) for every $y\in(0,1)$ and every N. Therefore, we can use the dominated convergence theorem to prove our claim. □

5 Concluding remarks

In our main result, we obtained an asymptotic expression for the variance of the rth smallest eigenvalue of a Laguerre random matrix in the freezing regime which is given by the integral of a Bessel function. This is natural, as it is well-known that the hard edge is ruled by Bessel statistics [13]. In this sense, we expect a similar result for the extreme eigenvalues of a Jacobi random matrix in the freezing regime, which are also characterized by Bessel statistics.

The results in this paper are only a small sample of the many aspects that can be studied in the freezing limit of random matrix ensembles. We expect many more developments in the future, such as the calculation of eigenvalue density profiles, correlation functions and other quantities of interest.

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